Electromagnetic Self-Force on a Hemispherical Cavity

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1 Problem

That a moving charge interacting with thermal radiation should feel a radiation pressure was anticipated by Stewart in 1871-3 [1], who inferred that both the energy and the momentum of the charge would be affected.[1] In 1873, Maxwell discussed the pressure of light/electromagnetic waves on conducting media at rest, and on "the medium in which waves are propagated" (Arts. 792-793 of [3]).

In 1876, Crookes demonstrated his famous radiometer (aka "light mill"), and speculated that it was driven by the pressure of light [4]. However, it was observed by Schuster [5, 6] that the rotation of the radiometer was opposite to that consistent with radiation pressure, and instead was due to thermal effects in the residual gas inside the device.[2]

The possibility of electromagnetic propulsion was revived in 1949 by Slepian in a delightful mock-proposal of an "electromagnetic spaceship" [8, 9] that appeared to violate the known laws of physics.[3] Of course, propulsion is possible for a system that emits a directed beam of electromagnetic waves/photons, but the thrust to power ratio is low, $F/P = (dp/dt)/(dE/dt) = p/E = 1/c = 3.33 \times 10^{-9}$ N/W, where p is the momentum of the emitted photons, $E = pc$ is their energy, and c is the speed of light in vacuum.[4]

There remains a hope for some people that propulsion systems could be developed that violate the known laws of physics,[5] and in particular closed cavities containing electromagnetic waves could exhibit a self force, in contradiction to Newton's third law and conservation of energy. One such speculation is based on a tapered cavity, for which the supposed self force would be towards the smaller end of the cavity [12].[6] Various experiments purport to confirm this concept (for example, [13, 14, 15]), although as for the case of Crookes' radiometer, thermal effects may well be the source of the tiny observed thrust [16].

Analytic computation of the fields and force on a tapered cavity are not possible, and numerical simulations will always predict a small nonzero force due to the limited accuracy of such modeling [17].[7] Consider instead a hemispherical cavity with perfectly conducting

[1] In 1871, light and thermal radiation were considered to be wave phenomena. In a corpuscular theory of light, as per Newton, it was natural to suppose that light exerted a pressure. For a review, see [2].

[2] For a review of experiments that eventually demonstrated the radiation pressure of light, see [7].

[3] Slepian's point was that such a scheme would not work, contrary to expectations from a naïve analysis.

[4] For a review of efforts on "photon rockets", see [10].

[5] For a review of "bootstrap spaceships", see [11].

[6] The logic here is not clear, in that if electromagnetic waves exert a pressure analogous to water pressure, the self force on a tapered electromagnetic cavity should be like the total force of water pressure on a tapered container of water, *i.e.*, zero. If the cavity walls were very thin, of order the 1.5-μm skin depth at, say, 2 GHz, the cavity would radiate externally, and a tapered cavity might then provide a small net propulsive force.

[7] Likewise, no experiment can ever "prove" that a quantity with a continuous distribution of values is zero. The challenge in measurements of such quantities is to convince the reader that the nonzero result is significant, and due to an hypothesized effect. Refs. [13, 14, 15] are deficient in the latter respect.

walls, and compute the total force on these walls when the cavity is excited in its lowest mode, to confirm that this is zero according to Maxwell's equations.

2 Solution

2.1 General Discussion

In Maxwell's theory, the force on the cavity walls can be computed by integrating the Maxwell stress tensor (Arts. 639-646 of [3]), once the electromagnetic fields are known. Here, we review a derivation of the Maxwell stress tensor, starting from the Lorentz force density $\mathbf{f}$ on charge and current densities ρ and $\mathbf{J}$ (in Gaussian units),[8]

$$\mathbf{f} = \frac{d\mathbf{p}_{\rm mech}}{dt} = \rho\mathbf{E} + \frac{\mathbf{J}}{c} \times \mathbf{B}, \tag{1}$$

where $\mathbf{p}_{\rm mech}$ is the density of mechanical momentum in a volume V where the matter is subject only to electromagnetic forces. Assuming the volume to be vacuum outside the charge and current densities, we can use Maxwell's equations to replace ρ and $\mathbf{J}$ in favor of the electromagnetic fields,

$$\begin{aligned}
\mathbf{f} = \frac{d\mathbf{p}_{\rm mech}}{dt} &= \frac{\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E})}{4\pi} - \frac{\mathbf{B}}{4\pi} \times (\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B} \times \frac{1}{4\pi c}\frac{\partial\mathbf{E}}{\partial t} \\
&= -\frac{\partial}{\partial t}\frac{\mathbf{E}\times\mathbf{B}}{4\pi c} + \left[\mathbf{E}\frac{\boldsymbol{\nabla}\cdot\mathbf{E}}{4\pi} + \frac{\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{B}}{4\pi} - \frac{\mathbf{E}}{4\pi}\times(\boldsymbol{\nabla}\times\mathbf{E}) - \frac{\mathbf{B}}{4\pi}\times(\boldsymbol{\nabla}\times\mathbf{B})\right] \\
&\equiv -\frac{\partial\mathbf{p}_{\rm EM}}{\partial t} + \boldsymbol{\nabla}\cdot\mathsf{T}_{\rm EM},
\end{aligned} \tag{2}$$

where,

$$\mathbf{p}_{\rm EM} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi c} \tag{3}$$

is the density of momentum associated with the electromagnetic field, and in vacuum,

$$T_{{\rm EM},ij} = \frac{1}{4\pi}\left(E_iE_j + B_iB_j - \delta_{ij}\frac{E^2+B^2}{2}\right) \tag{4}$$

is the symmetric Maxwell stress 3-tensor associated with the electromagnetic fields. To arrive at eq. (4) we note that,

$$[\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E}) - \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E})]_i = E_i\frac{\partial E_j}{\partial x_j} - E_j\frac{\partial E_j}{\partial x_i} + E_j\frac{\partial E_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left[E_iE_j - \delta_{ij}\frac{E^2}{2}\right]. \tag{5}$$

The total force $\mathbf{F}_V = \int_V \mathbf{f}\, d{\rm Vol}$ on the charges inside volume V is,

$$\begin{aligned}
\mathbf{F}_V &= \int_V \frac{d\mathbf{p}_{\rm mech}}{dt}\, d{\rm Vol} = \frac{d\mathbf{P}_{\rm mech}}{dt} = -\frac{d}{dt}\int_V \mathbf{p}_{\rm EM}\, d{\rm Vol} + \int_V \boldsymbol{\nabla}\cdot\mathsf{T}_{\rm EM}\, d{\rm Vol} \\
&= -\frac{d\mathbf{P}_{\rm EM}}{dt} + \int_S \mathsf{T}\cdot d\mathbf{Area},
\end{aligned} \tag{6}$$

[8]Maxwell discussed the "Lorentz" force law in Art. 599 of [3], but did not use it when deducing his stress tensor. The first derivation in the manner used here may be that on p. 24 of [18]. See also sec. 8.2.2 of [19].

where $\mathbf{P}_{\rm EM} = \int_V \mathbf{E} \times \mathbf{B}\, d{\rm Vol}/4\pi c$ is the total electromagnetic field momentum in volume V, whose bounding surface is S, and the area element is directed out of the volume.

