Dielectric (and Magnetic) Image Methods

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1 Problem

The method of images is most often employed in electrostatic examples with point or line charges in vacuum outside conducting planes, cylinders or spheres [1]. Develop similar prescriptions for the electric scalar potential in examples where the "conductor" is a linear, isotropic dielectric medium with relative permittivity ϵ.

Discuss also media with relative permeability μ when a line current or "point" magnetic dipole is present.

Use the results to discuss the forces on these static systems.

2 Image Solutions[1]

2.1 Dielectric Media

In a linear isotropic dielectric medium with relative permittivity ϵ, the electric field $\mathbf{E}$ and the displacement field $\mathbf{D}$ are related by $\mathbf{D} = \epsilon\mathbf{E}$ (in Gaussian units). We assume that there are no free charges in/on the dielectric medium, such that $\boldsymbol{\nabla}\cdot\mathbf{D} = 0$, and hence $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ within the dielectric medium. The net polarization charge density, if any, resides only on the surface of the dielectric medium.

Of course, $\boldsymbol{\nabla}\times\mathbf{E} = 0$ in static examples, so the electric field can be related to a scalar potential V according to $\mathbf{E} = -\boldsymbol{\nabla}V$. Thus, inside a linear, isotropic dielectric medium, the scalar potential obeys Laplace's equation, $\nabla^2 V = 0$, and the scalar potential can be represented by familiar Fourier series in rectangular, cylindrical and spherical coordinates (and 8 other coordinate systems as well).

If the interface between the dielectric medium and vacuum (or another dielectric medium) supports no free charge, then the normal component of the displacement field $\mathbf{D}$ and the tangential component of the electric field $\mathbf{E}$ are continuous across that interface. The latter condition is equivalent to the requirement that the potential V be continuous across the interface.

2.1.1 Point Charge Outside a Dielectric Half Space

This section is a variant of sec. 4.4 of [3].[2]

We first consider the case of a dielectric medium with relative permittivity ϵ in the half space $z < 0$ with a point charge q at $(x, y, z) = (0, 0, a)$, where otherwise the region $z > 0$ has relative permittivity ϵ'. The image method is to suppose that the potential in the region

[1]For a review, see [2].

[2]For a point charge outside an infinite, dielectric plate of finite thickness, see [4, 5].

$z > 0$ is that due to the original point charge q at $(0, 0, a)$ plus an image charge q' at $(0, 0, -b)$, and that the potential in the region $z < 0$ is that due to the original point charge plus a point charge q'' at $(0, 0, c)$.

We first recall that a "free" electric charge q in an infinite dielectric medium of relative permittivity ϵ' has fields $\mathbf{D} = \epsilon'\mathbf{E} = q\,\hat{\mathbf{r}}/r^2$, because charge $-q(\epsilon - 1)/\epsilon$ is induced on the adjacent surface of the medium surrounding the charge. This suggests that for the proposed image method, the electric scalar potential at $(x, 0, z > 0)$ has the form,

$$V(x, 0, z > 0) = \frac{q}{\epsilon'[x^2 + (z - a)^2]^{1/2}} + \frac{q'}{\epsilon'[x^2 + (z + b)^2]^{1/2}}\,, \tag{1}$$

and that at $(x, 0, z < 0)$ could be written as,

$$V(x, 0, z < 0) = \frac{q}{\epsilon'[x^2 + (z - a)^2]^{1/2}} + \frac{q''}{\epsilon'[x^2 + (z - c)^2]^{1/2}}\,. \tag{2}$$

Then, continuity of the potential V across the plane $z = 0$ requires that,

$$b = c, \qquad \text{and} \qquad q'' = q'. \tag{3}$$

Continuity of D_z across the plane $z = 0$ requires that $\epsilon' E_z(x, 0, 0^+) = \epsilon E_z(x, 0, 0^-)$, *i.e.*,

$$\epsilon' \frac{\partial V(x, 0, 0^+)}{\partial z} = \epsilon \frac{\partial V(x, 0, 0^-)}{\partial z}\,, \tag{4}$$

$$\frac{qa}{[x^2 + a^2]^{3/2}} - \frac{q'b}{[x^2 + b^2]^{3/2}} = \frac{\epsilon}{\epsilon'}\left(\frac{qa}{[x^2 + a^2]^{3/2}} + \frac{q'b}{[x^2 + b^2]^{3/2}}\right), \tag{5}$$

which implies that,[3]

$$a = b\ (= c), \qquad \text{and} \qquad (q'' =)\ q' = -q\frac{\epsilon - \epsilon'}{\epsilon + \epsilon'}\,. \tag{6}$$

The potential and electric field $\mathbf{E} = \mathbf{D}/\epsilon$ in the region $z < 0$ are as if the media were vacuum and the original charge q were replaced by charge $(q + q'')/\epsilon' = 2q/(\epsilon + \epsilon')$. In the region $z > 0$ the potential and $\mathbf{E}$ field are as for the original effective charge q/ϵ' plus an effective image charge $q'/\epsilon' = q''/\epsilon'$, both in vacuum.[4]

In the limit that $a = 0$ (such that charge q lies on the interface between the two dielectric media) the electric field is as if the media were vacuum but the charge were $2q/(\epsilon + \epsilon')$. Note that the electrical field is radial, and the same in both media despite their differing

[3]For a metamaterial with relative permeability $\epsilon' = -\epsilon$, eq. (6) diverges (as could its magnetic equivalents (37) and (43) for certain negative relative permeabilities). However, metamaterials can have negative permittivity (and/or permeability) only for nonzero frequencies [6], so technically this divergence cannot occur. It remains possible that metamaterials could lead to large "image" forces at low frequencies [7].

[4]The electric field of the image charge q' at $z = -a$ does not vary as $q'/\epsilon r^2$ as might be expected for an actual charge at this location, because the image charge represents effects at $z > 0$ of induced charges at the interface $z = 0$. Likewise, the electric field for $z < 0$ can be written as $2q\,\hat{\mathbf{r}}_q/(\epsilon' + \epsilon)r_q^2 = q\,\hat{\mathbf{r}}_q/\epsilon_{\mathrm{ave}}r_q^2$.

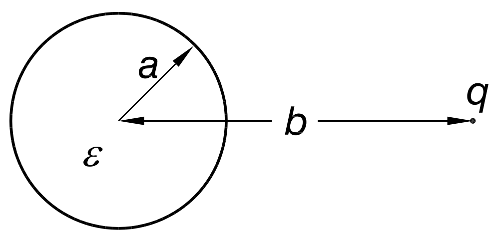
permittivities, as is consistent with the requirement that the tangential electric field be continuous at the interface $z = 0$.[5]

In the limit that $\epsilon \to \infty$, while $\epsilon' \to 1$, we obtain the image prescription for a point charge above a grounded, conducting plane; the potential above the plane is that due to the original charge plus an image charge $-q$ at $(0, 0, -a)$, and the potential below the plane is zero.

