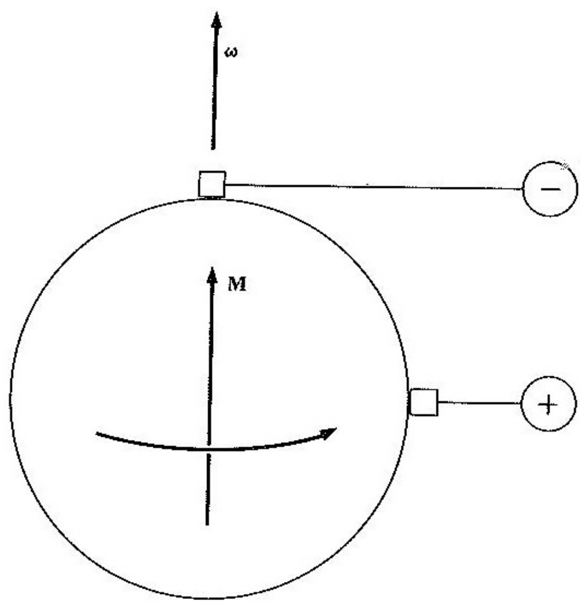
Unipolar Induction
via a Rotating Magnetized Sphere

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(November 13, 2012; updated July 30, 2019)

1 Problem

A neutral, conducting sphere of radius R with permanent magnetization density $\mathbf{M}_0 = M_0\,\hat{\mathbf{z}}$ parallel to its axis when at rest is rotated about that axis with angular velocity $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}$ with respect to the lab frame. Deduce the electric-charge distribution, the electric potential and the electric field in the lab frame. Compare with the case that the sphere is nonconducting.



This configuration is a variant of unipolar induction, as discussed by Faraday in 1851 [1], who also considered the case of the magnetized cylinder at rest while the voltmeter and contact wires rotated around the axis of the cylinder.[1]

2 Solution

The case of a magnetized sphere was perhaps first considered by Swann [6]. See also, [7, 8, 9]. For the related cases of a rotating magnetized cylinder, and a conducting sphere rotating in an external magnetic field, see [10, 11, 12].

A uniformly magnetized sphere of radius R that is at rest in an inertial frame is well known to have electric fields $\mathbf{E} = 0 = \mathbf{D}$ (and hence zero electric-polarization density $\mathbf{P}$),

[1] See also, for example, [2]-[5].

while the magnetic fields are given (in Gaussian units) by,

$$\mathbf{B}(r > R) = \mathbf{H}(r > R) = \frac{3(\mathbf{m}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_0}{r^3}, \quad \mathbf{B}(r < R) = \frac{8\pi}{3}\mathbf{M}_0, \quad \mathbf{H}(r < R) = -\frac{4\pi}{3}\mathbf{M}_0, \tag{1}$$

where the magnetic moment $\mathbf{m}_0$ is given by,[2]

$$\mathbf{m}_0 = \frac{4\pi R^3}{3}\mathbf{M}_0. \tag{2}$$

However, it is not self-evident that these forms hold in the lab frame of the present problem.

2.1 Analysis Using a Comoving Inertial Frame

As discussed in [14, 15, 16, 17], the best approach to an understanding of lab-frame electrodynamics of a rotating system is via a comoving inertial frame corresponding to some point in the rotating system.

We follow Minkowski [14] in arguing that the local magnetization at a point P in the rotating sphere equals the rest value $\mathbf{M}_0$ according to an observer in the inertial frame that is instantaneously comoving with point P. That is $\mathbf{M}^\star = \mathbf{M}_0$, where the superscript * indicates quantities observed in the comoving inertial frame.

Similarly, we expect that the electric polarization $\mathbf{P}^\star$ near point P in the comoving inertial frame equals that of the magnetized sphere in an inertial rest frame, namely $\mathbf{P}^\star = 0$.

Writing $\mathbf{v}$ as the velocity of point P in the lab frame, the field transformations to the comoving inertial frame are [18] (see also [19]), to order v/c where c is the speed of light in vacuum,

$$\mathbf{B}^\star = \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}, \qquad \mathbf{D}^\star = \mathbf{D} + \frac{\mathbf{v}}{c} \times \mathbf{H}, \qquad \mathbf{E}^\star = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \qquad \mathbf{H}^\star = \mathbf{H} - \frac{\mathbf{v}}{c} \times \mathbf{D},$$
$$\mathbf{M}^\star = \mathbf{M} + \frac{\mathbf{v}}{c} \times \mathbf{P}, \qquad \mathbf{P}^\star = \mathbf{P} - \frac{\mathbf{v}}{c} \times \mathbf{M}, \tag{3}$$

and the inverse transformations are,

$$\mathbf{B} = \mathbf{B}^\star + \frac{\mathbf{v}}{c} \times \mathbf{E}^\star, \qquad \mathbf{D} = \mathbf{D}^\star - \frac{\mathbf{v}}{c} \times \mathbf{H}^\star, \qquad \mathbf{E} = \mathbf{E}^\star - \frac{\mathbf{v}}{c} \times \mathbf{B}^\star, \qquad \mathbf{H} = \mathbf{H}^\star + \frac{\mathbf{v}}{c} \times \mathbf{D}^\star,$$
$$\mathbf{M} = \mathbf{M}^\star - \frac{\mathbf{v}}{c} \times \mathbf{P}^\star, \qquad \mathbf{P} = \mathbf{P}^\star + \frac{\mathbf{v}}{c} \times \mathbf{M}^\star. \tag{4}$$