One consequence of eq. (5) is that if volume V extends to infinity, where the electromagnetic fields fall off sufficiently quickly, then $\int_S \mathsf{T} \cdot d\mathbf{Area} \to 0$, and the total momentum $\mathbf{P}_{\rm mech} + \mathbf{P}_{\rm EM}$ is constant. For example, if the charges and currents emit electromagnetic radiation that carries $\mathbf{P}_{\rm EM}$, then the matter of the system takes on momentum $\mathbf{P}_{\rm mech} = -\mathbf{P}_{\rm EM}$, which is the principle of the "photon rocket".[9]

Another consequence of eq. (5) is that the total force $\mathbf{F}_S$ on the bounding surface S (whether or not that surface is a perfect conductor) can be written as,

$$\mathbf{F}_S = -\int_S \mathsf{T} \cdot d\mathbf{Area} = -\mathbf{F}_V - \frac{d\mathbf{P}_{\rm EM}}{dt}\,. \tag{7}$$

In particular, if there are no charges or currents in volume V, and the time-average of the electromagnetic fields is constant, then the time-average force on the bounding surface is zero. These are the supposed conditions of the experiments [13, 14, 15], so according to the laws of mechanics and electromagnetism, the electromagnetic self-force on the rf cavity in those experiments is zero when the fields are steady.[10]

The rest of this note considers the particular example of a hemispherical cavity with perfectly conducting walls. Appendix A reviews the fact, "well known to those who know well", that the $\mathbf{E}$ and $\mathbf{B}$ fields of a standing wave inside a cavity are 90° out of phase, while Appendix B reviews the modes of a rectangular cavity.

2.2 Electromagnetic Fields of a Resonant Hemispherical Cavity

The electromagnetic fields can be deduced by a solution to the Helmholtz equation in spherical coordinates (see, for example, [22] and sec. 9.24 of [23]),[11] but we adopt a different approach, based on the superposition of the advanced and retarded fields of a "point" (Hertzian) electric dipole. For the latter, our discussion parallels sec. 9.1-2 of [25] and secs. 62, 66-67 of [26].

We consider a hemispherical cavity of radius a, operated at vacuum, and desire the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ with time dependence $e^{-i\omega t}$ for the lowest resonant angular frequency ω. Assuming the cavity walls to perfectly conducting, then the fields at the cavity's interior surface obey,

$$\mathbf{E}_{\parallel} = 0 = \mathbf{B}_{\perp}. \tag{8}$$

We work in Gaussian units, and a spherical coordinate system (r, θ, ϕ), such that the interior of the cavity is at $r < a$, $0 < \theta < \pi/2$.

[9] For a review, see [10]. For a discussion of possible "rocket propulsion" in AC circuits, see [20].

[10] For zero charge and current density within volume V, the force on the cavity surface is $F_S = -d\mathbf{P}_{\rm EM}/dt$ and the mechanical momentum of the cavity (if subject to no external forces) is $\mathbf{P}_{\rm mech}(t) = -\mathbf{P}_{\rm EM}(t)$. If the steady-state value of the stored field momentum $\mathbf{P}_{\rm EM}$ is nonzero, then there is a net transient force on the system when the fields are built up from zero. The character of the associated net mechanical momentum of the system is delicate, and sometimes is called "hidden" mechanical momentum. Even the "simple" example of a coaxial cable with a battery at one end and a resistor at the other is quite subtle [21].

[11] For a "textbook" discussion, see sec. 19.6.3 of [24].

We first deduce the lowest mode for a spherical cavity, finding that the fields also satisfy the boundary conditions for a hemispherical cavity.

In general, the electromagnetic fields can be deduced from scalar and vector potentials V and $\mathbf{A}$ according to,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{9}$$

where c is the speed of light in vacuum. In case of time dependence $e^{-i\omega t}$, the Ampère-Maxwell equation in vacuum becomes,

$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = -\frac{i\omega}{c}\mathbf{E}, \qquad \mathbf{E} = \frac{i}{k}\boldsymbol{\nabla} \times \mathbf{B}, \tag{10}$$

where $k = \omega/c$ is the wave number; both $\mathbf{E}$ and $\mathbf{B}$ can be deduced from the vector potential $\mathbf{A}$.

As first noted by Lorenz [27], if the potentials satisfy the (Lorenz) gauge condition,

$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{1}{c}\frac{\partial V}{\partial t}, \tag{11}$$

then the vector potential can be related to the source current density $\mathbf{J}$ by,

$$\mathbf{A}_{\pm}(\mathbf{x}, t) = \int \frac{\mathbf{J}(\mathbf{x}', t'_{\pm} = t \pm R/c)}{cR}\, d^3\mathbf{x}', \qquad R = |\mathbf{x} - \mathbf{x}'|\,. \tag{12}$$

The form $\mathbf{A}_+$ is called the advanced potential and is seldom used, although it is a valid mathematical solution. The form $\mathbf{A}_-$ is the more familiar retarded potential.

For time dependence $\mathbf{J}(\mathbf{x}, t) = \mathbf{J}(\mathbf{x})\, e^{-i\omega t}$ the vector potentials (12) becomes,

$$\mathbf{A}_{\pm}(\mathbf{x}, t) = \int \frac{\mathbf{J}(\mathbf{x}')\, e^{-i\omega t'_{\pm}}}{cR}\, d^3\mathbf{x}' = \int \frac{\mathbf{J}(\mathbf{x}')\, e^{\mp ikR}}{cR}\, d^3\mathbf{x}'\, e^{-i\omega t}, \qquad \mathbf{A}_{\pm}(\mathbf{x}) = \int \frac{\mathbf{J}(\mathbf{x}')\, e^{\mp ikR}}{cR}\, d^3\mathbf{x}'. \tag{13}$$

We now specialize to the case of an oscillating, "point" electric dipole at the origin, consisting of charges $\pm q$ at positions $\mathbf{z}_{\pm} = \pm(z_0/2)\, e^{-i\omega t}\, \hat{\mathbf{z}}$, so the dipole moment is $\mathbf{p} = \mathbf{p}_0\, e^{-i\omega t}$, with $\mathbf{p}_0 = p_0\, \hat{\mathbf{z}}$ and $p_0 = qz_0$. The velocities of the charges $\pm q$ are $\mathbf{v}_{\pm} = \mp i\omega(z_0/2)\, e^{-i\omega t}\, \hat{\mathbf{z}}$, and the current density associated with the oscillating dipole is,

$$\mathbf{J}(\mathbf{x}, t) = q\mathbf{v}_+ + (-q)\mathbf{v}_- = -i\omega \mathbf{p} = -i\omega \mathbf{p}_0\, e^{-i\omega t}, \qquad \mathbf{J}(\mathbf{x}) = -i\omega \mathbf{p}_0. \tag{14}$$