2.1.2 Line Charge and Dielectric Cylinder

This section follows secs. 4.04-4.06 of [9]. A solution based on the use of a vector potential $\mathbf{A}$ *outside the line charge, where* $\boldsymbol{\nabla} \cdot \mathbf{D} = 0$ *so we can write* $\mathbf{D} = \boldsymbol{\nabla} \times \mathbf{A}$ *there, is given in prob. 3, sec. 7 of [10]. See [11] for an extension to case of a time-harmonic line charge.*

We next consider the case of a dielectric cylinder of radius a and relative permittivity ϵ when a thin wire that carries free charge q per unit length is located in a medium with relative permittivity ϵ' at distance $b > a$ from the center of the cylinder.



In this two-dimensional problem we take the axis of the cylinder to be z axis, and the position of the wire to be $(r, \theta) = (b, 0)$ in a cylindrical coordinate system (r, θ, z). Then, the

[5] If we suppose the "point" charge is actually a conducting sphere, some interesting considerations arise. See, for example, prob. 4.39 of [8].

We take the origin of a spherical coordinate system (r, θ, ϕ) to be at the center of the conducting sphere and the z-axis perpendicular to the planar interface between the regions of dielectric constant ϵ and ϵ'. Then, the two regions of dielectric material do not include the origin, and both extend to infinity. Within each of these two regions, the electric scalar potential V obeys Laplace's equation, $\nabla V = 0$, and is azimuthally symmetric about the z-axis, so the potential in the two regions can be expressed as series expansions,

$$V_\epsilon = \sum_n \frac{A_n}{r^{n+1}} P_n(\cos\theta), \qquad V_{\epsilon'} = \sum_n \frac{B_n}{r^{n+1}} P_n(\cos\theta), \tag{7}$$

where $P_n(\cos\theta)$ is the Legrendre polynomial of order n.

If R is the radius of the conducting sphere, we have that $V(R, \theta, \phi) = V_0$, a constant.

The potential must be continuous across the interface between the two regions, which is satisfied by $A_n = B_n$. If so, then the potential outside the conducting sphere has the form $V(r > R) = \sum_n (A_n/r^{n+1}) P_n(\cos\theta)$. The requirement that $V(R) = V_0$ tells us that $A_n = 0$ for $n \geq 1$ and $A_0 = V_0 R$. That is, $V(r > R) = V_0 R/r$, and the lines of electric field $\mathbf{E}$ are radial with respect to the center of the charged sphere.

However, this conflicts with the image solution found above!

The issue is that the above analysis has not satisfied the condition that $\mathbf{D}_\perp$ be continuous across the interface between the two dielectric regions (at which there is no free charge), except in the special case that the center of the conducting sphere lies in the plane of the interface for which $\mathbf{D}_\perp = 0$.

In general, the potential can be continuous across the interface with $A_n \neq B_n$, although this is perhaps counterintuitive.

We infer from the image method that if the conducting sphere is entirely within the region of dielectric constant ϵ', then the electric field in that region is not radial, but the electric field in the region of dielectric constant ϵ is radial, although not with respect to the center of the conducting sphere.

potential has the symmetry $V(r,-\theta) = V(r,\theta)$, so the Fourier expansion for the potential contains terms in $\cos n\theta$, but not $\sin n\theta$.

The potential due to the wire in the absence of the dielectric cylinder has the general form,

$$V_{\rm wire}(r,\theta) = \begin{cases} a_0 + \sum_{n=1} a_n \left(\frac{r}{b}\right)^n \cos n\theta & (r < b), \\ a_0 + b_0 \ln\frac{r}{b} + \sum_{n=1} a_n \left(\frac{b}{r}\right)^n \cos n\theta & (r > b), \end{cases} \tag{8}$$

since the potential should not blow up at the origin, should be continuous at $r = b$, and can have a logarithmic divergence at infinity. For large r the electric field due to the wire is $\mathbf{E}_{\rm wire} = 2q\,\hat{\mathbf{r}}/\epsilon' r$, and the corresponding asymptotic potential is defined to be $V_{\rm wire} = -(2q/\epsilon')\ln r$. Thus,

$$a_0 = -\frac{2q}{\epsilon'}\ln b, \qquad b_0 = -\frac{2q}{\epsilon'}\,. \tag{9}$$

The remaining coefficients a_n are determined the Maxwell equation $\boldsymbol{\nabla}\cdot\mathbf{D} = 4\pi\rho_{\rm free}$ by considering a Gaussian surface (of unit length in z) that surrounds the cylindrical shell (b,θ),

$$4\pi q_{\rm free,in} = \int \mathbf{D}\cdot d\mathbf{Area} = b\epsilon' \int d\theta\ (E_{r^+} - E_{r^-}). \tag{10}$$

From this we learn that,

$$\begin{aligned} 4\pi q\delta(\theta) &= b\epsilon'(E_{r^+} - E_{r^-}) = b\epsilon'\left(-\frac{\partial V_{\rm wire}(b^+)}{\partial r} + \frac{\partial V_{\rm wire}(b^-)}{\partial r}\right) \\ &= 2q + 2\epsilon' \sum_n n a_n \cos n\theta. \end{aligned} \tag{11}$$

Multiplying eq. (11) by $\cos n\theta$ and integrating over θ we find that,

$$a_n = \frac{2q}{\epsilon' n}. \tag{12}$$

Then, the potential due to the wire can be written as,

$$V_{\rm wire}(r,\theta) = \begin{cases} \frac{2q}{\epsilon'}[-\ln b + \sum_{n=1} \frac{1}{n}\left(\frac{r}{b}\right)^n \cos n\theta] & (r < b), \\ \frac{2q}{\epsilon'}[-\ln r + \sum_{n=1} \frac{1}{n}\left(\frac{b}{r}\right)^n \cos n\theta] & (r > b). \end{cases} \tag{13}$$

When the dielectric cylinder is present it supports a polarization charge density that results in an additional scalar potential which can be expanded in a Fourier series similar to eq. (8),

$$V_{\rm cylinder}(r,\theta) = \begin{cases} \sum_{n=1} A_n \left(\frac{r}{a}\right)^n \cos n\theta & (r < a), \\ \sum_{n=1} A_n \left(\frac{a}{r}\right)^n \cos n\theta & (r > a), \end{cases} \tag{14}$$

noting that the dielectric cylinder has zero total charge, so its potential goes to zero at large r. The coefficients A_n can be evaluated by noting that the radial component of the total electric displacement field $\mathbf{D} = \epsilon\mathbf{E}$ is continuous across the boundary $r = a$,

$$\epsilon\frac{\partial V(a^-)}{\partial r} = \epsilon'\frac{\partial V(a^+)}{\partial r}\,, \tag{15}$$

$$\epsilon \sum_{n=1} \left[\frac{2q}{\epsilon' a} \left(\frac{a}{b} \right)^n + \frac{n A_n}{a} \right] \cos n\theta = \epsilon' \sum_{n=1} \left[\frac{2q}{\epsilon' a} \left(\frac{a}{b} \right)^n - \frac{n A_n}{a} \right] \cos n\theta, \tag{16}$$

$$A_n = -\frac{2q}{\epsilon' n} \frac{\epsilon - \epsilon'}{\epsilon + \epsilon'} \left(\frac{a}{b} \right)^n, \tag{17}$$

$$V_{\rm cylinder}(r,\theta) = \begin{cases} -\frac{2q}{\epsilon'} \frac{\epsilon-\epsilon'}{\epsilon+\epsilon'} \sum_{n=1} \frac{1}{n} \left(\frac{r}{b} \right)^n \cos n\theta & (r < a), \\ \frac{2q}{\epsilon'} \frac{\epsilon-\epsilon'}{\epsilon+\epsilon'} (-\ln r) - \frac{2q}{\epsilon'} \frac{\epsilon-\epsilon'}{\epsilon+\epsilon'} \left[-\ln r + \sum_{n=1} \frac{1}{n} \left(\frac{a^2/b}{r} \right)^n \cos n\theta \right] & (r > a), \end{cases} \tag{18}$$