We now find the lab-frame polarization and magnetization densities inside the rotating sphere (at point $P = r\,\hat{\mathbf{r}} = \mathbf{r}$ in spherical coordinates in the lab frame) to be,

$$\mathbf{P}(r < R) = \frac{\mathbf{v}}{c} \times \mathbf{M}_0 = \frac{\boldsymbol{\omega} \times \mathbf{r}}{c} \times \mathbf{M}_0 = \frac{\omega M_0}{c}\,\mathbf{r}_\perp = \frac{r\omega M_0}{c}\left(\sin^2\theta\,\hat{\mathbf{r}} + \frac{\sin 2\theta}{2}\,\hat{\boldsymbol{\theta}}\right), \tag{5}$$

$$\mathbf{M}(r < R) \;=\; \mathbf{M}_0. \tag{6}$$

[2]See, for example, sec. 5.10 of [13].

Hence, there is a uniform bound volume-charge density inside the sphere given by,

$$\rho_{\rm bound}(r < R) = -\boldsymbol{\nabla} \cdot \mathbf{P} = -\frac{2\omega M_0}{c}\,. \tag{7}$$

Likewise, there is a bound surface-charge density,

$$\sigma_{\rm bound}(r = R) = \mathbf{P}(R^-) \cdot \hat{\mathbf{r}} = \frac{\omega R M_0 \sin^2\theta}{c}\,, \tag{8}$$

where θ is the polar angle with respect to the z-axis. The total bound charge is zero,

$$Q_{\rm bound} = \int \rho_{\rm bound}\, d{\rm Vol} + \int \sigma_{\rm bound}\, d{\rm Area} = -\frac{8\pi R^3\omega M_0}{3c} + \frac{2\pi R^3\omega M_0}{c}\int_{-1}^{1}(1-\cos^2\theta)\, d\cos\theta = 0. \tag{9}$$

2.2 Analysis in the Rotating Frame

The principles of electrodynamics in a rotating frame are summarized in the Appendix.

We cannot assume without question that the magnetization of the cylinder is $\mathbf{M}_0$ according to an observer at rest in the rotating frame. The best strategy is to use the comoving analysis of sec. 2.1 to identify the fields in the lab frame, and then use the transformations (39)-(40) to find the fields in the rotating frame, which we designate with a $'$,

$$\mathbf{B}' = 4\pi\mathbf{M}_0, \quad \mathbf{D}' = 0, \quad \mathbf{E}' = 0, \quad \mathbf{H}' = 0, \quad \mathbf{P}' = 0, \quad \mathbf{M}' = \mathbf{M}_0. \tag{10}$$

As the rotating frame is the rest frame of the magnetized sphere, we might have naïvely assumed these results to be obvious.

We can now consider Maxwell's equations (45)-(48) for $\mathbf{D}'$ and $\mathbf{H}'$ in the rotating frame. In the present example there are no free sources for $\mathbf{D}'$ or $\mathbf{H}'$, and also no "other" sources according to eqs. (49)-(50). Thus, it is consistent with Maxwell's equations in the rotating frame that $\mathbf{D}' = 0 = \mathbf{H}'$. Then, using $\mathbf{P}' = 0$ and $\mathbf{M}' = \mathbf{M}_0$ we have that $\mathbf{E}' = 0$ and $\mathbf{B}' = 4\pi\mathbf{M}_0$.

Alternatively, we can consider Maxwell's equations (51)-(52) for $\mathbf{E}'$ and $\mathbf{B}'$. On examining the extensive list (53)-(56) of possible sources in the rotating frame, we see that the eqs. (51)-(52) reduce to,

$$\boldsymbol{\nabla}' \cdot \mathbf{E}' = 0, \qquad \boldsymbol{\nabla}' \times \mathbf{B}' = \boldsymbol{\nabla}' \times 4\pi\mathbf{M}_0, \tag{11}$$

so that we again find $\mathbf{E}' = 0$ and $\mathbf{B}' = 4\pi\mathbf{M}_0$.

Transforming the fields from the rotating frame back to the lab frame we again obtain the results of eqs. (5)-(6).[3]

Although the electric polarization, $\mathbf{P}' = 0$, vanishes in the rotating frame (since this could only be due to a moving magnetization in this example), the bound charge density (41) is nonzero,

$$\rho'_{\rm bound} = -\boldsymbol{\nabla}' \cdot \mathbf{P}' - \frac{2\boldsymbol{\omega} \cdot \mathbf{M}'}{c} + \frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}' \times \mathbf{M}' = -\frac{2\omega M_0}{c} = \rho_{\rm bound}\,, \tag{12}$$

[3] In particular, we find it completely consistent to use the transformation $\mathbf{P} = \mathbf{P}' + \mathbf{v}/c \times \mathbf{M}'$ from the rotating frame to the lab frame, despite a claim to the contrary in [23].

recalling eqs. (7)-(8). Similarly, there is a bound surface charge density on the outer circumference of the sphere in the rotating frame given by,

$$\sigma'_{\rm bound} = \frac{\omega R M'}{c} = \frac{\omega R M_0}{c} = \sigma_{\rm bound}\,, \tag{13}$$

also recalling eq. (8).