Using this in eq. (13), R becomes simply the radial coordinate r of the observation point $\mathbf{x}$, such that,

$$\mathbf{A}_{\pm}(\mathbf{x}, t) = -ik\mathbf{p}_0 \frac{e^{i(\mp kr - \omega t)}}{r}. \tag{15}$$

The electromagnetic fields are then,

$$\begin{aligned} \mathbf{B}_{\pm} &= \boldsymbol{\nabla} \times \mathbf{A}_{\pm} = -ik\boldsymbol{\nabla}\frac{e^{i(\mp kr - \omega t)}}{r} \times \mathbf{p}_0 = -ik\frac{e^{i(\mp kr - \omega t)}}{r}\left(\mp ik\hat{\mathbf{r}} - \frac{\hat{\mathbf{r}}}{r}\right) \times \mathbf{p}_0 \\ &= k^2\, \hat{\mathbf{r}} \times \mathbf{p}_0 \frac{e^{i(\mp kr - \omega t)}}{r}\left(\mp 1 + \frac{i}{kr}\right), \end{aligned} \tag{16}$$

and,

$$
\begin{aligned}
\mathbf{E}_\pm &= \frac{i}{k}\boldsymbol{\nabla}\times\mathbf{B}_\pm = -\boldsymbol{\nabla}\times\left[e^{i(\mp kr-\omega t)}\left(\frac{\pm ik}{r^2}+\frac{1}{r^3}\right)\mathbf{r}\times\mathbf{p}_0\right] \\
&= -\boldsymbol{\nabla}e^{i(\mp kr-\omega t)}\left(\frac{\pm ik}{r^2}+\frac{1}{r^3}\right)\times(\mathbf{r}\times\mathbf{p}_0) - e^{i(\mp kr-\omega t)}\left(\frac{\pm ik}{r^2}+\frac{1}{r^3}\right)\boldsymbol{\nabla}\times(\mathbf{r}\times\mathbf{p}_0) \\
&= e^{i(\mp kr-\omega t)}\left[\mp\frac{k^2}{r}+3\left(\frac{ik}{r^2}\pm\frac{1}{r^3}\right)\right]\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{p}_0)+2\mathbf{p}_0\,e^{i(\mp kr-\omega t)}\left(\pm\frac{ik}{r^2}+\frac{1}{r^3}\right) \\
&= e^{i(\mp kr-\omega t)}\left\{\left[\mp\frac{k^2}{r}+3\left(\frac{ik}{r^2}\pm\frac{1}{r^3}\right)\right](\mathbf{p}_0\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\pm\left[\frac{k^2}{r}\mp\frac{ik}{r^2}-\frac{1}{r^3}\right]\mathbf{p}_0\right\}. \qquad (17)
\end{aligned}
$$

Since Maxwell's equations are linear, the combinations $\mathbf{B}_- - \mathbf{B}_+$ and $\mathbf{E}_- - \mathbf{E}_+$ are also possible solutions, and these correspond to total source electric dipoles moments of $\mathbf{p}-\mathbf{p}=0$. If these combined fields satisfy the boundary conditions (8), then they are possible fields inside a spherical cavity.

Suppressing the common factor $e^{-i\omega t}$, the combined fields are,

$$
\begin{aligned}
\mathbf{B} &= \mathbf{B}_- - \mathbf{B}_+ = k^2\,\hat{\mathbf{r}}\times\mathbf{p}_0\left(\frac{e^{ikr}+e^{-ikr}}{r}+\frac{i(e^{ikr}-e^{-ikr})}{kr^2}\right) \\
&= 2k^2\,\hat{\mathbf{r}}\times\mathbf{p}_0\left(\frac{\cos kr}{r}-\frac{\sin kr}{kr^2}\right), \qquad (18)
\end{aligned}
$$

and,

$$
\begin{aligned}
\mathbf{E} &= \mathbf{E}_- - \mathbf{E}_+ = \left[\left(\frac{k^2}{r}-\frac{3}{r^3}\right)(e^{ikr}-e^{-ikr})+3\frac{ik}{r^2}(e^{ikr}+e^{-ikr})\right](\mathbf{p}_0\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} \\
&\quad -\left[\left(\frac{k^2}{r}-\frac{1}{r^3}\right)(e^{ikr}-e^{-ikr})+\frac{ik}{r^2}(e^{ikr}+e^{-ikr})\right]\mathbf{p}_0 \\
&= 2i\left[\left(\frac{k^2}{r}-\frac{3}{r^3}\right)\sin kr+3\frac{k}{r^2}\cos kr\right](\mathbf{p}_0\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} \\
&\quad -2i\left[\left(\frac{k^2}{r}-\frac{1}{r^3}\right)\sin kr+\frac{k}{r^2}\cos kr\right]\mathbf{p}_0. \qquad (19)
\end{aligned}
$$

Finally, we take $\mathbf{p}_0 = 3i(E_0/4k^3)\,\hat{\mathbf{z}}$ to write,

$$
\mathbf{B} = -\frac{3iE_0}{2}\left(\frac{\sin kr}{k^2r^2}-\frac{\cos kr}{kr}\right)\sin\theta\,\hat{\boldsymbol{\phi}}, \qquad (20)
$$

$$
\begin{aligned}
\mathbf{E} &= -\frac{3E_0}{2}\left[\left(\frac{1}{kr}-\frac{3}{k^3r^3}\right)\sin kr+\frac{3\cos kr}{k^2r^2}\right]\cos\theta\,\hat{\mathbf{r}} \\
&\quad +\frac{3E_0}{2}\left[\left(\frac{1}{kr}-\frac{1}{k^3r^3}\right)\sin kr+\frac{\cos kr}{k^2r^2}\right](\cos\theta\,\hat{\mathbf{r}}-\sin\theta\,\hat{\boldsymbol{\theta}}) \\
&= 3E_0\left(\frac{\sin kr}{k^3r^3}-\frac{\cos kr}{k^2r^2}\right)\cos\theta\,\hat{\mathbf{r}}-\frac{3E_0}{2}\left(\frac{\sin kr}{kr}-\frac{\sin kr}{k^3r^3}+\frac{\cos kr}{k^2r^2}\right)\sin\theta\,\hat{\boldsymbol{\theta}}. \qquad (21)
\end{aligned}
$$

As $r\to 0$, $\mathbf{B}\to 0$ and $\mathbf{E}\to E_0\,\hat{\mathbf{z}}$, so the combined fields are well behaved at the origin even though the partial fields (16)-(17) diverge there.