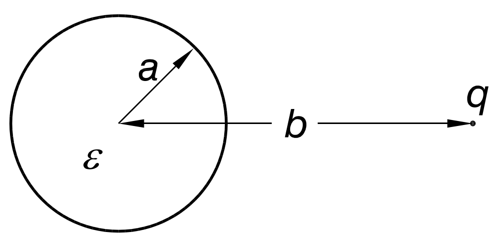
Outside the dielectric cylinder of radius a the total potential $V_{\rm wire} + V_{\rm cylinder}$ is the same as if the media were vacuum and the cylinder were replaced by a wire at the origin of charge $q(\epsilon - \epsilon')/\epsilon'(\epsilon + \epsilon')$ per unit length, and an oppositely charged wire at $(r, \theta) = (a^2/b, 0)$. Inside the cylinder, the potential is, to within a constant, as if charge per unit length $-q(\epsilon - \epsilon')/\epsilon'(\epsilon + \epsilon')$ had been added to the effective charge q/ϵ' per unit length of the original wire, bringing its effective linear charge density to $2q/(\epsilon + \epsilon')$. That is, an image-method holds for this example.

In the limit $\epsilon \to \infty$, while $\epsilon' \to 1$, we obtain the image prescription for a charged wire outside a neutral, conducting cylinder; the potential outside the cylinder is as if it were replaced by a wire of linear charge density $-q$ at $(r, \theta) = (a^2/b, 0)$ plus a wire of line charge density q along the z-axis, and the potential inside the cylinder is constant.

Another limit of interest is that the radius a of the dielectric cylinder goes to infinity while distance $d = b - a$ remains constant, such that the dielectric cylinder becomes a half space. For the potential outside the dielectric, the line charge is at depth $a - a^2/b \to d$ below the planar surface of the dielectric, as previously found in sec. 2.1.

2.1.3 Point Charge Outside a Dielectric Sphere

We finally consider the case of a dielectric sphere of radius a and relative permittivity ϵ when a point charge q per unit length is located in vacuum at distance $b > a$ from the center of the cylinder .



In this three-dimensional problem we take the center of the sphere to be the origin, and take the position of the point charge to be $(r, \theta, \phi) = (b, 0, 0)$ in a spherical coordinate system. The geometry is azimuthally symmetric, so the potential does not depend on coordinate ϕ.

The potential due to the point charge q in the absence of the dielectric sphere has the general form,

$$V_q(r,\theta,\phi.) = \begin{cases} \sum_{n=0} a_n \left(\frac{r}{b} \right)^n P_n(\cos\theta) & (r < b), \\ \sum_{n=0} a_n \left(\frac{b}{r} \right)^{n+1} P_n(\cos\theta) & (r > b), \end{cases} \tag{19}$$

where the P_n are Legendre polynomials of integer order. The coefficients a_n are determined the Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{D} = 4\pi\rho_{\rm free}$ by considering a Gaussian surface that surrounds the spherical shell (b, θ, ϕ),

$$4\pi q_{\rm free,in} = \int \mathbf{D} \cdot d\mathbf{Area} = 2\pi b^2 \epsilon' \int_{-1}^{1} d\cos\theta \ (E_{r^+} - E_{r^-}). \tag{20}$$

From this we learn that,

$$\begin{aligned} 2q\,\delta(\cos\theta) &= b^2\epsilon'(E_{r^+} - E_{r^-}) = b^2\epsilon' \left(-\frac{\partial V_q(b^+)}{\partial r} + \frac{\partial V_q(b^-)}{\partial r} \right) \\ &= b \sum_n (2n+1) a_n P_n(\cos\theta). \end{aligned} \tag{21}$$

Multiplying eq. (21) by $P_n(\cos\theta)$ and integrating over $\cos\theta$, we find,

$$a_n = \frac{q}{\epsilon' b}\,. \tag{22}$$

Then, the potential due to the charge q can be written as,

$$V_q(r,\theta) = \begin{cases} \frac{q}{\epsilon' b} \sum_{n=0} \left(\frac{r}{b}\right)^n P_n(\cos\theta) & (r < b), \\ \frac{q}{\epsilon' b} \sum_{n=0} \left(\frac{b}{r}\right)^{n+1} P_n(\cos\theta) = \frac{q}{\epsilon' r} \sum_{n=0} \left(\frac{b}{r}\right)^n P_n(\cos\theta) & (r > b), \end{cases} \tag{23}$$

When the dielectric sphere is present it supports a surface polarization charge density that results in an additional scalar potential which can be expanded in a Fourier series similar to eq. (19),

$$V_{\rm sphere}(r,\theta) = \begin{cases} \sum_{n=0} A_n \left(\frac{r}{a}\right)^n P_n(\cos\theta) & (r < a), \\ \sum_{n=0} A_n \left(\frac{a}{r}\right)^{n+1} P_n(\cos\theta) & (r > a). \end{cases} \tag{24}$$

The coefficients A_n can be evaluated by noting that the radial component of the total electric displacement field $\mathbf{D} = \epsilon\mathbf{E}$ is continuous across the boundary $r = a$,

$$\epsilon \frac{\partial V(a^-)}{\partial r} = \epsilon' \frac{\partial V(a^+)}{\partial r}\,, \tag{25}$$

$$\epsilon \sum_{n=0} \left[\frac{nq}{\epsilon' ab} \left(\frac{a}{b}\right)^n + \frac{nA_n}{a} \right] P_n(\cos\theta) = \epsilon' \sum_{n=0} \left[\frac{nq}{\epsilon' ab} \left(\frac{a}{b}\right)^n - \frac{(n+1)A_n}{a} \right] P_n(\cos\theta), \tag{26}$$

$$A_n = -\frac{q}{\epsilon' b}(\epsilon - 1) \frac{n}{n(\epsilon+1)+1} \left(\frac{a}{b}\right)^n, \tag{27}$$

$$V_{\rm sphere}(r,\theta) = \begin{cases} -\frac{q}{\epsilon' b}(\epsilon - 1) \sum_{n=0} \frac{n}{n(\epsilon+1)+1} \left(\frac{r}{b}\right)^n P_n(\cos\theta) & (r < a), \\ -\frac{qa/b}{\epsilon' a^2/b}(\epsilon - 1) \sum_{n=0} \frac{n}{n(\epsilon+1)+1} \left(\frac{a^2/b}{r}\right)^{n+1} P_n(\cos\theta) & (r > a), \end{cases} \tag{28}$$