Thus, investigations of electromagnetism on the Earth's surface are not subject to "fictitious" charge and current densities (55)-(55) as can occur in some examples of physics in a rotating frame.

2.3 Nonconducting Sphere

If the sphere is nonconducting there are no free charges or currents.[4]

The electric field $\mathbf{E}$ can be deduced from a scalar potential V,

$$\mathbf{E} = -\boldsymbol{\nabla} V = -\boldsymbol{\nabla} V_\rho - \boldsymbol{\nabla} V_\sigma, \tag{14}$$

where the potential V_ρ (continuous at $r = R$) due to the bound volume-charge density (7) is,

$$V_\rho(r < R) = -\frac{4\pi\omega M_0}{3c}(3R^2 - r^2), \qquad V_\rho(r > R) = -\frac{8\pi R^3 \omega M_0}{3cr}\,, \tag{15}$$

and the potential V_σ (also continuous at $r = R$, and which obeys $\nabla^2 V_\sigma = 0$ except at $r = R$) due to the bound surface-charge density (8) can be written in the form,

$$V_\sigma(r < R) = \sum_n A_n \frac{r^n}{R^n} P_n(\cos\theta), \qquad V_\sigma(r > R) = \sum_n A_n \frac{R^{n+1}}{r^{n+1}} P_n(\cos\theta), \tag{16}$$

where P_n is the Legendre polynomial of order n. Noting that,

$$P_0 = 1, \qquad \text{and} \qquad P_2 = \frac{3\cos^2\theta - 1}{2} = 1 - \frac{3\sin^2\theta}{2}\,, \tag{17}$$

the bound surface-charge density (8) is related to the potential V_σ by,

$$\begin{aligned}
\sigma_{\rm bound} &= \frac{\omega R M_0 \sin^2\theta}{c} = \frac{2\omega R M_0}{3c}(P_0 - P_2) \\
&= \frac{E_{\sigma,r}(R^+) - E_{\sigma,r}(R^-)}{4\pi} = \frac{1}{4\pi}\left(\frac{\partial V_\sigma(R^-)}{\partial r} - \frac{\partial V_\sigma(R^+)}{\partial r}\right) \\
&= \sum_n (2n+1)\frac{A_n}{4\pi R} P_n.
\end{aligned} \tag{18}$$

Hence, the Fourier coefficients A_n are all zero except that,

$$A_0 = \frac{8\pi R^2 \omega M_0}{3c}\,, \qquad \text{and} \qquad A_2 = -\frac{8\pi R^2 \omega M_0}{15c}\,. \tag{19}$$

[4]This case has been discussed in [24, 25].

Thus,

$$V_\sigma(r < R) = \frac{8\pi\omega M_0}{3c}\left(R^2 - \frac{r^2}{5}P_2\right), \qquad V_\sigma(r > R) = \frac{8\pi\omega M_0}{3c}\left(\frac{R^3}{r} - \frac{R^5}{5r^3}P_2\right). \tag{20}$$

and the total electric potential is,

$$V(r < R) = -\frac{4\pi R^2\omega M_0}{3c} + \frac{4\pi r^2\omega M_0}{5c}(2 - \cos^2\theta), \qquad V(r > R) = -\frac{4\pi R^5\omega M_0}{15cr^3}(3\cos^2\theta - 1). \tag{21}$$

The difference ΔV in potential between points on the equator and on the "north" pole of the sphere is,

$$\Delta V = V(\theta = 90^\circ) - V(\theta = 0) = \frac{4\pi R^2\omega M_0}{5c}. \tag{22}$$

Note also that inside the sphere, $V(r < R) \neq V_0 + V_1 r^2 \sin^2\theta = V^0 + V_1 r_\perp^2$, so that the electric field is not in the $\mathbf{r}_\perp$ direction, $\mathbf{E}(r < R) \neq -2V_1 r_\perp \hat{\mathbf{r}_\perp}$.

The radial electric field is,

$$E_r(r < R) = -\frac{8\pi r\omega M_0}{5c}(2 - \cos^2\theta), \qquad E_r(r > R) = -\frac{4\pi R^5\omega M_0}{5cr^4}(3\cos^2\theta - 1), \tag{23}$$

which obeys $\sigma_{\rm bound} = [E_r(R^+) - E_r(R^-)]/4\pi$, and the θ-field is,

$$E_\theta(r < R) = -\frac{4\pi r\omega M_0}{5c}\sin 2\theta, \qquad E_\theta(r > R) = -\frac{4\pi R^5\omega M_0}{5cr^4}\sin 2\theta. \tag{24}$$

The electric-displacement field is $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$, so outside the sphere $\mathbf{D}(r > R) = \mathbf{E}(r > R)$, while inside it,[5]

$$D_r(r < R) = -\frac{4\pi r\omega M_0}{5c}(3\cos^2\theta - 1), \qquad D_\theta(r < R) = \frac{6\pi r\omega M_0}{5c}\sin 2\theta. \tag{25}$$

The radial component of $\mathbf{D}$ is continuous at $r = R$, as expected.

