

What is i^i ?

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*This amusing problem is posed by R.C. Henry at <http://henry.pha.jhu.edu/rch.html>*¹

Henry's solution invokes the facts that $(a^b)^c = a^{bc}$ and that one can write $i = e^{(4n+1)i\pi/2}$, for any real integer n , to find the infinite set of answers,²

$$i^i = (e^{(4n+1)i\pi/2})^i = e^{-(4n+1)\pi/2}. \quad (1)$$

Henry favors setting $n = 0$, $\Rightarrow i^i = e^{-\pi/2} \approx 0.208$.

Following Henry's advice to try the problem before looking at his solution, I proceeded differently,

$$i^i = x = e^{\ln x} = e^{-(4n+1)\pi/2}, \quad (2)$$

as in eq. (1), using,

$$\ln x = i \ln i = i \ln(1 e^{i\pi/2}) = i[0 + (4n + 1)i\pi/2] = -(4n + 1)\pi/2, \quad (3)$$

which follows from the definition of the logarithm of a complex number,

$$\ln z = \ln r + i(\phi + 2n\pi), \quad \text{where} \quad z = r e^{i\phi}. \quad (4)$$

¹July 19, 2022. For a related puzzler, see <http://kirkmcd.princeton.edu/examples/1tox.pdf>

²July, 23, 2022. As noted itox.pdf of footnote 1, use of $(e^x)^y = e^{xy}$ for complex x and y can lead to trouble, but Henry's analysis is OK.