The potential of the dielectric sphere does not quite have the form of the potential due to point charges. Hence, there is no general image method for the case of a point charge outside a dielectric sphere.[6]

In the limit $\epsilon \to \infty$ while $\epsilon' \to 1$ we do obtain the usual image prescription for a point charge outside a neutral, conducting sphere; the potential outside the sphere is as if it were replaced by a point charge $-qa/b$ at $(r, \theta, \phi) = (a^2/b, 0, 0)$ plus a charge qa/b at the origin, and the potential inside the sphere is constant.[7]

2.2 Conducting Dielectric Media

In general a dielectric medium has a small electrical conductivity σ. Then, if an electric field exists inside the medium, a free-current density flows according to,

$$\mathbf{J}_{\rm free} = \sigma \mathbf{E} = \frac{\sigma \mathbf{D}}{\epsilon}, \tag{29}$$

where ϵ is the (relative) permittivity.[8] Conservation of free charge can then be expressed as,

$$\frac{\partial \rho_{\rm free}}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{J}_{\rm free} = -\frac{\sigma \boldsymbol{\nabla} \cdot \mathbf{D}}{\epsilon} = -\frac{4\pi\sigma}{\epsilon} \rho_{\rm free} \,, \tag{30}$$

such that in the absence of an energy flow to maintain the electric field, the free charge distribution decays according to,

$$\rho_{\rm free}(t) = \rho_0 \, e^{-t/\tau}. \tag{31}$$

with time constant $\tau = \epsilon/4\pi\sigma$. For metals this time constant is so short that the above calculation should be modified to include wave motion [20], but for dielectric with low conductivity this approximation is reasonable.

If an external charge is brought near a conducting dielectric medium, the potentials and associated charge distributions found in sec. 2 apply only for times small compared to the relaxation time τ. At longer times the medium behaves like a static conductor, with zero internal electric field, for which the potentials and charge distributions are the familiar versions for good conductors. For example, glass has (relative) dielectric constant ≈ 4 and electrical conductivity $\approx 10^{-4}$ Gaussian units, and hence $\tau_{\rm glass} \approx 1$ hour. Rock has conductivity $\sigma \approx 10^6$ Gaussian units, and hence $\tau_{\rm rock} \approx 1$ μs.

[6]It was found by Neumann in 1883 [12] that the problem of a point charge inside/outside a dielectric sphere can be solved by an image point charge together with an image line charge outside/inside the sphere. However, this result was little known until rediscovered by Lindell (1992) [13, 14], and then re-rediscovered by Norris in 1995 [15].

[7]Dec. 12, 2023. A curiosity is associated with the example of a point charge q at $(r, \theta, \phi) = (b, 0, 0)$ outside a grounded conducting sphere of radius a (see, for example, sec. 2.2 of [3]). The field outside the sphere due to the surface charge density on it, $\sigma(r, \theta) = -q(b^2 - r^2)/[4\pi r(b^2 + r^2 - 2br\cos\theta)^{3/2}]$, is equivalent to that of an image point charge $-qa/b$ at $(r, \theta, \phi) = (a^2/b, 0, 0)$, which field falls off outside the sphere as $1/R^2$ where R is the distance from the image charge. However, the charge density σ that produces this field is not spherically symmetric. See also Chap. 5, p. 57 of [16]. This example goes beyond the considerations in [17], which were restricted to fields that fall off as $1/r^2$ from the center of a spherical shell.

[8]For a steady-state example of a conducting dielectric, see [18]. For examples of steady currents in two- and three-dimensional conducting media in which image methods are relevant, see [19].

If the time structure of the external charges and currents is known, it is preferable to perform a full time-domain analysis. For a review, see [21]. For an incident wave of angular frequency ω the medium can be approximated as a good conductor if $\omega \ll 1/\tau$ and as a nonconductor if $\omega \gg 1/\tau$. For frequencies high enough that the medium is a "good conductor", the waves are attenuated as they penetrate into the medium over the skin depth, $d = c/\sqrt{2\pi\mu\omega\sigma}$, where μ if the relative permeability of the medium. When $\omega \approx 1/\tau$, $\delta \approx \sqrt{\epsilon/\mu}\, c/2\pi\sigma$, which provides an estimate of the maximum distance a transient field can penetrate into the medium.[9] For glass, this maximum depth is very large, but for rock is of order 1 m.

2.3 Permeable Media

So far as is presently known, (Gilbertian) magnetic charges do not exist in Nature,[10] and all magnetic fields can be associated with (Ampèrian) electrical currents. The magnetic moments of elementary particles, nuclei and atoms are quantum effects, not well described in "classical" electrodynamics, where one characterizes their presence in macroscopic media by the density $\mathbf{M}$ of magnetic moments. Noting that $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ and that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, one can define an effective magnetic charge density according to $\rho_{m,\mathrm{eff}} = -\boldsymbol{\nabla} \cdot \mathbf{M}$, such that $\boldsymbol{\nabla} \cdot \mathbf{H} = 4\pi\rho_{m,\mathrm{eff}}$, but isolated effective magnetic charges do not exist and the effective magnetic charge density is only an alternative representation of Ampèrian currents in bulk matter.[11] Furthermore, $\rho_{m,\mathrm{eff}} = 0$ inside linear, isotropic magnetic media, where $\mathbf{B} = \mu\mathbf{H}$, and one can speak only of an surface density of effective magnetic charges.

We first consider (secs. 2.3.1-2) image methods for permeable media in two-dimensional cases with a line conduction current I parallel to the z-axis, and then we turn (sec. 2.3.3) to three-dimensional examples with permanent magnetism.[12]

Away from the line current, $\boldsymbol{\nabla} \times \mathbf{H} = 0$, so we could write $\mathbf{H} = -\boldsymbol{\nabla}\Phi_M$ where Φ_M is a scalar potential. However, when conduction currents are present it seems better to work with the vector potential $\mathbf{A}$, such that $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. We recall that for a current I along the z-axis in a medium of relative permeability μ', the magnetic field is $\mathbf{B} = \mu'\mathbf{H} = 2\mu' I\,\hat{\boldsymbol{\theta}}/cr$, so the vector potential is $\mathbf{A} = -(2\mu' I/c) \ln r\, \hat{\mathbf{z}}$.

2.3.1 Line Current Outside a Permeable Half Space

We first consider the case of a medium with relative permeability μ in the half space $x < 0$ with a line current I at $(x, y) = (a, 0)$, where otherwise the region $x > 0$ has relative permeability μ'. The image method is to suppose that the vector potential in the region $x > 0$ is that due to the original current I at $(a, 0)$ plus an image current I' at $(-a, 0)$, and that the potential in the region $x < 0$ is that due to the original point current plus additional current I'' also at $(a, 0)$.