The rotating bound-charge densities (which are proportional to ω/c) generate current densities proportional to ω^2/c, which generate magnetic fields proportional to $\omega^2 R^2/c^2$. We neglect these tiny magnetic fields as this analysis is accurate only to order $v/c \approx \omega R/c$. In this approximation, the lab-frame magnetic fields are entirely due to the magnetization $\mathbf{M} \approx \mathbf{M}_0$, so the magnetic fields are given by eq. (1).

2.4 Conducting Sphere

As for the nonconducting sphere, any currents in/on a conducting sphere, other than those directly associated with the magnetization density $\mathbf{M} \approx \mathbf{M}_0$, are of order $\omega^2 R^2/c^2$, and we neglect the magnetic field which they generate. Then, the magnetic field in the lab frame is again given by eq. (1).

[5]This case of a nonconducting, rotating sphere is an example where the free charge density is zero but the electric-displacement field $\mathbf{D}$ is nonzero.

The bound-charge distributions are the same for a conducting and a nonconducting sphere, so the electric field has contributions (17)-(24) in both cases. These contributions appear to have been ignored in all previous analyses, such as [6, 7, 8, 9]. However, as discussed further below, once it is established that the magnetic field inside the sphere in the lab frame is uniform (and parallel to the axis of rotation) the total electric-charge distribution and the electric field $\mathbf{E}$ are the same as for a conducting sphere, with zero magnetization, that rotates in a uniform, external magnetic field [12].

The free charges inside the sphere must be at rest relative to the rotating sphere. That is, the electric field in the comoving frame must be zero at points inside the sphere; $\mathbf{E}^\star = 0 = \mathbf{E} + \mathbf{v}/c \times \mathbf{B}$, recalling eq. (3), so the lab-frame electric field for $r < R$ is given by,

$$\mathbf{E}(r < R) = -\frac{\mathbf{v}}{c} \times \mathbf{B} = -\frac{\boldsymbol{\omega} \times \mathbf{r}}{c} \times \frac{8\pi \mathbf{M}_0}{3} = -\frac{8\pi\omega M_0}{3c}\mathbf{r}_\perp. \tag{26}$$

The total electric-charge density inside the sphere is therefore,

$$\rho_{\rm total} = \frac{\boldsymbol{\nabla} \cdot \mathbf{E}}{4\pi} = -\frac{4\omega M_0}{3c} = \rho_{\rm bound} + \rho_{\rm free}, \tag{27}$$

and hence the free-charge density is, recalling eq. (7),

$$\rho_{\rm free} = \frac{2\omega M_0}{3c}\,. \tag{28}$$

From eq. (26) we deduce that the electric potential inside the sphere has the form,

$$V(r < R) = V_0 + \frac{4\pi\omega M_0}{3c} r_\perp^2 = V_0 + \frac{4\pi\omega M_0}{3c} r^2 \sin^2\theta = V_0 + \frac{8\pi r^2 \omega M_0}{9c}[1 - P_2(\cos\theta)], \tag{29}$$

recalling eq. (17). As in eq. (16) the electric potential outside the sphere can be written as,

$$V(r > R) = \sum_n A_n \frac{R^{n+1}}{r^{n+1}} P_n(\cos\theta), \tag{30}$$

The potential is continuous at $r = R$, so we learn that all A_n vanish except A_0 and A_2, and that,

$$A_2 = -\frac{8\pi R^2 \omega M_0}{9c}\,. \tag{31}$$

Assuming that the sphere is electrically neutral, coefficient $A_0 = 0$. Hence the electric potential outside the sphere is,

$$V(r > R) = -\frac{8\pi R^5 \omega M_0}{9cr^3} P_2(\cos\theta), \tag{32}$$

and the electric field outside the sphere is,

$$E_r(r > R) = -\frac{8\pi R^5 \omega M_0}{3cr^4} P_2(\cos\theta), \qquad E_\theta(r > R) = -\frac{4\pi R^5 \omega M_0}{3cr^4} \sin 2\theta. \tag{33}$$

The difference ΔV in the external potential between points on the equator and on the "north" pole of the sphere is,

$$\Delta V = V(\theta = 90^\circ) - V(\theta = 0) = \frac{4\pi R^2 \omega M_0}{3c}\,, \tag{34}$$

which is slightly larger that the result (22) for a nonconducting sphere.[6] This is the open-circuit voltage difference between the equator and the "north" pole. It is also the difference in the interior potential (29), such that for any closed loop that includes both the "north" pole and a point on the equation, we have that $\oint_{\rm loop} \mathbf{E} \cdot d\mathbf{l} = 0$.[7]

For completeness, we note that the surface-charge density is given by,

$$\sigma_{\rm total} = \frac{E_r(R^+) - E_r(R^-)}{4\pi} = \frac{\omega R M_0}{c}\frac{5\cos^2\theta - 2}{3}\,, \qquad \sigma_{\rm free} = \frac{2\omega R M_0 \cos^2\theta}{3c}\,. \tag{35}$$

recalling the bound surface charge of eq. (8). Finally, the electric-displacement field $\mathbf{D}$ inside the conducting, magnetized sphere is,

$$\mathbf{D}(r < R) = \mathbf{E} + 4\pi\mathbf{P} = \frac{4\pi\omega M_0}{3c}\mathbf{r}_\perp. \tag{36}$$