We also recognize that fields could be expressed in terms of the so-called spherical Bessel functions, as found in the more usual derivations [22, 23],

$$j_0(x) = \frac{\sin x}{x}, \quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \quad j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3\cos x}{x^2}, \quad ... \tag{22}$$

Indeed, the field components of eqs. (20)-(21) can be written as,

$$B_\phi = -\frac{3iE_0}{2} j_1(kr)\sin\theta, \tag{23}$$

$$E_r = 3E_0 \frac{j_1(kr)}{kr}\cos\theta, \tag{24}$$

$$E_\theta = -\frac{3E_0}{2}\left[j_0(kr) - \frac{j_1(kr)}{kr}\right]\sin\theta = -\frac{3E_0}{2kr}\left[kr\, j_1(kr)\right]'\sin\theta. \tag{25}$$

The fields (20)-(21) satisfy the boundary conditions (8) for both a sphere and hemisphere of radius a, provided $E_\theta(r = a) = 0$,

$$0 = \sin ka\left(\frac{1}{ka} - \frac{1}{k^2a^3}\right) + \frac{1}{k^2a^2}\cos ka, \qquad \cos ka = \sin ka\left(\frac{1}{ka} - ka\right) \qquad \Rightarrow \qquad ka = 2.744. \tag{26}$$

This TM (transverse magnetic) mode is the lowest-frequency mode of the sphere or hemisphere. The fields of this mode are illustrated in the figure below, from [22].[12]

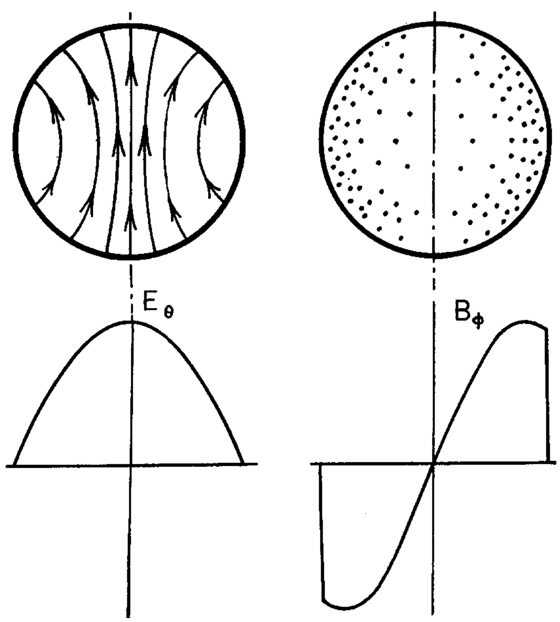

FIG. 1. The left-hand pair of drawings represent a possible field inside a spherical conducting shell at a time when B = 0. The upper picture is a cross section of the sphere, the lower is a graph showing the variation with *r* of the electric field in the equatorial plane. The right-hand drawings show the situation 1/4 cycle later when the electric field has disappeared and the magnetic field has taken up the energy. The vector B is entirely in the ϕ direction and its variation with *r* is shown in the graph below.

[12]The peak field at the "pole" $(x, y, z) = (0, 0, a)$ of the cavity is, from eqs (21) and (26), $3E_0 \sin ka/ka = 0.42E_0$. As such, thermionic emission of electrons is more probable from the base of the cavity than from its hemispherical surface. A current of electrons could flow inside the cavity from its base to its hemispherical surface. To satisfy overall momentum conservation, the momentum of the cavity would be equal and opposite to that of the thermionic current, such that the cavity would move in the $-z$ direction. This is opposite to the direction of the force observed in the experiments [13, 14, 15], and so the reaction force to a thermionic current appears not to explain these experiments.

*There exists a TE (*transverse electric*) mode for a spherical cavity with* $ka = 4.493$*, which can be found by a procedure similar to the above, but supposing the advanced and retarded fields are due to a "point" magnetic dipole. See prob. 8 of [28]. However, a hemispherical cavity does not support this mode.*

2.3 Stored Electromagnetic Field Momentum

The time-average electromagnetic field momentum stored in the hemispherical cavity is,

$$\mathbf{P}_{\rm EM} = Re \int_V \frac{\mathbf{E} \times \mathbf{B}^\star}{8\pi c}\, d{\rm Vol} = 0, \tag{27}$$

since the electric and magnetic fields (20)-(21) and (23)-(25) are 90° out of phase with one another.[13] Hence, there is no net mechanical momentum imparted to the cavity as the electromagnetic fields build up from zero.[14]

2.4 Force on the Cavity Walls

Given the forms (20)-(21) and (23)-(25) for the electromagnetic fields inside a hemispherical cavity with perfectly conducting walls, we can evaluate the resulting time-average force on these wall using the Maxwell stress tensor,

$$\mathsf{T}_{ij} = \frac{1}{8\pi} Re \left[E_i E_j^\star + B_i B_j^\star - \frac{\delta_{ij}}{2} \left(|E|^2 + |B|^2 \right) \right]. \tag{28}$$

On the base of the cavity, $z = 0$, $\theta = \pi/2$, $E_{r,{\rm base}} = 0$ and $E_{z,{\rm base}} = -E_{\theta,{\rm base}}$, while on the hemisphere, $r = a$, $E_{\theta,{\rm hemi}} = 0$. That is, on both the base and on the hemisphere, the fields $\mathbf{B}$ and $\mathbf{E}$ each have only a single nonzero component. The electromagnetic stress normal to the cavity base (in the inward direction) is given by,

$$\mathsf{T}_{zz,{\rm base}} = \mathsf{T}_{\theta\theta,{\rm base}} = \frac{1}{16\pi} \left(|E_{\theta,{\rm base}}|^2 - |B_{\phi,{\rm base}}|^2 \right), \tag{29}$$

where the electric stress is in the $+z$ direction, while the magnetic stress is in the $-z$ direction.[15] Similarly, the stress normal to the hemispherical surface of the cavity, where the electric field is purely radial, is,

$$\mathsf{T}_{rr,{\rm hemi}} = \frac{1}{16\pi} \left(|E_{r,{\rm hemi}}|^2 - |B_{\phi,{\rm hemi}}|^2 \right), \tag{30}$$

where the electric stress is inward and the magnetic stress is outward.

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[13] The steady-state electromagnetic fields inside any cavity can be regarded as the superposition of modes, with the $\mathbf{E}$ and $\mathbf{B}$ fields of each mode being 90° out of phase, as reviewed in Appendix A.

