Permeable Shell in a Uniform External Field

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(February 24, 2007)

1 Problem

Deduce the ratio of the external magnetic field H_0 to the magnetic field H_{in} inside a spherical shell of relative permeability μ.

2 Solution

We consider a spherical shell of inner radius a and outer radius b made of a material of relative permeability μ that is immersed in an otherwise uniform external magnetic field $\mathbf{B}_0 = \mathbf{H}_0 = -H_0\hat{\mathbf{z}}$, where we use Gaussian units. The magnetic field $\mathbf{H}$ can be deduced from a scalar potential Φ according to $\mathbf{H} = -\boldsymbol{\nabla}\Phi$. The scalar potential corresponding to the external field is,

$$\Phi_0 = H_0 r \cos\theta = B_0 r P_1, \tag{1}$$

in a spherical coordinate system (r, θ, ϕ) whose origin is at the center of the permeable sphere. The potential of the perturbed field will contain only angular functions $P_1(\cos\theta)$, and can be written as,

$$\Phi = \begin{cases} ArP_1 & (r < a), \\ BrP_1 + CP_1/r^2 & (a < r < b), \\ H_0 r P_1 + DP_1/r^2 & (r > b). \end{cases} \tag{2}$$

The magnetic field for $r < a$ is $H_{in} = A$, so we wish to relate this quantity to the external field H_0.

Continuity of the potential at $r = a$ and b requires that,

$$A = B + C/a^3, \tag{3}$$

$$B + C/b^3 = H_0 + D/b^3. \tag{4}$$

The Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ implies that the radial component of the magnetic field $\mathbf{B} = \mu\mathbf{H}$ is continuous at the boundaries at $r = a$ and b, and hence,

$$A = \mu(H_0 + B - 2C/a^3), \tag{5}$$

$$\mu(B - 2C/b^3) = H_0 - 2D/b^3. \tag{6}$$

From eqs. (3) and (5) we find, writing $A = H_{in}$,

$$B = \frac{2\mu + 1}{3\mu} H_{in}, \tag{7}$$

and then,

$$C = \frac{\mu - 1}{3\mu} a^3 H_{in}. \tag{8}$$

Equation (4) now gives,

$$D = \left[\frac{2\mu + 1}{3\mu} b^3 + \frac{\mu - 1}{3\mu} a^3\right] H_{in} - b^3 H_0. \tag{9}$$

Inserting eqs. (7)-(9) into eq. (6) we have,

$$\begin{aligned} 3H_0 &= H_{in}\left[\mu\left(\frac{2\mu+1}{3\mu} - 2\frac{\mu-1}{3\mu}\frac{a^3}{b^3}\right) + 2\frac{2\mu+1}{3\mu} + 2\frac{\mu-1}{3\mu}\frac{a^3}{b^3}\right] \\ &= \frac{H_{in}}{3}\left[5 + 4\frac{a^3}{b^3} + 2\left(\mu + \frac{1}{\mu}\right)\frac{b^3 - a^3}{b^3}\right]. \end{aligned} \tag{10}$$

As expected, this form implies that $H_{in} = H_0$ when either $\mu = 1$ or $a = b$.

For a thin, high-permeability shell with $b = a + \delta a$ and $\mu \gg 1$, eq. (10) becomes,[1]

$$\frac{H_0}{H_{in}} \approx 1 + \frac{2\mu}{3}\frac{\delta a}{a}. \tag{11}$$

For an application of eq. (11), see [3].

References

[1] A. Mager *Magnetic Shields*, IEEE Trans. Magn. **6**, 67 (1970), http://kirkmcd.princeton.edu/examples/detectors/mager_ieeetm_6_67_70.pdf

[2] R. Fitzpatrick, *Magnetic Shielding*, http://farside.ph.utexas.edu/teaching/jk1/lectures/node52.html

[3] V. Ghazikhanian and K.T. McDonald, *μ-Metal Wire Magnetic Shields for Large PMTs* (Feb. 24. 2007), http://kirkmcd.princeton.edu/dayabay/shield.pdf

[1] Equation (11) differs slightly from that quoted without derivation in [1], while it agrees with that given by [2].