2.5 Comments

In the above analysis we have ignored the correction to the magnetic field due to the rotating charge distributions, as this is of order $\omega^2 R^2/c^2$. We also ignore the effect of centrifugal forces on the charge distributions. Furthermore, if the material is paramagnetic rather then having strictly fixed magnetization, there results a small electric field even if the magnetization is zero when at rest [20]. *Some additional remarks on these topics appear at the end of [12].*

If the conductor (and magnetization, if any) has axial symmetry, and the magnetic field is uniform and parallel to the axis of rotation, the internal electric field will be in the $\mathbf{r}_\perp$-direction. Then for a neutral conductor, the electric-dipole moment is zero, and the lowest nonzero moment of the charge distribution is the quadrupole. Hence, the electric field falls off as $1/r^4$ at large distances, no matter what is the shape of the axially symmetric conductor.[8]

A Summary of the Principles of Electrodynamics in a Rotating Frame

For reference, we reproduce the principles of electrodynamics in the frame of a slowly rotating medium where ϵ and μ differ from unity.[9,10]

[6] Of course, a rotating nonconducting sphere could not be used as a unipolar generator.

[7] It remains disconcerting to some people that nonetheless the rotating, conducting, magnetized sphere could be used as a homopolar generator, driving a current through a load connected to the "north" pole and the equator via sliding contacts.

[8] The case of a right-circular-cylinder conductor is reviewed in [10].

[9] This Appendix is from sec. 2.2.5 of [17].

[10] This case is discussed most thoroughly by Ridgely [21, 22], but primarily for the interesting limit of steady charge and current distributions.

The (cylindrical) coordinate transformation is,

$$r' = r, \qquad \phi' = \phi - \omega t, \qquad z' = z, \qquad t' = t, \tag{37}$$

where quantities in observed in the rotating frame are labeled with a $'$. The transformations of charge and current density are,

$$\rho' = \rho, \qquad \mathbf{J}' = \mathbf{J} - \rho\,\mathbf{v}, \tag{38}$$

where $\mathbf{v}$ ($v \ll c$) is the velocity with respect to the lab frame of the observer in the rotating frame. The transformations of the electromagnetic fields are,

$$\mathbf{B}' = \mathbf{B}, \qquad \mathbf{D}' = \mathbf{D} + \frac{\mathbf{v}}{c} \times \mathbf{H}, \qquad \mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \qquad \mathbf{H}' = \mathbf{H}. \tag{39}$$

The transformations of the electric and magnetic polarizations are,

$$\mathbf{P}' = \mathbf{P} - \frac{\mathbf{v}}{c} \times \mathbf{M}, \qquad \mathbf{M}' = \mathbf{M}, \tag{40}$$

if we regard these polarizations as defined by $\mathbf{D}' = \mathbf{E}' + 4\pi\mathbf{P}'$ and, $\mathbf{B}' = \mathbf{H}' + 4\pi\mathbf{M}'$.

The lab-frame bound charge and current densities, $\rho_{\text{bound}} = -\boldsymbol{\nabla}\cdot\mathbf{P}$ and $\mathbf{J}_{\text{bound}} = \partial\mathbf{P}/\partial t + c\,\boldsymbol{\nabla}\times\mathbf{M}$ transform to

$$\rho'_{\text{bound}} = -\boldsymbol{\nabla}'\cdot\mathbf{P}' - \frac{2\boldsymbol{\omega}\cdot\mathbf{M}'}{c} + \frac{\mathbf{v}}{c}\cdot\boldsymbol{\nabla}'\times\mathbf{M}', \tag{41}$$

$$\mathbf{J}'_{\text{bound}} = \frac{\partial\mathbf{P}'}{\partial t'} + c\,\boldsymbol{\nabla}'\times\mathbf{M}' + \mathbf{v}(\boldsymbol{\nabla}'\cdot\mathbf{P}') + \frac{\mathbf{v}}{c}\times\frac{\partial\mathbf{M}'}{\partial t'} + (\mathbf{P}'\cdot\boldsymbol{\nabla})\mathbf{v} - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{P}'. \tag{42}$$

Force $\mathbf{F}$ is invariant under the transformation (37). In particular, a charge q with velocity $\mathbf{v}_q$ in the lab frame experiences a Lorentz force in the rotating frame given by,

$$\mathbf{F}' = q\left(\mathbf{E}' + \frac{\mathbf{v}'_q}{c}\times\mathbf{B}'\right) = q\left(\mathbf{E} + \frac{\mathbf{v}_q}{c}\times\mathbf{B}\right) = \mathbf{F}, \tag{43}$$

where $\mathbf{v}'_q = \mathbf{v}_q - \mathbf{v}$. Similarly, the Lorentz force density $\mathbf{f}'$ on charge and current densities in the rotating frame is,

$$\mathbf{f}' = \rho'\mathbf{E}' + \frac{\mathbf{J}'}{c}\times\mathbf{B}' = (\rho'_{\text{free}} + \rho'_{\text{bound}})\,\mathbf{E}' + \frac{\mathbf{J}'_{\text{free}} + \mathbf{J}'_{\text{bound}}}{c}\times\mathbf{B}'. \tag{44}$$