[14] There is also no net, time-average flow of energy inside the cavity, $\int \langle \mathbf{S} \rangle \, d{\rm Vol} = Re \int c\mathbf{E} \times \mathbf{B}^\star \, d{\rm Vol}/4\pi = 0$, as expected. When degenerate modes exist, it is possible that the time-average Poynting vector/flux of energy $\langle \mathbf{S} \rangle$ is nonzero inside the cavity, but the net (volume integral) flow of energy is zero, as illustrated in Appendix B for a cubical cavity.

[15] This follows from the memorable result that there exists a tension along field lines and a repulsion between them.

The electromagnetic force on the base of the cavity is,

$$
\begin{aligned}
F_{z,\text{base}} &= \int_0^a \mathsf{T}_{zz,\text{base}}\, 2\pi r\, dr = \frac{1}{8}\int_0^a (|E_{\theta,\text{base}}|^2 - |B_{\phi,\text{base}}|^2)\, r\, dr \\
&= \frac{1}{8}\left(\frac{3E_0}{2}\right)^2 \int_0^a \left\{ \left[\left(\frac{1}{kr} - \frac{1}{k^3r^3}\right)\sin kr + \frac{\cos kr}{k^2r^2}\right]^2 - \left(\frac{\sin kr}{k^2r^2} - \frac{\cos kr}{kr}\right)^2 \right\} r\, dr \\
&= \frac{9E_0^2}{32k^2}\int_0^{ka} \left\{ \left[\left(\frac{1}{x} - \frac{1}{x^3}\right)\sin x + \frac{\cos x}{x^2}\right]^2 - \left(\frac{\sin x}{x^2} - \frac{\cos x}{x}\right)^2 \right\} x\, dx \\
&= \frac{9E_0^2}{32k^2}\int_0^{ka} \left\{ \left[\left(\frac{1}{x^2} - \frac{2}{x^4} + \frac{1}{x^6}\right)\frac{1-\cos 2x}{2} + \left(\frac{1}{x^3} - \frac{1}{x^5}\right)\sin 2x + \frac{1+\cos 2x}{2x^4}\right] \right. \\
&\qquad \left. - \left[\frac{1-\cos 2x}{2x^4} - \frac{\sin 2x}{x^3} + \frac{1+\cos 2x}{2x^2}\right]\right\} x\, dx \\
&= \frac{9E_0^2}{32k^2}\int_0^{2ka} \left\{ \left[\left(\frac{1}{y} - \frac{8}{y^3} + \frac{16}{y^5}\right)\frac{1-\cos y}{2} + \left(\frac{2}{y^2} - \frac{8}{y^4}\right)\sin y + 2\frac{1+\cos y}{y^3}\right] \right. \\
&\qquad \left. - \left[\frac{2}{y^3}(1-\cos y) - \frac{2}{y^2}\sin y + \frac{1+\cos y}{2y}\right]\right\} dy \\
&= \frac{9E_0^2}{64k^2}\int_0^{2ka} \left[-\frac{8}{y^3} + \frac{16}{y^5} + \left(\frac{8}{y^2} - \frac{16}{y^4}\right)\sin y + \left(-\frac{2}{y} + \frac{16}{y^3} - \frac{16}{y^5}\right)\cos y\right] dy \\
&= \frac{9E_0^2}{64k^2}\left[\frac{4}{(2ka)^2} - \frac{4}{0^2} - \frac{4}{(2ka)^4} + \frac{4}{0^4} - \frac{8\sin 2ka}{2ka} + \frac{8\sin 0}{0} + 8\int_0^{2ka}\frac{\cos y}{y}\, dy \right. \\
&\qquad + \frac{16\sin 2ka}{3(2ka)^3} - \frac{16\sin 0}{3\cdot 0^3} + \frac{8\cos 2ka}{3(2ka)^2} - \frac{8}{3\cdot 0^2} - \frac{8\sin 2ka}{3(2ka)} + \frac{8\sin 0}{3\cdot 0} \\
&\qquad + \frac{8}{3}\int_0^{2ka}\frac{\cos y}{y}\, dy - 2\int_0^{2ka}\frac{\cos y}{y}\, dy \\
&\qquad - \frac{8\cos 2ka}{(2ka)^2} + \frac{8\cos 0}{0^2} + \frac{8\sin 2ka}{2ka} - \frac{8\sin 0}{0} - 8\int_0^{2ka}\frac{\cos y}{y}\, dy \\
&\qquad + \frac{4\cos 2ka}{(2ka)^4} - \frac{4\cos 0}{0^4} - \frac{4\sin 2ka}{3(2ka)^3} + \frac{4\sin 0}{3\cdot 0^3} \\
&\qquad \left. - \frac{2\cos 2ka}{3(2ka)^2} + \frac{2\cos 0}{3\cdot 0^2} + \frac{2\sin 2ka}{3(2ka)} - \frac{2\sin 0}{3\cdot 0} - \frac{2}{3}\int_0^{2ka}\frac{\cos y}{y}\, dy\right] \\
&= \frac{9E_0^2}{64k^2}\left[-\frac{1}{2} + \frac{1}{k^2a^2} - \frac{1}{4k^4a^4} + \cos 2ka\left(-\frac{3}{2k^2a^2} + \frac{1}{4k^4a^4}\right) - \frac{\sin 2ka}{ka} + \frac{\sin 2ka}{2k^3a^3}\right] \\
&= \frac{9E_0^2}{64k^2}\left[-\frac{1}{2} - \frac{1}{2k^2a^2} + \sin^2 ka\left(2 + \frac{1}{2k^4a^4}\right)\right], \qquad (31)
\end{aligned}
$$

where we have used Dwight 431.12, 431.13, 431.14, 441.13 and 441.13 [29]. Note that $\cos 0/0^2 = \lim_{y\to 0}(\cos y)/y^2 = 1/0^2 - 1/2$, $\sin 0/0^3 = \lim_{y\to 0}(\sin y)/y^3 = 1/0^2 - 1/6$,[16] while $\cos 0/0^4 = \lim_{y\to 0}(\cos y)/y^4 = 1/0^4 - 1/2\cdot 0^2 + 1/24$. Also, we have eliminated $\sin 2ka$

[16] We cannot use L'Hôpital's rule here, which would imply that $\sin 0/0^3 = \lim_{y\to 0}(\sin y)/y^3 = \lim_{y\to 0}(\cos y)/3y^2 = 1/3\cdot 0^2$, as this rule applies only when the limit is finite.

and $\cos 2ka$ in favor of $\sin^2 ka$ using eq. (26),

$$\sin 2ka = 2\sin ka \cos ka = 2\sin^2 ka \left(\frac{1}{ka} - ka\right). \tag{32}$$

