Maximum Energy of Circular Colliders

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1 Problem

In its final months of operation, the world's largest circular particle accelerator, the Large Electron-Positron (LEP) collider, was pushed to its maximum energy of 100 GeV per beam, yielding collision energies of 200 GeV. Two main limitations to circular-accelerator performance are energy loss due to synchrotron radiation, and the maximum value of fields that can be obtained in bending magnets. This problem explores these limitations.[1]

Although the LEP tunnel has a circumference of 26.7 km, the effective bending radius of the dipole magnets is only 3.1 km.

a) What is the magnetic field B in the bending magnets?

b) What fraction of the energy of an electron is lost to synchrotron radiation during one orbit around the LEP ring at 100 GeV beam energy?

Hint: Recall the Larmor expression for the power radiated by an accelerated charge with nonrelativistic velocity.

c) LEP will be converted to LHC, the Large Hadron Collider, and will accelerate protons instead of electrons. New magnets will be installed in the old tunnel. What is the highest possible beam energy of LHC in the future, based upon a physics estimate of the strength of materials used in the magnets?

2 Solution

a) For a collider of energy 200 GeV, the energy of each beam is 100 GeV. A 100 GeV electron beam is highly relativistic ($v \approx c$), with Lorentz factor,

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{E}{m_e c^2} \approx 2 \times 10^5, \tag{1}$$

since the electron mass is $m_e \approx 0.5\ \mathrm{MeV}/c^2$. The circular motion of the electron in the lab frame is due to the Lorentz force caused by the transverse magnetic field $\mathbf{B}$. The Lorentz force equals the relativistic mass γm_e times the radial acceleration,

$$\mathbf{F} = e\mathbf{v} \times \mathbf{B} = -evB\hat{\mathbf{r}} = -\gamma m_e \frac{v^2}{R}\hat{\mathbf{r}} \tag{2}$$

[1]See also K.T. McDonald, *The Maximal Energy Attainable in a Betatron*, http://kirkmcd.princeton.edu/examples/betatron.pdf

(in MKSA units). Thus, the magnetic field must be,

$$B = \frac{\gamma m_e v}{eR} \approx \frac{\gamma m_e c^2}{ceR} = \frac{E/e}{cR} \approx \frac{10^{11}\ \mathrm{V}}{3 \times 10^8\ \mathrm{m/s} \cdot 3.1 \times 10^3\ \mathrm{m}} = 0.112\mathrm{T}, \tag{3}$$

which is only 5% of the saturation field of an electromagnet with an iron flux return.

If one prefers to use Gaussian units, $F = ma$ becomes,

$$\mathbf{F} = e\frac{\mathbf{v}}{c} \times \mathbf{B} = -\frac{evB}{c}\hat{\mathbf{r}} = -\gamma m_e \frac{v^2}{R}\hat{\mathbf{r}}, \tag{4}$$

so that,

$$B = \frac{\gamma m_e c v}{eR} \approx \frac{\gamma m_e c^2}{eR} = \frac{E}{eR} \approx \frac{2 \times 10^{11}\ \mathrm{eV} \cdot 1.6 \times 10^{-12}\ \mathrm{erg/eV}}{4.8 \times 10^{-10}\ \mathrm{esu} \cdot 3.1 \times 10^5\ \mathrm{cm}} = 1120\mathrm{G}. \tag{5}$$

b) The Larmor expression for the power radiated by a particle of mass m and charge e with acceleration $a^\star$ in the frame where the particle is instantaneously at rest is (in Gaussian units),

$$P^\star = \frac{2e^2 a^{\star 2}}{3c^3} \tag{6}$$

Recall that the radiated power is a relativistic invariant, since $P = dE/dt$ and both energy and time are "time" components of a 4-vector, and so transform in the same way.

The acceleration $a^\star$ of the electron in its rest frame is due to the electric field $E^\star$ that is the transformation of the transverse magnetic field B in the lab frame,

$$E^\star = \gamma\beta B \approx \gamma B. \tag{7}$$

The rest-frame acceleration is therefore,

$$a^\star = \frac{eE^\star}{m_e} \approx \frac{\gamma B e}{m_e}, \tag{8}$$

and hence,

$$\begin{aligned}
\frac{dE}{dt} &= P = P^\star \approx \frac{2\gamma^2 e^4 B^2}{3m_e^2 c^3} = \frac{2\gamma^2 c r_0^2 B^2}{3} \\
&= 0.667 \cdot (2 \times 10^5)^2 \cdot 3 \times 10^{10} \cdot (2.8 \times 10^{-13})^2 \cdot 1120^2 \\
&\approx 75\ \mathrm{erg/s} = 7.5 \times 10^{-6}\ \mathrm{J/s} \approx 5 \times 10^{13}\ \mathrm{eV/s}.
\end{aligned} \tag{9}$$

using the classical electron radius $r_0 = e^2/m_e c^2 \approx 2.8 \times 10^{-13}$ cm, and the fact that 1 eV $= 1.6 \times 10^{-19}$ J. The time for an electron to make one turn around the LEP ring whose effective circumference C is $2\pi \cdot 3.1 = 19.5$ km is,

$$\Delta t = \frac{C}{c} = \frac{19.5 \times 10^5}{3 \times 10^{10}} \approx 6.5 \times 10^{-5}\ \mathrm{s}. \tag{10}$$

Thus, the energy radiated during one turn is,

$$\Delta E = P\Delta t \approx 5 \times 10^{13} \cdot 6.5 \times 10^{-5} = 3.3 \times 10^9\ \mathrm{eV} = 3.3\ \mathrm{GeV} = 0.033E. \tag{11}$$

c) Ultimately magnets blow up because the pressure due to the Lorentz force of the magnetic field on the conductors exceeds the tensile strength of the wires which make up the winding. For a quick estimate, we note that the total energy of the system, magnetic field energy + atomic binding energy, must be negative for stability.

The magnetic field energy density is $B^2/8\pi$ in Gaussian units, while the atomic binding energy of a typical metal is about 1 eV = 10^{-19} J = 10^{-12} erg, and whose atomic diameter is about 3 angstroms = 3×10^{-8}cm. Thus,

$$B_{\text{crit}}^2 \approx \frac{8\pi \cdot 10^{-12}}{(3 \times 10^{-8})^3} \approx 10^{16}, \tag{12}$$

and $B_{\text{crit}} \approx 10^6$ G = 100 T. Since the energy of particles that can be stored in a ring of fixed radius is proportional to the magnetic field, as shown in eq. (3), beams of 100 GeV $\cdot$ 100 T / 0.1 T = 100 TeV could be stored in the LEP tunnel with 100 T magnets (leading to 200 TeV collision energy)

However, magnets of 100 T last only one pulse. The highest field strength of magnets that can be pulsed "indefinitely" is about 50 T. These magnets are made of copper, and consume vast amounts of power. For economic plausibility, a large ring must be made of superconducting magnets, for which 10-12 T is the present limit of reliability for large magnets as are needed for a particle accelerator. Thus, the LHC might plausibly be upgraded to 10-12 TeV per beam, compared to the nominal design of 7 TeV per beam using 8-T magnets.