Maxwell's equations in the rotating frame can be written as,

$$\boldsymbol{\nabla}'\cdot\mathbf{B}' = 0, \tag{45}$$

$$\boldsymbol{\nabla}'\cdot\mathbf{D}' = 4\pi\rho'_{\text{free,total}} = 4\pi\left(\rho'_{\text{free}} + \rho'_{\text{other}}\right), \tag{46}$$

$$\boldsymbol{\nabla}'\times\mathbf{E}' + \frac{\partial\mathbf{B}'}{\partial ct'} = 0, \tag{47}$$

$$\boldsymbol{\nabla}'\times\mathbf{H}' - \frac{\partial\mathbf{D}'}{\partial ct'} = \frac{4\pi}{c}\mathbf{J}'_{\text{free,total}} = \frac{4\pi}{c}\left(\mathbf{J}'_{\text{free}} + \mathbf{J}'_{\text{other}}\right), \tag{48}$$

where $\rho'_{\rm free} = \rho_{\rm free}$ and $\mathbf{J}'_{\rm free} = \mathbf{J}_{\rm free} - \rho_{\rm free}\mathbf{v}$ are the free charge and current densities, and the "other" charge and current densities that appear to an observer in the rotating frame are,

$$\begin{aligned}
\rho'_{\rm other} &= -\frac{\mathbf{v}\cdot\mathbf{J}'_{\rm free}}{c^2} + \frac{\boldsymbol{\omega}\cdot\mathbf{H}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c}\cdot\frac{\partial\mathbf{D}'}{\partial ct'}\,, && (49)\\
\mathbf{J}'_{\rm other} &= \rho'_{\rm free}\mathbf{v} + \left(\frac{\mathbf{D}'}{4\pi}\cdot\boldsymbol{\nabla}\right)\mathbf{v} - (\mathbf{v}\cdot\boldsymbol{\nabla})\frac{\mathbf{D}'}{4\pi} - \frac{\mathbf{v}}{4\pi c}\times\frac{\partial\mathbf{H}'}{\partial t'}\,. && (50)
\end{aligned}$$

The "other" charge and current distributions are sometimes called "fictitious" [26], but we find this term ambiguous. For an example with an "other" charge density $\boldsymbol{\omega}\cdot\mathbf{H}'/2\pi c$ in the rotating frame, see [27].

Maxwell's equations can also be expressed only in terms of the fields $\mathbf{E}'$ and $\mathbf{B}'$ and charge and current densities associated with free charges as well as with electric and magnetic polarization,

$$\boldsymbol{\nabla}'\cdot\mathbf{E}' = 4\pi\rho'_{\rm total}, \qquad (51)$$

and,

$$\boldsymbol{\nabla}'\times\mathbf{B}' - \frac{\partial\mathbf{E}'}{\partial ct'} = \frac{4\pi}{c}\mathbf{J}'_{\rm total}, \qquad (52)$$

where,

$$\begin{aligned}
\rho'_{\rm total} &= \rho'_{\rm free} - \frac{\mathbf{v}}{c^2}\cdot\mathbf{J}'_{\rm free} - \boldsymbol{\nabla}'\cdot\mathbf{P}' + \frac{\boldsymbol{\omega}\cdot\mathbf{H}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c}\cdot\frac{\partial\mathbf{D}'}{\partial ct'}\\
&= \rho'_{\rm free,total} - \boldsymbol{\nabla}'\cdot\mathbf{P}'\\
&= \rho'_{\rm free} + \rho'_{\rm bound} + \rho'_{\rm more}\,, && (53)\\
\rho'_{\rm more} &= -\frac{\mathbf{v}}{c^2}\cdot\left(\mathbf{J}'_{\rm free} + \frac{\partial\mathbf{P}'}{\partial t'} + c\boldsymbol{\nabla}'\times\mathbf{M}'\right) + \frac{\boldsymbol{\omega}\cdot\mathbf{B}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c}\cdot\frac{\partial\mathbf{E}'}{\partial ct'}\,, && (54)\\
\mathbf{J}'_{\rm total} &= \mathbf{J}'_{\rm free} + \frac{\partial\mathbf{P}'}{\partial t'} + c\boldsymbol{\nabla}'\times\mathbf{M}' + \rho'_{\rm free}\mathbf{v} + \left(\frac{\mathbf{D}'}{4\pi}\cdot\boldsymbol{\nabla}\right)\mathbf{v} - (\mathbf{v}\cdot\boldsymbol{\nabla})\frac{\mathbf{D}'}{4\pi} - \frac{\mathbf{v}}{4\pi c}\times\frac{\partial\mathbf{H}'}{\partial t'}\\
&= \mathbf{J}'_{\rm free,total} + \frac{\partial\mathbf{P}'}{\partial t'} + c\boldsymbol{\nabla}'\times\mathbf{M}'\\
&= \mathbf{J}'_{\rm free} + \mathbf{J}'_{\rm bound} + \mathbf{J}'_{\rm more}\,, && (55)\\
\mathbf{J}'_{\rm more} &= \mathbf{v}\left(\rho'_{\rm free} - \boldsymbol{\nabla}'\cdot\mathbf{P}' - \frac{2\boldsymbol{\omega}\cdot\mathbf{M}'}{c} + \frac{\mathbf{v}}{c}\cdot\boldsymbol{\nabla}'\times\mathbf{M}'\right)\\
&\quad + \left(\frac{\mathbf{E}'}{4\pi}\cdot\boldsymbol{\nabla}\right)\mathbf{v} - (\mathbf{v}\cdot\boldsymbol{\nabla})\frac{\mathbf{E}'}{4\pi} - \frac{\mathbf{v}}{4\pi c}\times\frac{\partial\mathbf{B}'}{\partial t'}\,. && (56)
\end{aligned}$$

The contribution of the polarization densities to the source terms in Maxwell's equations in much more complex in the rotating frame than in the lab frame.Because of the "other" source terms that depend on the fields in the rotating frame, Maxwell's equations cannot be solved directly in this frame. Rather, an iterative approach is required in general.

