3×3 Magic Squares with Duplicate Digits Allowed

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
Nancy C. Schaefer
Littlebrook Elementary School, Princeton, NJ 08540
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1 Problem

Magic squares are $n \times n$ array of integers for which the sum of the numbers in the columns, rows and diagonals are all the same. The classic 3×3 magic square that incorporates the digits 1-9 is shown below:

$$\begin{array}{ccc} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{array} \tag{1}$$

Give all possible 3×3 magic squares whose elements are the single digits 1-9, but with duplicate digits allowed. Thus,

$$\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \tag{2}$$

is the simplest magic square according to the present problem.

Squares that are related by reflection about a horizontal, vertical or diagonal axis are not considered as distinct. That is, the square

$$\begin{array}{ccc} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{array} \tag{3}$$

is considered the same as square (1).

2 Solution

Extensive web sites related to magic squares can be found starting with [1].

We solved the present problem by the method of exhaustion, using a computer program to carry out the search for 3×3 magic squares. While sums of 3 digits from 1 to 9 range between 3 and 27, we find that only sums that are multiples of 3 are associated with magic squares, and that there are 35 distinct 3×3 magic squares with duplicate digits allowed.

2.1 Sum = 3

There is only 1 magic square for this case:

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix} \tag{4}$$

2.2 Sum = 6

There are 2 distinct magic square for this case (with a total of 5 if reflected squares are counted as different):

$$\begin{matrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{matrix} \qquad \begin{matrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{matrix} \tag{5}$$

2.3 Sum = 9

There are 4 distinct magic square for this case (with a total of 13 if reflected squares are counted as different):

$$\begin{matrix} 1 & 5 & 3 \\ 5 & 3 & 1 \\ 3 & 1 & 5 \end{matrix} \qquad \begin{matrix} 2 & 3 & 4 \\ 5 & 3 & 1 \\ 2 & 3 & 4 \end{matrix} \qquad \begin{matrix} 2 & 4 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 4 \end{matrix} \qquad \begin{matrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{matrix} \tag{6}$$

2.4 Sum = 12

There are 6 distinct magic square for this case (with a total of 25 if reflected squares are counted as different):

$$\begin{matrix} 1 & 7 & 4 \\ 7 & 4 & 1 \\ 4 & 1 & 7 \end{matrix} \qquad \begin{matrix} 2 & 5 & 5 \\ 7 & 4 & 1 \\ 3 & 3 & 6 \end{matrix} \qquad \begin{matrix} 2 & 6 & 4 \\ 6 & 4 & 2 \\ 4 & 2 & 6 \end{matrix} \qquad \begin{matrix} 3 & 4 & 5 \\ 6 & 4 & 2 \\ 3 & 4 & 5 \end{matrix} \qquad \begin{matrix} 3 & 5 & 4 \\ 5 & 4 & 3 \\ 4 & 3 & 5 \end{matrix} \qquad \begin{matrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{matrix} \tag{7}$$

2.5 Sum = 15

There are 9 distinct magic square for this case (with a total of 41 if reflected squares are counted as different). The second of these is the classic 3×3 magic square.

$$\begin{matrix} 1 & 9 & 5 \\ 9 & 5 & 1 \\ 5 & 1 & 9 \end{matrix} \qquad \begin{matrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{matrix} \qquad \begin{matrix} 2 & 8 & 5 \\ 8 & 5 & 2 \\ 5 & 2 & 8 \end{matrix} \qquad \begin{matrix} 3 & 5 & 7 \\ 9 & 5 & 1 \\ 3 & 5 & 7 \end{matrix} \qquad \begin{matrix} 3 & 6 & 6 \\ 8 & 5 & 2 \\ 4 & 4 & 7 \end{matrix} \qquad \begin{matrix} 3 & 7 & 5 \\ 7 & 5 & 3 \\ 5 & 3 & 7 \end{matrix}$$

$$\begin{matrix} 4 & 5 & 6 \\ 7 & 5 & 3 \\ 4 & 5 & 6 \end{matrix} \qquad \begin{matrix} 4 & 6 & 5 \\ 6 & 5 & 4 \\ 5 & 4 & 6 \end{matrix} \qquad \begin{matrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{matrix} \tag{8}$$

2.6 Sum = 18

There are 6 distinct magic square for this case (with a total of 25 if reflected squares are counted as different):

$$\begin{matrix} 3 & 9 & 6 \\ 9 & 6 & 3 \\ 6 & 3 & 9 \end{matrix} \qquad \begin{matrix} 4 & 7 & 7 \\ 9 & 6 & 3 \\ 5 & 5 & 8 \end{matrix} \qquad \begin{matrix} 4 & 8 & 6 \\ 8 & 6 & 4 \\ 6 & 4 & 8 \end{matrix} \qquad \begin{matrix} 5 & 6 & 7 \\ 8 & 6 & 4 \\ 5 & 6 & 7 \end{matrix} \qquad \begin{matrix} 5 & 7 & 6 \\ 7 & 6 & 5 \\ 6 & 5 & 7 \end{matrix} \qquad \begin{matrix} 6 & 6 & 6 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \end{matrix} \tag{9}$$

2.7 Sum = 21

There are 4 distinct magic square for this case (with a total of 13 if reflected squares are counted as different):

$$\begin{matrix} 5 & 9 & 7 \\ 9 & 7 & 5 \\ 7 & 5 & 9 \end{matrix} \qquad \begin{matrix} 6 & 7 & 8 \\ 9 & 7 & 5 \\ 6 & 7 & 8 \end{matrix} \qquad \begin{matrix} 6 & 8 & 7 \\ 8 & 7 & 6 \\ 7 & 6 & 8 \end{matrix} \qquad \begin{matrix} 7 & 7 & 7 \\ 7 & 7 & 7 \\ 7 & 7 & 7 \end{matrix} \tag{10}$$

2.8 Sum = 24

There are 2 distinct magic square for this case (with a total of 5 if reflected squares are counted as different):

$$\begin{matrix} 7 & 9 & 8 \\ 9 & 8 & 7 \\ 8 & 7 & 9 \end{matrix} \qquad \begin{matrix} 8 & 8 & 8 \\ 8 & 8 & 8 \\ 8 & 8 & 8 \end{matrix} \tag{11}$$

2.9 Sum = 27

There is only 1 magic square for this case:

$$\begin{matrix} 9 & 9 & 9 \\ 9 & 9 & 9 \\ 9 & 9 & 9 \end{matrix} \tag{12}$$

3 References

[1] http://forum.swarthmore.edu/alejandre/magic.square.html