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[9] Compare eq. (7.70) of [3]. This argument seems to have disappeared from the 3$^{\mathrm{rd}}$ edition.

[10] For a discussion of electrodynamics if magnetic charges existed, see, for example, [22].

[11] However, one should not associate a magnetization or effective magnetic charges with "free" conduction currents. Attempts to do this lead to confusions as exhibited in [23]. See also [24, 25].

[12] The earliest discussion of magnetic image methods may be that by Thompson (1895) [26], soon followed by Searle [27].

According to the suggested image method, the vector potential at $x > 0$ can be written as,

$$A_z(x>0) = -\frac{\mu' I}{c}\ln[(x-a)^2+y^2] - \frac{\mu' I'}{c}\ln[(x+a)^2+y^2]\,, \tag{32}$$

and that at $x < 0$ as,

$$A_z(x<0) = -\frac{\mu'(I+I'')}{c}\ln[(x-a)^2+y^2]. \tag{33}$$

Then, continuity of the potential across the plane $x = 0$ requires that,

$$I'' = I'. \tag{34}$$

Continuity of H_y across the plane $x = 0$ requires that,

$$H_y(x=0^+) = \frac{B_y(x=0^+)}{\mu'} = -\frac{1}{\mu'}\frac{\partial A_z(x=0^+)}{\partial x} = H_y(x=0^-) = -\frac{1}{\mu}\frac{\partial A_z(x=0^-)}{\partial x}\,, \tag{35}$$

$$\mu(I-I') = \mu'(I+I''), \tag{36}$$

which implies that,

$$I'' = I' = I\frac{\mu-\mu'}{\mu+\mu'}\,. \tag{37}$$

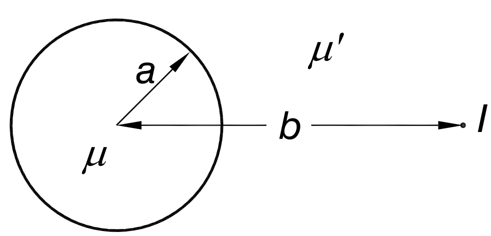
The vector potential and **B** field in the region $x < 0$ are as if that region were vacuum and the original current I were replaced by an effective current $\mu'(I + I'') = 2\mu\mu' I/(\mu + \mu')$. In the region $x > 0$ the potential and **B** field are as if due to an effective current $\mu' I$ at $x = a$ and an effective image current $\mu' I'$ at $x = -a$, both in vacuum.

In the limit that $a = 0$ (such that current q lies on the interface between the two permeable media) the **H** field is as if the media were vacuum but the current were $2\mu' I/(\mu + \mu')$. The magnetic field is radial, and **H** is the same in both media despite their differing permittivities, as is consistent with the requirement that the tangential **H** field be continuous at the interface $x = 0$.

In the limit that $\mu \to 0$, while $\mu' \to 1$, we obtain the image prescription for a line current at $x = a$ above a grounded, conducting plane $x = 0$; the potential above the plane is that due to the original current plus an image current $-I$ at $x = -a$, and the potential below the plane is zero. *Note that a perfect conductor is a kind of limit of a permeable medium as $\mu \to 0$, while being a kind of limit of a dielectric medium as $\epsilon \to \infty$.*

2.3.2 Line Current and Permeable Cylinder

We next consider the case of a cylinder of radius a and relative permittivity μ when a thin wire that carries charge I per unit length is located in a medium with relative permeability μ' distance $b > a$ from the center of the cylinder.



Following the success of the image method in secs. 2.1.1-2 and 2.3.1, we anticipate that outside the cylinder the vector potential and **B** field are the same as if all media were vacuum and the cylinder is replaced by a wire at the origin of effective current $-\mu' I(\mu-\mu')/(\mu+\mu')$, and an opposite current at $(r,\theta) = (a^2/b, 0)$. Of course, the effective current due to the original wire is $\mu' I$ when the media are imagined to be vacuum. Inside the cylinder, the potential is as if current $I(\mu-\mu')/(\mu+\mu')$ had been added to that of the original wire, bringing its effective current (as if in vacuum) to $2\mu I/(\mu+\mu')$.

In the limit $\mu \to 0$ with $\mu' \to 1$ we obtain the image prescription for a linear current I outside a perfectly conducting cylinder; the vector potential and magnetic fields outside the cylinder is as if it were replaced by a current $-I$ at $(r,\theta) = (a^2/b, 0)$ plus a current I along the z-axis, and the potential inside the cylinder is constant.[13,14]

2.3.3 Point Magnetic Charge, and Point Magnetic Dipole, Outside a Permeable Half Space

In this section we consider permanent magnets in vacuum that are outside a permeable half space $z < 0$ in which the (relative) permeability is μ.

We begin with the case of a point magnetic charge q_M (even though these don't seem to exist in Nature) at $(x,y,z) = (0,0,a)$. As in sec. 2.1.1, the image method is to suppose that the magnetic scalar potential Φ_M in the region $z > 0$ is that due to the original point charge q_M at $(0,0,a)$ plus an image charge q'_M at $(0,0,-b)$,

$$\Phi_M(x,0,z>0) = \frac{q_M}{[x^2+(z-a)^2]^{1/2}} + \frac{q'_M}{[x^2+(z+b)^2]^{1/2}}\,, \tag{38}$$

and that the potential in the region $z < 0$ (inside the permeable half space) is that due to the original point charge plus a point charge q''_M at $(0,0,c)$,

$$\Phi_M(x,0,z<0) = \frac{q_M}{[x^2+(z-a)^2]^{1/2}} + \frac{q''_M}{[x^2+(z-c)^2]^{1/2}}\,, \tag{39}$$

Continuity of the potential Φ_M across the plane $z = 0$ requires that,

$$b = c, \qquad \text{and} \qquad q''_M = q'_M. \tag{40}$$

Continuity of B_z across the plane $z = 0$ requires that $B_z(x,0,0^+) = H_z(x,0,0^+) = B_z(x,0,0^-) = \mu H_z(x,0,0^-)$, *i.e.*,

$$\frac{\partial \Phi_M(x,0,0^+)}{\partial z} = \mu \frac{\partial \Phi_M(x,0,0^-)}{\partial z}\,, \tag{41}$$

$$\frac{q_M a}{[x^2+a^2]^{3/2}} - \frac{q'_M b}{[x^2+b^2]^{3/2}} = \mu\left(\frac{q_M a}{[x^2+a^2]^{3/2}} + \frac{q'_M b}{[x^2+b^2]^{3/2}}\right), \tag{42}$$

[13] This justifies the choice of signs for the image currents inside the cylinder, as made in the previous paragraph.