The constitutive equations for linear isotropic media at rest in the rotating frame are,

$$\mathbf{D}' = \epsilon\mathbf{E}', \qquad \mathbf{B}' = \mu\mathbf{H}' - (\epsilon\mu - 1)\frac{\mathbf{v}}{c}\times\mathbf{E}', \qquad (57)$$

in the rotating frame, and,

$$\mathbf{D} = \epsilon\mathbf{E} + (\epsilon\mu - 1)\frac{\mathbf{v}}{c} \times \mathbf{H}, \qquad \mathbf{B} = \mu\mathbf{H} - (\epsilon\mu - 1)\frac{\mathbf{v}}{c} \times \mathbf{E}, \tag{58}$$

in the lab frame. The lab-frame constitutive equations (58) are the same as for a nonrotating medium that moves with constant velocity $\mathbf{v}$ with respect to the lab frame.

We can also write the constitutive equations (57) for a linear isotropic medium in terms of the fields $\mathbf{B}'$, $\mathbf{E}'$, $\mathbf{P}'$ and $\mathbf{M}'$ by noting that $\mathbf{D}' = \mathbf{E}' + 4\pi\mathbf{P}'$ and $\mathbf{H}' = \mathbf{B}' - 4\pi\mathbf{M}'$, so that,

$$\begin{aligned}
\mathbf{P}' &= \frac{\epsilon - 1}{4\pi}\mathbf{E}', \\
\mathbf{M}' &= \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}'}{4\pi} - \left(\epsilon - \frac{1}{\mu}\right)\frac{\mathbf{v}}{c} \times \frac{\mathbf{E}'}{4\pi} = \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}'}{4\pi} - \frac{\epsilon\mu - 1}{\mu(\epsilon - 1)}\frac{\mathbf{v}}{c} \times \mathbf{P}'.
\end{aligned} \tag{59}$$

Similarly, the constitutive equations (58) in the lab frame can be written to order v/c as,

$$\begin{aligned}
\mathbf{P} &= \frac{\epsilon - 1}{4\pi}\mathbf{E} + \left(\epsilon - \frac{1}{\mu}\right)\frac{\mathbf{v}}{c} \times \frac{\mathbf{B}}{4\pi} \qquad = \frac{\epsilon - 1}{4\pi}\mathbf{E} + \frac{\epsilon\mu - 1}{\mu - 1}\frac{\mathbf{v}}{c} \times \mathbf{M}, \\
\mathbf{M} &= \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}}{4\pi} - \left(\epsilon - \frac{1}{\mu}\right)\frac{\mathbf{v}}{c} \times \frac{\mathbf{E}}{4\pi} = \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}}{4\pi} - \frac{\epsilon\mu - 1}{\mu(\epsilon - 1)}\frac{\mathbf{v}}{c} \times \mathbf{P}.
\end{aligned} \tag{60}$$

Ohm's law for the conduction current $\mathbf{J}_C$ has the same form for a medium with velocity $\mathbf{u}'$ relative to the rotating frame as it does for a medium with velocity $\mathbf{u}$ relative to the lab frame,

$$\mathbf{J}'_C = \sigma\left(\mathbf{E}' + \frac{\mathbf{u}'}{c} \times \mathbf{B}'\right) = \sigma\left(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}\right) = \mathbf{J}_C, \tag{61}$$

where σ is the electric conductivity of a medium at rest.

References

[1] M. Faraday, *On Lines of Magnetic Force*, Phil. Trans. Roy. Soc. London **142**, 25 (1852), http://kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_142_25_52.pdf

[2] M.J. Crooks *et al.*, *One-piece Faraday generator: A paradoxical experiment from 1851*, Am. J. Phys. **46**, 729 (1978), http://kirkmcd.princeton.edu/examples/EM/crooks_ajp_46_729_78.pdf

[3] P. Lorrain, *Electrostatic charges in* $\mathbf{v} \times \mathbf{B}$ *fields: the Faraday disk and the rotating sphere*, Eur. J. Phys. **11**, 94 (1990), http://kirkmcd.princeton.edu/examples/EM/lorrain_ejp_11_94_90.pdf

[4] P. Lorrain, J. McTavish and F. Lorrain, *Magnetic fields in moving conductors: four simple examples*, Eur. J. Phys. **19**, 451 (1998), http://kirkmcd.princeton.edu/examples/EM/lorrain_ejp_19_451_98.pdf

[5] K.T. McDonald, *Is Faraday's Disk Dynamo a Flux-Rule Exception?* (July 27, 2019), http://kirkmcd.princeton.edu/examples/faradaydisk.pdf