Finally, from the square of eq. (26) we find

$$\sin^2 ka = \frac{k^2a^2}{1 - k^2a^2 + k^4a^4}, \tag{33}$$

such that eq. (31) can also be written as,

$$F_{z,\mathrm{base}} = \frac{9E_0^2}{64k^2}\frac{4k^2a^2 - k^4a^4}{1 - k^2a^2 + k^4a^4}. \tag{34}$$

Turning to the forces on the hemispherical surface, by symmetry these can only lead to a nonzero z-component. Recalling that T_{rr} is positive for inward stress on the hemisphere, the magnetic force on the hemisphere is,

$$\begin{aligned} F^B_{z,\mathrm{hemi}} &= -\int_0^1 \mathsf{T}^B_{rr,\mathrm{hemi}} \cos\theta\,(2\pi a^2)\,d\cos\theta = \frac{a^2}{8}\int_0^1 |B_{\phi,\mathrm{hemi}}|^2 \cos\theta\,d\cos\theta \\ &= \frac{a^2}{8}\left(\frac{3E_0}{2}\right)^2\left(\frac{\sin ka}{k^2a^2} - \frac{\cos ka}{ka}\right)^2\int_0^1 \sin^2\theta\cos\theta\,d\cos\theta \\ &= \frac{9E_0^2}{128k^2}\left(\frac{\sin ka}{ka} - \cos ka\right)^2 = \frac{9E_0^2}{128k^2}k^2a^2\sin^2 ka, \end{aligned} \tag{35}$$

noting that from eq. (26), $(\sin ka)/ka - \cos ka = ka \sin ka$. Similarly, the electric force on the hemisphere is,

$$\begin{aligned} F^E_{z,\mathrm{hemi}} &= -\int_0^1 \mathsf{T}^E_{rr,\mathrm{hemi}} \cos\theta\,(2\pi a^2)\,d\cos\theta = -\frac{a^2}{8}\int_0^1 |E_{r,\mathrm{hemi}}|^2 \cos\theta\,d\cos\theta \\ &= -\frac{a^2}{8}(3E_0)^2\left(\frac{\sin ka}{k^3a^3} - \frac{\cos ka}{k^2a^2}\right)^2\int_0^1 \cos^2\theta\cos\theta\,d\cos\theta \\ &= -\frac{9E_0^2}{32k^4a^2}\left(\frac{\sin ka}{ka} - \cos ka\right)^2 = -\frac{9E_0^2}{32k^2}\sin^2 ka. \end{aligned} \tag{36}$$

The total electromagnetic force on the hemisphere is,

$$F_{z,\mathrm{hemi}} = F^B_{z,\mathrm{hemi}} + F^E_{z,\mathrm{hemi}} = -\frac{9E_0^2}{64k^2}\sin^2 ka\left(2 - \frac{k^2a^2}{2}\right) = -\frac{9E_0^2}{64k^2}\frac{4k^2a^2 - k^4a^4}{1 - k^2a^2 + k^4a^4}, \tag{37}$$

which is equal and opposite to the total electromagnetic force (34) on the base.

Hence, the total electromagnetic (self-)force on the hemispherical cavity is zero, as expected.

A Appendix: E and B of Cavity Fields Need Not Be 90° Out of Phase, but They Are for Standing Waves

It seems to be well known, but little discussed, that the steady-states, standing-wave modes of a cavity, at a given angular frequency ω, can be written in terms of $\mathbf{E}$ and $\mathbf{B}$ that are 90° out of phase. This is stated to be almost self-evident in sec. 19.6 of [24], but deserves some clarification, as given below.[17]

We consider electromagnetic fields of the form $\mathbf{E}(\mathbf{x},t) = (E_x(\mathbf{x}), E_y(\mathbf{x}), E_z(\mathbf{x}))\, e^{-i\omega t}$ and $\mathbf{B}(\mathbf{x},t) = (B_x(\mathbf{x}), B_y(\mathbf{x}), B_z(\mathbf{x}))\, e^{-i\omega t}$. Each of the six spatial functions $E_j(\mathbf{x})$ and $B_j(\mathbf{x})$ can be written as $f(\mathbf{x})\, e^{-i\phi(\mathbf{x})}$ where f and ϕ are real functions. Of course, the physical field components are just the real part of the notation used here, so each physical field component has the form $Re[f\, e^{-i(\omega t+\phi)}] = f\, \cos(\omega t + \phi)$.

We seek standing-wave modes of a cavity at rest, supposing that these cannot have spatial dependence of the phase factor ϕ, such that each field component is the product of a spatial function $f(\mathbf{x})$ and a time-dependent function $\cos(\omega t + \phi)$ where the constant ϕ (and the function f) can vary from one component to another.

Faraday's law, $\boldsymbol{\nabla} \times \mathbf{E} = -(1/c)\partial\mathbf{B}/\partial t$, for these standing waves can be written as,

$$\frac{\partial E_z}{\partial y}\cos(\omega t + \phi_{E_z}) - \frac{\partial E_y}{\partial z}\cos(\omega t + \phi_{E_y}) = kB_x \sin(\omega t + \phi_{B_x}), \tag{38}$$

$$\frac{\partial E_x}{\partial z}\cos(\omega t + \phi_{E_x}) - \frac{\partial E_z}{\partial x}\cos(\omega t + \phi_{E_z}) = kB_y \sin(\omega t + \phi_{B_y}), \tag{39}$$

$$\frac{\partial E_y}{\partial x}\cos(\omega t + \phi_{E_y}) - \frac{\partial E_x}{\partial y}\cos(\omega t + \phi_{E_x}) = kB_z \sin(\omega t + \phi_{B_z}), \tag{40}$$

where $k = \omega/c$ for cavities operated at vacuum. If all of the derivatives of the electric field components are nonzero, and all magnetic field components are nonzero, we must have that,

$$\phi_{E_x} = \phi_{E_y} = \phi_{E_z} = \phi, \qquad \text{and} \qquad \phi_{B_x} = \phi_{B_y} = \phi_{B_z} = \phi + \frac{\pi}{2}, \tag{41}$$

for a single constant phase ϕ so that all 9 terms in eqs. (38)-(40) have the same time dependence, $\cos(\omega t + \phi)$. In this case, the electric field components differ in phase by $\pi/2$, *i.e.*, by 90°, from the magnetic field components, as claimed.

However, it could be that some of the electric and magnetic field components are zero. For example, the fields could be the sum of two plane, standing waves, such as,

$$\mathbf{E} = \cos kz \cos\omega t\, \hat{\mathbf{x}} - \cos kx \cos(\omega t + \pi/2)\, \hat{\mathbf{y}}, \tag{42}$$

$$\mathbf{B} = -\cos kz \cos(\omega t + \pi/2)\, \hat{\mathbf{y}} - \sin kx \cos\omega t\, \hat{\mathbf{z}}, \tag{43}$$

whose traveling-wave component propagate along z and x, with $E_z = 0 = B_y$. The electric and magnetic fields (42)-(43) do not each have a single phase factor, and do not obey the form claimed for cavity modes.

