Ladder + Rope

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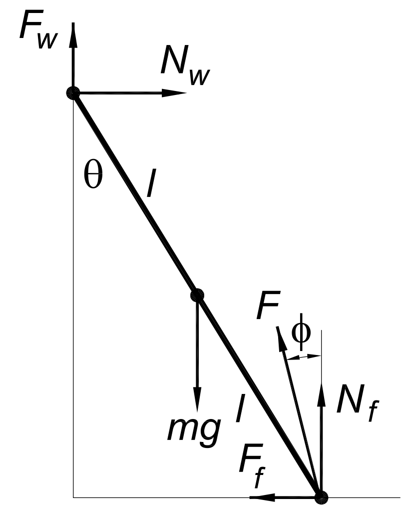
(November 21, 2012)

1 Problem

In a variant of the famous problem of static equilibrium of a ladder, what minimum force F must be exerted on the foot of the ladder, via a rope at angle ϕ to the vertical, such that the normal force N_f of the floor on the ladder goes to zero?



2 Solution

In general there will be nonzero horizontal and vertical force components at the points of contact of the ladder with the floor and wall, as sketched above. The conditions of static equilibrium, that the sum of the (vector) forces be zero and that the torque about any point be zero, give only three relations among the four unknown force components. The usual procedure is to assume that the vertical force, F_w, of the wall on the ladder is zero, in which case $N_f = mg$, $N_w = F_f = (mg/2)\tan\theta$, and $\tan\theta \le 2\mu_f$ where μ_f is the coefficient of static friction of the ladder with the floor. Since this assumption leads to a solution, it might be argued that Nature prefers this solution, and F_w is zero even if the wall has a nonzero coefficient of static friction.

In the present example we will find that when the forces, N_f and F_f, of the floor on the ladder go to zero, the vertical force F_w is nonzero in general. If the forces on the ladder change continuously as the tension F in the rope is increased, it must be that Nature does not always choose that F_w be zero when the mathematics of static equilibrium gives indeterminant solutions.

Rather, the view expressed on p. 332 of the text of Halliday and Resnick [1] is more correct; that details of the elastic behavior of the floor and ladder (along with initial conditions) resolve the indeterminancy, with different force components in cases with differing details.

In the rest of this note we consider that the forces of the floor on the ladder have just gone to zero, *i.e.*, $F_f = 0 = N_f$.

Then, vanishing of the total horizontal force on the ladder implies that,

$$N_w = F \sin\phi, \tag{1}$$

vanishing of the total vertical force implies,

$$mg = F\cos\phi + F_w, \tag{2}$$

and vanishing of the torque about the top of the ladder (of length $2l$),

$$\frac{mgl\sin\theta}{2} = Fl\sin(\theta - \phi), \qquad i.e., \qquad F = \frac{mg\sin\theta}{2\sin(\theta-\phi)} = \frac{mg}{2\cos\phi}\,\frac{\tan\theta}{\tan\theta - \tan\phi}\,. \tag{3}$$

Since the tension F in the rope, and the normal force N_w, cannot be negative, we must have that,

$$0 < \phi < \theta. \tag{4}$$

From eqs. (2)-(3) we find that the vertical force of the wall on the ladder is,

$$F_w = \frac{mg}{2}\frac{\sin\theta\cos\phi - 2\cos\theta\sin\phi}{\sin(\theta-\phi)} = \frac{mg}{2}\frac{\tan\theta - 2\tan\phi}{\tan\theta - \tan\phi}\,, \tag{5}$$

which can either positive or negative (although negative values are more readily achievable).

If the force F_w of the wall on the ladder can be characterized by a coefficient μ_w of static friction, we must have that,

$$\mu_w \geq \frac{|F_w|}{N_w} = \frac{|2\sin\theta\cos\phi - \cos\theta\sin\phi|}{\sin\theta\sin\phi} = \frac{|2\tan\phi - \tan\theta|}{\tan\theta\tan\phi}\,. \tag{6}$$

In principle, the ladder could rest stably against the wall with $\theta \approx 45^\circ$ when $F = 0$, in which case it would suffice that $\mu_w \lesssim 1$ for the ladder not to slip at the wall when its foot loses contact with the floor for $\phi \lesssim \theta$ and F very large. Since coefficients of static friction larger than unity are hard to achieve, the likely behavior for small initial θ is that the ladder starts to slip at the wall before its lower end loses contact with the floor, which leads to another famous ladder problem (see, for example, [2]).

Thanks to Juan Uson for discussions of this problem.

References

[1] D. Halliday and R. Resnick, *Physics* (Wiley, 1960),
http://kirkmcd.princeton.edu/examples/mechanics/resnick_halliday_p332.pdf

[2] K.T. McDonald, *Torque Analyses of a Sliding Ladder* (May 6, 2007),
http://kirkmcd.princeton.edu/examples/ladder.pdf