[14] *June 3, 2022.* Engineered metamaterials can have $\mu = -1$, which leads to unusual behavior. The case of a cylindrical shell with $\mu = -1$, and $\mu' = 1$ inside and outside it, is discussed in [28], where a kind of image solution holds when a current-carrying wire is placed in the cylindrical cavity of the shell.

which implies that,

$$a = b = c, \qquad \text{and} \qquad q''_M = q'_M = -q_M \frac{\mu - 1}{\mu + 1}. \tag{43}$$

The potential and magnetic field $\mathbf{H} = \mathbf{B}/\mu$ in the permeable region $z < 0$ are as if the media were vacuum and the original magnetic charge q_M were replaced by charge $q_M + q''_M = 2q/(\mu+1)$. In the region $z > 0$ the potential and $\mathbf{B} = \mathbf{H}$ fields are as for the original charge q_M plus an image charge $-q_M(\mu - 1)/(\mu + 1)$, both in vacuum.

In the limit that $a = 0$ (such that charge q_M lies on the surface of the permeable medium the $\mathbf{H}$ field is as if the media were vacuum but the charge were $2q_M/(\mu + 1)$. Note that the $\mathbf{H}$ field is radial, and the same in both media despite their differing permeabilities, as is consistent with the requirement that the tangential component of $\mathbf{H}$ be continuous at the interface $z = 0$.

In the limit that $\mu \to \infty$, the potential above the plane is that due to the original charge plus an image charge $-q_M$ at $(0, 0, -a)$, and the potential below the plane is zero. However, this is not the image prescription for a point magnetic charge above a perfectly conducting plane, where the $\mathbf{B}$ field must be tangential to the surface of the conduction plane such that the image charge is $+q_M$ rather than $-q_M$. That is, a magnetic charge is attracted to a half space of infinite permeability, but repelled from a perfectly conducting (or superconducting) plane.

A magnetic dipole $\mathbf{m}$ can be regarded (insofar as we are only considered with its effect outside the dipole) as due to a pair of equal and opposite magnetic charges $\pm q_M$ separated by distance $\mathbf{d} = \mathbf{m}/q_M$. If this dipole is in vacuum outside a permeable half space, then the image method of point magnetic charges tells us that the potential and fields of the dipole in vacuum are as if the permeable medium were also vacuum but with an image magnetic dipole,

$$\mathbf{m}' = (\mathbf{m}_\perp - \mathbf{m}_\parallel) \frac{\mu - 1}{\mu + 1}, \qquad \text{where} \qquad \mathbf{m} = \mathbf{m}_\perp + \mathbf{m}_\parallel. \tag{44}$$

And, the $\mathbf{H}$ field inside the permeable medium is as if the medium were vacuum but the original magnetic dipole had strength $2\mathbf{m}/(\mu + 1)$.[15]

For a high-permeability medium ($\mu \gg 1$) the image dipole is $\mathbf{m}' = \mathbf{m}_\perp - \mathbf{m}_\parallel$, such that for a permanent magnet with magnetization perpendicular to the surface of the permeable medium, the image magnet has the same magnetization $\mathbf{m} = \mathbf{m}_\perp$ as the actual magnet.

3 Forces

When an object is embedded in a surrounding medium there is an ambiguity as to whether or not forces on the adjacent surface of the medium surrounding the object should be counted at part of the force on the object itself. So, we first consider the dielectric examples of sec. 2.1 where $\epsilon' = 1$.

[15] These results are discussed in [2, 29].

3.1 Point Charge Outside a Dielectric Half Space

3.1.1 $\epsilon' = 1$

It seems obvious that the force on the charge q at $z = a$ in the presence of a medium of relative permittivity ϵ at $z < 0$ can be computed as that due to the electric field of the image charge q',

$$\mathbf{F}_q = q\mathbf{E}_{q'} = -\frac{q^2}{4a^2}\frac{\epsilon - 1}{\epsilon + 1}\,\hat{\mathbf{z}}, \tag{45}$$

recalling eq. (6) (for $\epsilon' = 1$).

This force should be equal and opposite to that on the dielectric half space $z < 0$.

The medium at $z < 0$ has no free charge, and the bulk bound charge density $\rho_{\rm bound} = -\boldsymbol{\nabla}\cdot\mathbf{P} = -[(\epsilon - 1)/4\pi\epsilon]\boldsymbol{\nabla}\cdot\mathbf{D} = 0$ is also zero for a linear, isotropic medium. However, there exists a nonzero bound charge density on the surface $z = 0^-$ given by,

$$\sigma_{\rm bound}(z = 0^-) = \mathbf{P}(z = 0^-)\cdot\hat{\mathbf{n}} = \frac{\epsilon - 1}{4\pi}E_z(z = 0^-) = -\frac{\epsilon - 1}{4\pi}\frac{2q}{\epsilon + 1}\frac{a}{(r^2 + a^2)^{3/2}}\,, \tag{46}$$

recalling that the field inside the dielectric is that due to an effective charge $2q/(\epsilon + 1)$ at $z = a$,

$$\mathbf{E}(z = 0^-) = \frac{2q}{(\epsilon + 1)(r^2 + a^2)^{3/2}}(r\,\hat{\mathbf{r}} - a\,\hat{\mathbf{z}}). \tag{47}$$

The force on the bound surface charge density is not, however, $\sigma_{\rm bound}E_z(z = 0^-)$, but rather that due to the average electric field on the surface charge layer, which for $\epsilon' = 1$ is simply,

$$E_{{\rm ave},z}(z = 0^-) \;=\; \frac{E_z(z = 0^-) + E_z(z = 0^+)}{2}\,. \tag{48}$$

Recalling from sec. 2.1.1 that the field for $z > 0$ (with $\epsilon' = 1$) is that due to charge q at $z = a$ and the image charge $q' = -q(\epsilon - 1)/(\epsilon + 1)$ at $z = -a$,

$$\begin{aligned}\mathbf{E}(z = 0^+) &= \frac{q}{(r^2 + a^2)^{3/2}}(r\,\hat{\mathbf{r}} - a\,\hat{\mathbf{z}}) - \frac{q}{(r^2 + a^2)^{3/2}}\frac{\epsilon - 1}{\epsilon + 1}(r\,\hat{\mathbf{r}} + a\,\hat{\mathbf{z}}) \\ &= \frac{2q}{(\epsilon + 1)(r^2 + a^2)^{3/2}}(r\,\hat{\mathbf{r}} - \epsilon a\,\hat{\mathbf{z}}).\end{aligned} \tag{49}$$

Then,

$$E_{{\rm ave},z} = \frac{1}{2}\left(-\frac{2q}{\epsilon + 1} - \frac{2q\epsilon}{\epsilon + 1}\right)\frac{a}{(r^2 + a^2)^{3/2}} = -\frac{qa}{(r^2 + a^2)^{3/2}}, \tag{50}$$

and,

$$\begin{aligned}F_z(z < 0) &= \sigma_{\rm bound}(z = 0^-)E_{{\rm ave},z}(z = 0^-) = \frac{q^2a^2}{2\pi}\frac{\epsilon - 1}{\epsilon + 1}\int\frac{d{\rm Area}}{(r^2 + a^2)^3} \\ &= \frac{q^2a^2}{2}\frac{\epsilon - 1}{\epsilon + 1}\int_0^\infty\frac{d\,r^2}{(r^2 + a^2)^3} = \frac{q^2}{4a^2}\frac{\epsilon - 1}{\epsilon + 1}\,,\end{aligned} \tag{51}$$

which is indeed equal and opposite to the force found in eq. (45).

