Relativistic Angular Momentum of a Spinning Disk

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(October 29, 2022)

1 Problem

What is the angular momentum of a uniform disk of rest mass m_0 and radius r when it spins with angular velocity ω about its axis, which is at rest in an inertial frame? Of course, the tangential velocity, $v = \omega r$, of its rim is less than the speed c of light in vacuum. You may neglect a possible increase in the radius of the disk as it spins.

2 Solution

For low angular velocity ω, the angular momentum of the uniform disk is,

$$L_0 = I_0 \omega = \frac{m_0 r^2 \omega}{2}, \qquad I_0 = \frac{m_0 r^2}{2}, \tag{1}$$

where I_0 is the moment of inertia of the uniform disk (for small ω).

2.1 Relativistic Mass

We first compute the "relativistic mass" of the spinning disk,

$$\begin{aligned} m &= \int dm = \int \frac{dm_0}{\sqrt{1 - v^2/c^2}} = \int_0^r 2\pi r'\, dr' \, \frac{\rho_0}{\sqrt{1 - \omega^2 r'^2/c^2}} = \int_0^{r^2} \pi\, dr'^2 \, \frac{m_0}{\pi r^2 \sqrt{1 - \omega^2 r'^2/c^2}} \\ &= \frac{m_0}{r^2} \left[-\frac{2c^2}{\omega^2} \sqrt{1 - \omega^2 r'^2/c^2} \right]_0^{r^2} = \frac{2 m_0 c^2}{\omega^2 r^2} \left(1 - \sqrt{1 - \omega^2 r^2/c^2} \right) \approx \frac{2 m_0 c^2}{\omega^2 r^2} \left(\frac{\omega^2 r^2}{2c^2} + \frac{\omega^4 r^4}{8c^4} \right) \\ &= m_0 \left(1 + \frac{\omega^2 r^2}{4c^2} \right) = m_0 \left(1 + \frac{v^2}{4c^2} \right), \end{aligned} \tag{2}$$

where $\rho_0 = m/\pi r^2$ is the rest-mass density per unit area of the disk, and we have used Dwight 191.01 [1].

2.2 Relativistic Angular Momentum

The angular momentum $\mathbf{L}$ of the disk, about its center, is,

$$\mathbf{L} = \int \mathbf{r}' \times dm\, \mathbf{v}', \tag{3}$$

which is along the axis of the disk, with value L,

$$
\begin{aligned}
L = \int dm\, r' v' = \int \frac{dm_0}{\sqrt{1 - v^2/c^2}}\, r' v' &= \int_0^r 2\pi r'\, dr'\, \frac{\rho_0}{\sqrt{1 - \omega^2 r'^2/c^2}}\, r'\, \omega r' \\
&= \int_0^{r^2} \pi\, dr'^2\, \frac{m_0}{\pi r^2 \sqrt{1 - \omega^2 r'^2/c^2}}\, \omega r'^2 \\
&= \frac{m_0\, \omega}{r^2} \frac{2c^4}{\omega^4} \left[\frac{(\sqrt{1 - \omega^2 r'^2/c^2})^3}{3} - \sqrt{1 - \omega^2 r'^2/c^2} \right]_0^{r^2} \\
&= \frac{2 m_0 c^4}{\omega^3 r^2} \left(\frac{(\sqrt{1 - \omega^2 r^2/c^2})^3}{3} - \sqrt{1 - \omega^2 r^2/c^2} + \frac{2}{3} \right) \\
&\approx \frac{2 m_0 c^4}{\omega^3 r^2} \left(\frac{\omega^4 r^4}{4c^4} + \frac{\omega^6 r^6}{6c^6} \right) = \frac{m_0 r^2 \omega}{2} \left(1 + \frac{\omega^2 r^2}{3c^2} \right),
\end{aligned} \tag{4}
$$

using Dwight 191.11. For $\omega r/c \ll 1$, we recover the nonrelativistic result that $L = I_0\, \omega$ where $I_0 = m_0 r^2/2$ is the (nonrelativistic) moment of inertia of a uniform disk. However, in the relativistic case, the angular momentum L is not linearly proportional to the angular velocity ω, so we cannot define a relativistic moment of inertia in the usual manner.

2.2.1 Comment

A rapidly rotating disk is unstable against the centrifugal force, and will fly apart if the tangential velocity, $v = \omega r$, of the rim exceeds (roughly) the speed of sound in the material of the disk. See, for example, [2]. Since $v_{\rm sound}/c \approx 10^{-5}$ for typical materials, the relativistic corrections to the mass and angular momentum, eqs. (2) and (4), are of relative order 10^{-10} or less for an intact spinning disk, and are of little concern from an "engineering" perspective.[1]

Thanks to Derek Abbott for e-diskussions of the problem.

References

[1] H.B. Dwight, *Tables of Integrals and Other Mathematical Data*, 4th ed. (Macmillan, 1961), http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf

[2] K.T. McDonald, *Rayleigh's Spinning Ring* (July 12, 2017), http://kirkmcd.princeton.edu/examples/ring.pdf

[1] A separate issue is that the atoms in the spinning disk are Lorentz contracted in the direction of their velocity, so the disk would crack at high enough angular velocity. However, the disk cracks due to centrifugal force at much lower angular velocity.