[17] A resonant cavity is often regarded as a resonant L-C circuit, and since the stored energy in the latter oscillates between "electric" and "magnetic" terms, it is natural to suppose that $\mathbf{E}$ and $\mathbf{B}$ in a resonant cavity are 90° out of phase.

The fields (42)-(43) do not satisfy the boundary conditions for a cavity with perfectly conducting walls, that the fields at the walls obey $\mathbf{E}_{\parallel} = 0 = \mathbf{B}_{\perp}$ for any possible geometry of a cavity. Hence, the claim does not hold for standing waves in general, but could be true for standing waves inside a cavity with perfectly conducting walls.

Note that if in the original coordinate system it happened that some derivatives of the electric field were zero, and/or some components of the magnetic field were zero, we could rotate the coordinates axes, or switch to a curvilinear coordinate system, such that all derivatives of $\mathbf{E}$ and all components of $\mathbf{B}$ would be nonzero, and we could conclude that $\mathbf{E}$ is 90° out of phase with $\mathbf{B}$ for standing-wave modes.

We also show that in the original (rectangular) coordinate system, more detailed considerations confirm that $\mathbf{E}$ is 90° out of phase with $\mathbf{B}$ for standing-wave modes of a cavity with perfectly conducting walls.

A.1 TM Modes, $B_z = 0$

The magnetic field lines inside a finite cavity form closed loops, so at most one component of $\mathbf{B}$, say B_z, can be zero. In this case the mode is called transverse magnetic, TM. Equation (40) then tells us that either E_x and E_y are both zero, or $\phi_{E_x} = \phi_{E_y}$ with E_x and E_y are both nonzero.

A.1.1 E_x and E_y are both zero

In this case, lines of $\mathbf{E}$ are parallel to the z-axis, so the cavity surface must be either perpendicular or parallel to this axis for the electric field to satisfy the perfect-conductor boundary conditions. That is, the cavity is a cylindrical prism. We define $\phi = \phi_{E_z}$, such that eqs. (38)-(39) reduce to,

$$\frac{\partial E_z}{\partial y}\cos(\omega t + \phi) = kB_x \sin(\omega t + \phi_{B_x}), \qquad -\frac{\partial E_z}{\partial x}\cos(\omega t + \phi) = kB_y \sin(\omega t + \phi_{B_y}), \quad (44)$$

which are sufficient to determine that $\phi_{B_x} = \phi_{B_y} = \phi + \pi/2$, and hence that $\mathbf{E}$ is 90° out of phase with $\mathbf{B}$.

A.1.2 $\phi_{E_x} = \phi_{E_y}$, E_x and E_y are both nonzero

Since a TEM (transverse electromagnetic) wave cannot exist inside a closed cavity, E_z is also nonzero.

It is useful to consider the Ampère-Maxwell equation inside the cavity, $\boldsymbol{\nabla}\times\mathbf{B} = (1/c)\partial\mathbf{B}/\partial t$, where its components are,

$$\frac{\partial B_z}{\partial y}\cos(\omega t + \phi_{B_z}) - \frac{\partial B_y}{\partial z}\cos(\omega t + \phi_{B_y}) = -kE_x \sin(\omega t + \phi_{E_x}), \quad (45)$$

$$\frac{\partial B_x}{\partial z}\cos(\omega t + \phi_{B_x}) - \frac{\partial B_z}{\partial x}\cos(\omega t + \phi_{B_z}) = -kE_y \sin(\omega t + \phi_{E_y}), \quad (46)$$

$$\frac{\partial B_y}{\partial x}\cos(\omega t + \phi_{B_y}) - \frac{\partial B_x}{\partial y}\cos(\omega t + \phi_{B_x}) = -kE_z \sin(\omega t + \phi_{E_z}). \quad (47)$$

We now define $\phi = \phi_{E_x} = \phi_{E_y}$. Then, eqs. (45)-(46) tell us that $\phi_{B_x} = \phi_{B_y} = \phi + \pi/2$, after which eq. (47) tells us that $\phi_{E_z} = \phi$. Hence, **E** is 90° out of phase with **B** in this case as well.

A.2 All Components of B Are Nonzero, but Some Derivatives of E Are Zero

Can a derivative $\partial E_i/\partial x_j$ be zero for $i \neq j$ while E_i is nonzero? This is possible in free space, but not inside a cavity with perfectly conducting walls, where the tangential component of the electric field must be zero. If $\partial E_i/\partial x_j = 0$, then E_i is independent of x_j. However, for a closed surface there must be some x_j where the surface is parallel to the i-axis, at which E_i must be zero. Hence, it must be that E_i is zero everywhere.[18]

So, a condition that a derivative of a component of **E** be zero implies that the component is zero. Taking that component to be the z-component, the mode can be called transverse electric, TE.

One can now make an argument for such a TE mode similar to that in sec. A.1 for a TM mode, to conclude that for the TE mode, **E** and **B** are 90° out of phase.

A.3 Comments

In sum, standing-wave modes of a cavity can always be written in a form with **E** and **B** are 90° out of phase.

However, many cavity modes are degenerate, meaning that more than one field pattern is possible at a given frequency. For example, the modes of a spherical cavity are infinitely degenerate, while the modes of a rectangular cavity are sixfold degenerate. Such cavities can be excited in a superposition of degenerate standing-wave modes at a given frequency, with arbitrary phases of the different modes, such that the total fields cannot be characterized by a single phase of **E** and a single phase for **B**. These fields do not have **E** and **B** 90° out of phase, and are not standing waves by our narrow definition.

In Appendix B we consider the related example of a cubical cavity, and display possible fields that are not a standing-wave mode.

For many cavity geometries (such as the hemispherical cavity considered in this note) the lowest-frequency mode is nondegenerate, in which case **E** is 90° out of phase with **B** for fields at this frequency. However, a spherical cavity can have its lowest-frequency mode with fields axially symmetric about an arbitrary axis through the cavity center, and hence this mode is infinitely degenerate, such that fields can be excited in a superposition of modes with different symmetry axes, for which **E** need not be 90° out of phase with **B**.