We can also compute the forces using the Maxwell stress tensor, which for linear, isotropic dielectric media has the form,

$$T_{ij} = \frac{1}{4\pi}\left(E_i D_j - \frac{\delta_{ij}}{2}\mathbf{E}\cdot\mathbf{D}\right). \tag{52}$$

The force on the charge q and on the dielectric medium at $z < 0$ can be computed by considering surfaces at $z = 0^+$ (in vacuum) that are completed by infinite hemispheres towards positive and negative z, respectively. Since the stress tensor varies as $1/r^4$ on these infinite hemispheres there is no contribution to the forces from these surfaces, so the force on the charge q is equal and opposite to that on the medium at $z < 0$. The force on the latter is,

$$\begin{aligned}
F_z(z<0) &= \int_{z=0^+} T_{zz}\, d\mathbf{Area}_z = \frac{1}{4\pi}\int_{z=0^+}\left(E_z^2(z=0^+) - \frac{E^2(z=0^+)}{2}\right) d\text{Area} \\
&= \frac{1}{4\pi}\int_0^\infty \pi\, dr^2\left(\frac{4q^2\epsilon^2 a^2}{(\epsilon+1)^2(r^2+a^2)^3} - \frac{2q^2(r^2+\epsilon^2 a^2)}{(\epsilon+1)^2(r^2+a^2)^3}\right) \\
&= \frac{q^2}{2(\epsilon+1)^2}\int_0^\infty dr^2\,\frac{\epsilon^2 a^2 - r^2}{(r^2+a^2)^3} = \frac{q^2}{2(\epsilon+1)^2}\left[\frac{\epsilon^2 a^2}{2a^4} - \left(\frac{1}{a^2} - \frac{a^2}{2a^4}\right)\right] \\
&= \frac{q^2}{4a^2}\frac{\epsilon-1}{\epsilon+1},
\end{aligned} \tag{53}$$

using Dwight 91.3 [31], which is the same result as found in eq. (51).[16] This reinforces the author's view that the stress tensor is the most reliable method of computation of forces in electromagnetic examples, in that only the total electromagnetic fields need be used (without need for discussion of effective charge densities and effective average fields as in eqs. (46)-(51)).[17]

3.1.2 $\epsilon' \neq 1$

When both ϵ and ϵ' are different from 1 the electric field on both sides of the interface $z = 0$ are, recalling sec. 2.1.1,

$$\mathbf{E}(z=0^-) = \frac{q}{(r^2+a^2)^{3/2}}\frac{2}{\epsilon'+\epsilon}(r\,\hat{\mathbf{r}} - a\,\hat{\mathbf{z}}), \tag{54}$$

and,

$$\begin{aligned}
\mathbf{E}(z=0^+) &= \frac{q}{\epsilon'(r^2+a^2)^{3/2}}(r\,\hat{\mathbf{r}} - a\,\hat{\mathbf{z}}) - \frac{q}{\epsilon'(r^2+a^2)^{3/2}}\frac{\epsilon-\epsilon'}{\epsilon+\epsilon'}(r\,\hat{\mathbf{r}} + a\,\hat{\mathbf{z}}) \\
&= \frac{2q}{\epsilon'(\epsilon+\epsilon')(r^2+a^2)^{3/2}}(\epsilon' r\,\hat{\mathbf{r}} - \epsilon a\,\hat{\mathbf{z}}),
\end{aligned} \tag{55}$$

[16] A peculiar use of the stress tensor was made in sec. IIIC of [30], which led to the erroneous conclusion that a new form of the stress tensor must be invented to compute forces on dielectrics.

[17] For other examples that illustrate this view, see [32, 33].

so that E_r is continuous across the interface. The bound surface charge densities are then,

$$\sigma_{\rm bound}(z=0^-) = \mathbf{P}(z=0^-)\cdot\hat{\mathbf{n}}^- = \frac{\epsilon-1}{4\pi}\mathbf{E}(z=0^-)\cdot\hat{\mathbf{z}} = -\frac{qa(\epsilon-1)}{2\pi(\epsilon+\epsilon')(r^2+a^2)^{3/2}}\,, \tag{56}$$

and,

$$\sigma_{\rm bound}(z=0^+) = \mathbf{P}(z=0^+)\cdot\hat{\mathbf{n}}^+ = -\frac{\epsilon'-1}{4\pi}\mathbf{E}(z=0^+)\cdot\hat{\mathbf{z}} = \frac{qa\epsilon(\epsilon'-1)}{2\pi\epsilon'(\epsilon+\epsilon')(r^2+a^2)^{3/2}}\,. \tag{57}$$

The electric field component E_z at $z = 0$ can be defined as,

$$\begin{aligned} E_z(z=0) &= E_z(z=0^-) + 4\pi\sigma_{\rm bound}(z=0^-) = E_z(z=0^+) - 4\pi\sigma_{\rm bound}(z=0^+) \\ &= -\frac{2qa\epsilon}{(\epsilon+\epsilon')(r^2+a^2)^{3/2}}\,, \end{aligned} \tag{58}$$

so the electric field exactly at the interface is,

$$\mathbf{E}(z=0) = \frac{2q}{(\epsilon+\epsilon')(r^2+a^2)^{3/2}}(r\,\hat{\mathbf{r}} - \epsilon a\,\hat{\mathbf{z}}). \tag{59}$$

The force on the medium at $z < 0$ can be computed as,

$$\begin{aligned} F_z(z<0) &= \sigma_{\rm bound}(z=0^-)\frac{E_z(z=0^-)+E_z(z=0)}{2} = \frac{q^2a^2}{2\pi}\frac{(\epsilon-1)(\epsilon+1)}{(\epsilon+\epsilon')^2}\int\frac{d\text{Area}}{(r^2+a^2)^3} \\ &= \frac{q^2a^2}{2}\frac{\epsilon^2-1}{(\epsilon+\epsilon')^2}\int_0^\infty\frac{d\,r^2}{(r^2+a^2)^3} = \frac{q^2}{4a^2}\frac{\epsilon^2-1}{(\epsilon+\epsilon')^2}\,, \end{aligned} \tag{60}$$

and also via the stress tensor at the interface (where $\mathbf{D} = \mathbf{E}$) as,

$$\begin{aligned} F_z(z<0) &= \int_{z=0} T_{zz}\,d\mathbf{Area}_z = \frac{1}{4\pi}\int_{z=0^+}\left(E_z^2(z=0) - \frac{E^2(z=0)}{2}\right)d\text{Area} \\ &= \frac{1}{4\pi}\int_0^\infty \pi\,dr^2\left(\frac{4q^2\epsilon^2a^2}{(\epsilon+\epsilon')^2(r^2+a^2)^3} - \frac{2q^2(r^2+\epsilon^2a^2)}{(\epsilon+\epsilon')^2(r^2+a^2)^3}\right) \\ &= \frac{q^2}{2(\epsilon+\epsilon')^2}\int_0^\infty dr^2\,\frac{\epsilon^2a^2-r^2}{(r^2+a^2)^3} = \frac{q^2}{2(\epsilon+\epsilon')^2}\left[\frac{\epsilon^2a^2}{2a^4} - \left(\frac{1}{a^2}-\frac{a^2}{2a^4}\right)\right] \\ &= \frac{q^2}{4a^2}\frac{\epsilon^2-1}{(\epsilon+\epsilon')^2}\,. \end{aligned} \tag{61}$$

