A Gentler Loop-the-Loop

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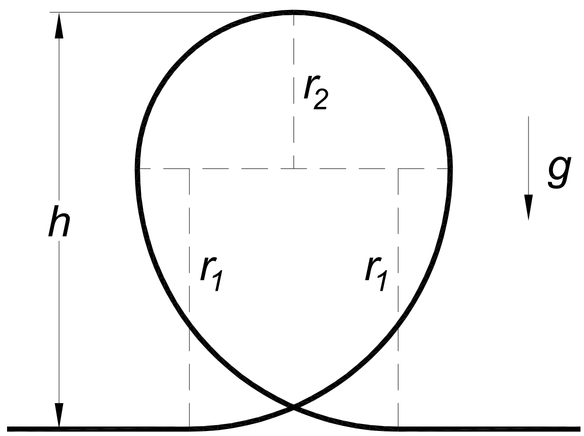
1 Problem

A classic mechanics problem is a bead of mass m that slides on a vertical, circular loop of wire, without friction. If the bead experiences zero normal force when at the top of the loop, then the normal force (and apparent weight of the bead) at the bottom is $6mg$, where g is the acceleration due to gravity.

In case of a vertical loop in a roller coaster a maximum apparent weight of $6mg$ is considered excessive, and loops of noncircular shapes are typically used to lower the peak force.[1] Consider a loop the consists of two quarter circles of radii r_1 and a half circle of radius r_2 with total vertical height $h = r_1 + r_2$ as sketched below. What values of r_1 and r_2 minimize the peak normal force on a sliding bead if the normal force is zero at the top of the loop?



2 Solution

If the normal force vanishes when the bead is at the top of the loop, then its velocity v_t there is,

$$v_t^2 = g r_2, \tag{1}$$

[1]For a description of some of the shapes used in roller coaster loops see, for example,
A.-M. Pendrill, *Rollercoaster loop shapes*, Phys. Ed. **40**, 517 (2005),
http://kirkmcd.princeton.edu/examples/mechanics/pendrill_pe_40_517_05.pdf
See also, A.B. Nordmark and H. Essén, *The comfortable roller coaster—on the shape of tracks with a constant normal force*, Eur. J. Phys. **31**, 1307 (2010),
http://kirkmcd.princeton.edu/examples/mechanics/nordmark_ejp_31_1307_10.pdf
A.-M. Pendrill, *Roller coaster loop shapes revisited*, Phys. Ed. **51**, 517 (2016),
http://kirkmcd.princeton.edu/examples/mechanics/pendrill_pe_51_030106_16.pdf

such that the force of gravity, mg provides the required centripetal force mv_t^2/r_2 for circular motion.

The normal force when the bead is on the arc of radius r_2 is maximum when it is at the lowest point on that arc, where the motion is instantaneously vertical and gravity does not contributes to the (horizontal) centripetal force. The velocity v_l at this point is (assuming no energy is lost to friction) related by,

$$v_l^2 = v_t^2 + 2gr_2 = 3gr_2, \tag{2}$$

and the normal force there is just the required centripetal force,

$$N_l = \frac{mv_l^2}{r_2} = 3mg, \tag{3}$$

independent of the value of r_2.

When the bead is on either of the arcs of radius r_1 the normal force is maximum when the bead is at the lowest point, *i.e.*, at the bottom of the loop. Its velocity v_b there is related by,

$$v_b^2 = v_t^2 + 2gh = g(r_2 + 2h), \tag{4}$$

and the normal force N_b is related by,

$$N_b = \frac{mv_b^2}{r_1} + mg = mg\left(\frac{r_2 + 2h}{r_1} + 1\right) = \frac{3mgh}{r_1} \geq 3mg\,. \tag{5}$$

Thus, N_b is the peak normal force for any choice of r_1 and r^2, and this peak value is minimized when $r_1 = h$ and $r_2 = 0$, in which case the peak normal force is $3mg$.

Of course, the case of $r_2 = 0$ is nonphysical. Use of a practical, nonzero value of r_2 implies a peak normal force larger than $3mg$. For example, with $r_1 = 3h/4$, $r_2 = h/4$, the peak normal force is $N_b = 4mg$. *Typical loops in roller coasters are designed for peak normal force (peak apparent weight) of about* $4mg$.