B Appendix: Modes of a Rectangular Cavity

For reference, we explicitly display the fields for a rectangular cavity (sec. 19.20) of [30]) that extends over $0 < x < a$, $0 < y < b$, $0 < z < c$. The allowed angular frequencies ω are related

[18] This argument does not prohibit $\partial E_i/\partial x_i$ from being zero, with E_i then independent of x_i in that the closed surface could be a cylindrical prism along the i-axis.

by,

$$\omega^2 = (k_x^2 + k_y^2 + k_z^2)c^2, \qquad k_x = \frac{l\pi}{a}, \quad k_y = \frac{m\pi}{b}, \quad k_z = \frac{n\pi}{c}, \tag{48}$$

where l, m and n are integers. In general, there are six distinct mode for each allowed frequency, three so-called TE modes with the electric field transverse to one of the three coordinate axes, and three TM modes with the magnetic field transverse to a coordinate axis. The fields for modes with $\mathbf{E}$ or $\mathbf{B}$ transverse to the z-axis are,

$$\begin{aligned}
\mathbf{E}_{\mathrm{TE},z} &= \omega C_{\mathrm{TE},z}[k_y \cos k_x x \sin k_y y\,\hat{\mathbf{x}} - k_x \sin k_x x \cos k_y y\,\hat{\mathbf{y}}] \sin k_z z \sin(\omega t + \phi_{\mathrm{TE},z}), && (49)\\
\mathbf{B}_{\mathrm{TE},z} &= c C_{\mathrm{TE},z}\{k_z[k_y \sin k_x x \cos k_y y\,\hat{\mathbf{x}} + k_x \cos k_x x \sin k_y y\,\hat{\mathbf{y}}] \cos k_z z \\
&\qquad -(k_x^2 + k_y^2) \cos k_x x \cos k_y y \sin k_z z\,\hat{\mathbf{z}}\} \cos(\omega t + \phi_{\mathrm{TE},z}), && (50)\\
\mathbf{E}_{\mathrm{TM},z} &= c C_{\mathrm{TM},z}\{k_z[k_y \cos k_x x \sin k_y y\,\hat{\mathbf{x}} + k_x \sin k_x x \cos k_y y\,\hat{\mathbf{y}}] \sin k_z z \\
&\qquad -(k_x^2 + k_y^2) \sin k_x x \sin k_y y \cos k_z z\,\hat{\mathbf{z}}\} \sin(\omega t + \phi_{\mathrm{TM},z}), && (51)\\
\mathbf{B}_{\mathrm{TM},z} &= \omega C_{\mathrm{TM},z}[k_y \sin k_x x \cos k_y y\,\hat{\mathbf{x}} - k_x \cos k_x x \sin k_y y\,\hat{\mathbf{y}}] \cos k_z z \cos(\omega t + \phi_{\mathrm{TM},z}). && (52)
\end{aligned}$$

While each of the six cavity modes of frequency ω has $\mathbf{E}$ 90° out of phase with $\mathbf{B}$, it is possible to excite a superposition of modes with arbitrary phase differences, in which case the total cavity fields no longer have $\mathbf{E}$ 90° out of phase with $\mathbf{B}$.

We illustrate this for a cubical cavity of edge length a which is excited in a superposition of the $\mathrm{TM}_z(110)$ and a $\mathrm{TM}_x(011)$ mode, for which $k_x = k_y = k_z \equiv k = \pi/a$, $\omega = \sqrt{2}kc$. The fields for the $\mathrm{TM}_z(110)$ mode are, for $\phi_{\mathrm{TM},z} = 0$,

$$\begin{aligned}
\mathbf{E}_{\mathrm{TM},z} &= -\sqrt{2} \sin kx \sin ky \sin \omega t\,\hat{\mathbf{z}}, && (53)\\
\mathbf{B}_{\mathrm{TM},z} &= (\sin kx \cos ky\,\hat{\mathbf{x}} - \cos kx \sin ky\,\hat{\mathbf{y}}) \cos \omega t. && (54)
\end{aligned}$$

while those for the $\mathrm{TM}_x(011)$ mode are, for $\phi_{\mathrm{TM},x} = \pi/2$,

$$\begin{aligned}
\mathbf{E}_{\mathrm{TM},x} &= \sqrt{2} \sin ky \sin kz \cos \omega t\,\hat{\mathbf{x}}, && (55)\\
\mathbf{B}_{\mathrm{TM},x} &= -(\sin ky \cos kz\,\hat{\mathbf{y}} - \cos ky \sin kz\,\hat{\mathbf{z}}) \sin \omega t. && (56)
\end{aligned}$$

The total fields, for which $\mathbf{E}$ and $\mathbf{B}$ are not simply 90° out of phase, are,

$$\begin{aligned}
\mathbf{E} &= \sqrt{2} \sin ky \sin kz \cos \omega t\,\hat{\mathbf{x}} - \sqrt{2} \sin kx \sin ky \sin \omega t\,\hat{\mathbf{z}}, && (57)\\
\mathbf{B} &= \sin kx \cos ky \cos \omega t\,\hat{\mathbf{x}} \\
&\quad -(\cos kx \sin ky \cos \omega t + \sin ky \cos kz \sin \omega t)\,\hat{\mathbf{y}} \\
&\quad + \cos ky \sin kz \sin \omega t\,\hat{\mathbf{z}}. && (58)
\end{aligned}$$

All field components except B_y are standing waves.

The time-average Poynting vector is nonzero,

$$\langle \mathbf{S} \rangle = \frac{c}{4\pi} \langle \mathbf{E} \times \mathbf{B} \rangle = \frac{\sqrt{2}c \sin^2 ky}{8\pi} [-\sin kx \cos kz\,\hat{\mathbf{x}} + \cos kx \sin kz\,\hat{\mathbf{z}}]\,. \tag{59}$$

However, the volume integral of $\langle \mathbf{S} \rangle$ is zero, since $\int_0^a \cos kx\, dx = \int_0^a \cos(\pi x/a)\, dx = 0$, and hence the time-average stored field momentum is also zero,

$$\langle \mathbf{P} \rangle = \int \frac{\langle \mathbf{S} \rangle}{c^2}\, d\text{Vol} = 0. \tag{60}$$

There exists a time-average flow of energy in closed loops inside the cavity, but there is no net (volume integral) flow of energy, as expected for a cavity at rest in a steady state.

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This meeting featured an inspirational address by W. Thomson as a memorial to Herschel; among many other topics Thomson speculated on the size of atoms, on the origin of life on Earth as due to primitive organisms arriving in meteorites, and on how the Sun's source of energy cannot be an influx of matter as might, however, explain the small advance of the perihelion of Mercury measured by LeVerrier.
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Stewart argued that the radiation resistance felt by a charge moving through blackbody radiation should vanish as the temperature of the bath went to zero, just as he expected the electrical resistance of a conductor to vanish at zero temperature.
The 43rd meeting was also the occasion of reports by Maxwell on the exponential atmosphere as an example of statistical mechanics (pp. 29-32), by Rayleigh on the diffraction limit to the sharpness of spectral lines (p. 39), and (perhaps of greatest significance to the attendees) by A.H. Allen on the detection of adulteration of tea (p. 62).

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