The force on all material at $z > 0$, *i.e.*, the original charged wire plus the medium with relative permittivity ϵ', is equal and opposite to that of eqs. (60)-(61).

The issue of the force on the "point" charge q is delicate in macroscopic electrodynamics, as the size of a "point" charge is smaller than the length scale over which the macroscopic averages are taken. Following Lord Kelvin (for example, p. 397 of [34]), we can suppose the charge resides in a cavity that is otherwise vacuum and compute the electric field in this cavity due to the bound charge densities outside it. Of course, the result is famously dependent on the shape of the cavity. For a spherical cavity, the field inside is given by,[18]

$$\mathbf{E}_{\rm in} = \frac{3\epsilon'}{2\epsilon'+1}\mathbf{E}_{\rm out}, \tag{62}$$

[18] See, for example, sec. 4.4 of [3].

where $\mathbf{E}_{\rm out}$ is the field at the position of the cavity (in a medium with relative permittivity ϵ') in its absence.

In the present example, it seems reasonable to take $\mathbf{E}_{\rm out}$ to be the field at the position of the charge q due to the image charge q' at $z = -a$,

$$\mathbf{E}_{\rm out} = -\frac{q}{4\epsilon' a^2}\frac{\epsilon - \epsilon'}{\epsilon + \epsilon'}\,\hat{\mathbf{z}}, \tag{63}$$

in which case the force on the charge q is,

$$\mathbf{F}_q = q\mathbf{E}_{\rm in} = -\frac{3q^2}{4a^2}\frac{\epsilon - \epsilon'}{(\epsilon + \epsilon')(2\epsilon' + 1)}\,\hat{\mathbf{z}}, \tag{64}$$

which reduces to eq. (45) if $\epsilon' = 1$. However, this force is not of the form $\mathbf{F}_q = q\mathbf{E}_{\rm total}$ or $\mathbf{F}_q = q\mathbf{E}_{\rm out}$, as might be expected by a naïve extrapolation of the microscopic force law $\mathbf{F}_q = q\mathbf{E}$.

3.2 Line Charge Outside a Dielectric Cylinder

The force (per unit length) on a line charge q (per unit length) at $x = b$ in vacuum in the presence of a cylinder of relative permittivity ϵ at $r < a$ can be computed as that due to the electric field of the image charges $\pm q(\epsilon - 1)/(\epsilon + 1)$ at the origin and at $x = a^2/b$,

$$\mathbf{F}_q = q\mathbf{E}_{\rm image} = -q^2\frac{\epsilon - 1}{\epsilon + 1}\left(\frac{1}{b - a^2/b} - \frac{1}{b}\right)\hat{\mathbf{x}} = -q^2\frac{\epsilon - 1}{\epsilon + 1}\frac{a^2}{b(b^2 - a^2)}\,\hat{\mathbf{x}}, \tag{65}$$

recalling sec. 2.1.2 (for $\epsilon' = 1$).[19] The force on the cylinder is equal and opposite, as can be confirmed by use of the stress tensor at $r = a^+$.

3.3 Line Current Outside a Permeable Half Space

The force (per unit length) on a line current I at $x = a$ in vacuum in the presence of a medium of relative permeability μ at $x < 0$ can be computed as that due to the magnetic field of the image current $I(\mu - 1)/(\mu + 1)$ at $x = -a$,

$$\mathbf{F}_I = \frac{\mathbf{I} \times \mathbf{B}_{\rm image}}{c} = -\frac{I^2}{c^2 a}\frac{\mu - 1}{\mu + 1}\,\hat{\mathbf{x}}, \tag{66}$$

recalling sec. 2.3.1 (for $\mu' = 1$). The force on the cylinder is equal and opposite, as can be confirmed by use of the stress tensor at $x = 0^+$.

[19] The related problem of a line charge inside a cylindrical cavity in an otherwise infinite dielectric is solved in prob. 4, sec. 6 of [10], where the magnitude of the force is given as $2bq^2(\epsilon - 1)/(a^2 - b^2)(\epsilon + 1)$.

3.4 Line Current Outside a Permeable Cylinder

The force (per unit length) on a line current I at $x = b$ in vacuum in the presence of a cylinder of relative permeability μ at $r < a$ can be computed as that due to the magnetic field of the image currents $\mp I(\mu - 1)/(\mu + 1)$ at the origin and at $x = a^2/b$,

$$\mathbf{F}_I = \frac{\mathbf{I} \times \mathbf{B}_{\rm image}}{c} = -\frac{2I^2}{c^2}\frac{\mu - 1}{\mu + 1}\left(\frac{1}{b - a^2/b} - \frac{1}{b}\right)\hat{\mathbf{x}} = -\frac{2I^2}{c^2}\frac{\mu - 1}{\mu + 1}\frac{a^2}{b(b^2 - a^2)}\hat{\mathbf{x}}, \qquad (67)$$

recalling sec. 2.3.2 (for $\mu' = 1$).[20] The force on the cylinder is equal and opposite, as can be confirmed by use of the stress tensor at $r = a^+$.

3.5 Permanent Magnet in Contact with a High Permeability Medium (Refrigerator Magnet)

If the permanent magnet is magnetized perpendicular to the surface of the high permeability medium, then the magnetization of the image magnet is the same as that of the original, according to sec. 2.3.3. The magnet is attracted to the medium with a force $\int B^2\, d{\rm Area}/8\pi$, where $\mathbf{B}$ is the magnetic field at the symmetry plane of the original magnet plus its image. For example, if the original magnet is a hemisphere of radius a and uniform magnetization $\mathbf{M}$ perpendicular to its base, the combination of original plus image magnet is spherical, with well-known fields, and the attractive force is $\pi^2 a^2 M^2$ [35].

4 Time-Dependent Image Method (Oct. 25, 2023)

In general, image methods apply to static examples. However, the image methods is also valid for a arbitrary charge/current distributions in vacuum, outside an infinite perfectly conducting plane [36]. For other geometries, the time-dependent fields that would be implied by the image method do not satisfy the boundary conditions at the perfectly conduction surface.

Some examples of time-dependent image methods are given in [37].

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