[6] W.F.G. Swann, *Unipolar Induction*, Phys. Rev. **15**, 365 (1920),
http://kirkmcd.princeton.edu/examples/EM/swann_pr_15_365_20.pdf

[7] O. Rognley, *The Eelectric Field of a Magnetized Spheroid Rotating about the Axis of Magnetization*, Phys. Rev. **19**, 609 (1922),
http://kirkmcd.princeton.edu/examples/EM/rognley_pr_19_609_22.pdf

[8] L. Davis, Jr, *Stellar Electromagnetic Fields*, Phys. Rev. **47**, 632 (1947),
http://kirkmcd.princeton.edu/examples/EM/davis_pr_72_632_47.pdf

[9] A.I. Miller, *Unipolar Induction: A Case Study in the Interaction between Science and Technology*, Ann. Sci. **38**, 155 (1981),
http://kirkmcd.princeton.edu/examples/EM/miller_as_38_155_81.pdf

[10] K.T. McDonald, *Unipolar Induction via a Rotating Magnetized Cylinder* (Aug. 17, 2008), http://kirkmcd.princeton.edu/examples/magcylinder.pdf

[11] J.J. Thomson, *Recent Researches in Electricity and Magnetism* (Clarendon Press, 1893), secs. 434-440, http://kirkmcd.princeton.edu/examples/EM/thomson_recent_researches_in_electricity.pdf

[12] K.T. McDonald, *Conducting Sphere That Rotates in a Uniform Magnetic Field* (Mar. 13, 2002), http://kirkmcd.princeton.edu/examples/rotatingsphere.pdf

[13] J.D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, 1975), sec. 5.10,
http://kirkmcd.princeton.edu/examples/EM/jackson_ce2_75.pdf

[14] H. Minkowski, *Die Grundgleichen für die elektromagnetischen Vorlage in bewegten Körpern*, Göttinger Nachricthen, pp. 55-116 (1908),
http://kirkmcd.princeton.edu/examples/EM/minkowski_ngwg_53_08.pdf
http://kirkmcd.princeton.edu/examples/EM/minkowski_ngwg_53_08_english.pdf

[15] J. van Bladel, *Relativistic Theory of Rotating Disks*, Proc. IEEE **61**, 260 (1973),
http://kirkmcd.princeton.edu/examples/EM/vanbladel_pieee_61_260_73.pdf

[16] T. Shiozawa, *Phenomenological and Electron-Theoretical Study of the Electrodynamics of Rotating Systems*, Proc. IEEE **61**, 1694 (1973),
http://kirkmcd.princeton.edu/examples/EM/shiozawa_pieee_61_1694_73.pdf

[17] K.T. McDonald, *Electrodynamics of Rotating Systems* (Aug. 6, 2008),
http://kirkmcd.princeton.edu/examples/rotatingEM.pdf

[18] H.A. Lorentz, *Alte und Neue Fragen der Physik*, Phys. Z. **11**, 1234 (1910),
http://kirkmcd.princeton.edu/examples/EM/lorentz_pz_11_1234_10.pdf

[19] V. Hnizdo and K.T. McDonald, *Fields and Moments of a Moving Electric Dipole* (Nov. 29, 2011), http://kirkmcd.princeton.edu/examples/movingdipole.pdf

[20] S.J. Barnett, *Magnetization by Rotation*, Phys. Rev. **6**, 239 (1915),
http://kirkmcd.princeton.edu/examples/EM/barnett_pr_6_239_15.pdf

[21] C.T. Ridgely, *Applying relativistic electrodynamics to a rotating material medium*, Am. J. Phys. **66**, 114 (1998), http://kirkmcd.princeton.edu/examples/EM/ridgely_ajp_66_114_98.pdf

[22] C.T. Ridgely, *Applying covariant versus contravariant electromagnetic tensors to rotating media*, Am. J. Phys. **67**, 414 (1999), http://kirkmcd.princeton.edu/examples/EM/ridgely_ajp_67_414_99.pdf

[23] D.L. Webster, *Schiff's Charges and Currents in Rotating Matter*, Am. J. Phys. **31**, 590 (1963), http://kirkmcd.princeton.edu/examples/EM/webster_ajp_31_590_63.pdf

[24] W.F.G. Swann, *An Experiment on Electromagnetic Induction and Relative Motion*, Phys. Rev. **19**, 38 (1922), http://kirkmcd.princeton.edu/examples/EM/swann_pr_19_38_22.pdf

[25] A. Baños, Jr, and R.K. Golden, *The Electromagnetic Field of a Rotating Uniformly Magnetized Sphere*, J. Appl. Phys. **23**, 1294 (1952), http://kirkmcd.princeton.edu/examples/EM/banos_jap_23_1294_52.pdf

[26] L.I. Schiff, *A Question in General Relativity*, Proc. Nat. Acad. Sci. **25**, 391 (1939), http://kirkmcd.princeton.edu/examples/EM/schiff_proc_nat_acad_sci_25_391_39.pdf

[27] K.T. McDonald, *The Barnett Experiment with a Rotating Solenoid Magnet* (Apr. 6, 2003), http://kirkmcd.princeton.edu/examples/barnett.pdf