Helmholtz' Influence on *The Mikado* by Gilbert and Sullivan

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The first German edition of Helmholtz' great work *On the Sensations of Tone* appeared in 1864,[1] and the first English edition appear in 1875.[2] Section V.3 discussed *Musical Tones of Strings Excited by Striking*, and noted that while a very brief strike leads to an unpleasant tone, if the strike is spread out over a half period of the fundamental frequency of the stretched string, then a rather musical tone is obtained. This phenomenon attracted the interest of various British physicists in the years 1883-84, as mentioned in the 2nd edition (1885).[3]

In 1885, Gilbert and Sullivan premiered their comic opera *The Mikado*, which includes the line,
Awaiting the sensation of a short sharp shock,
from a cheap and chippy chopper on a big black block!
at the end of Act 1.

I conjecture that this line was inspired in part by an awareness of Gilbert as to Helmholtz' theory of the sensations caused by striking of stretched strings (to follow Gilbert's tendency towards alliteration).

I note also Gilbert's whimsical interest in mathematics, shown in the lines of the Major General's patter song, Act 1 of *The Pirates of Penzance* (1879),
I'm very well acquainted, too, with matters mathematical,
I understand equations both the simple and quadratical.
About binomial theorem I'm teeming with a lot o' news,
With many cheerful facts about the square of the hypotenuse.

[1] H. Helmholtz, *Die Lehre von den Tonempfindungen als Physiologische Grundlage für die Theorie der Musik* (Friedrich vieweg, 1863), http://kirkmcd.princeton.edu/examples/mechanics/helmholtz_63.pdf

[2] H.L.F. Helmholtz, *On the Sensations of Tone* (Longmans, Green, 1875), http://kirkmcd.princeton.edu/examples/mechanics/helmholtz_75.pdf

[3] H.L.F. Helmholtz, *On the Sensations of Tone*, 2nd English ed. (Longmans, Green, 1885), pp. 380-394, http://kirkmcd.princeton.edu/examples/mechanics/helmholtz_85.pdf. See also, pp. 74-80 (and pp. 545-546), particularly the footnotes, which recounts interest in England in Helmholtz' theories of the piano in the years 1883-1885.

A Appendix: The Piano as a Physics Problem

A.1 Problem

A piano wire is struck by a sharp blow from a hammer, and a fairly pure note is produced. This is perhaps surprising in view of the analysis on, for example, p. 229 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l21.pdf of the effect of an impulsive blow. Helmholtz suggested that a better approximation to the effect of the hammer is that it exerts a force,

$$F(x,t) = \begin{cases} F\,\delta(x-b)\sin\frac{2\pi t}{T} & (0 < t < T/2), \\ 0 & (\text{otherwise}). \end{cases} \tag{1}$$

That is, the force goes through one half period of s sinusoidal oscillation.

The force is applied at distance b from one end of a wire of length l and mass density ρ per unit length, which is fixed at both end and subject to a tension that makes the transverse wave velocity equal to c.

Consider a Fourier analysis of the vibrations, $s(x,t) = \sum_n \phi_n(t)\sin(n\pi x/l)$, and use Green's method[4] to solve the differential equations for the ϕ_n to show that,

$$s(x,t) = \frac{2FT}{\pi^2 c\,\rho}\sum_n \frac{1}{n(1-(\frac{ncT}{2l})^2)}\sin\frac{n\pi b}{l}\cos\frac{n\pi cT}{4l}\sin\frac{n\pi x}{l}\sin\frac{n\pi c(t-T/4)}{l}\,. \tag{2}$$

If we take $b = l/2$, the midpoint, and $T = 2l/c$, the fundamental period, then,

$$s(x,t) = \frac{Fl}{\pi^2 T}\sum_n \frac{\sin n\pi}{n(1-n^2)}\sin n\pi x\,\sin\frac{n\pi c(t-T/4)}{l}\,, \tag{3}$$

so all harmonics vanish except $n = 1$, since $\lim_{n\to 1}\frac{\sin n\pi}{1-n^2} = \frac{\pi\cos n\pi}{-2n} = \frac{\pi}{2}$.

A.2 Solution

The transverse displacement $s(x,t)$ of the string can be written as a sum of spatial modes, $\sin(n\pi x/l)$, whose time dependence $\phi_n(t)$ is to be determined,

$$s(x,t) = \sum_n \phi_n(t)\sin\frac{n\pi x}{l}\,. \tag{4}$$

The equation of motion of the string is,

$$\rho\ddot{s} = c^2\rho s'' + F(x,t), \tag{5}$$

$$\sum_n\left(\ddot{\phi}_n + \frac{n^2\pi^2c^2}{l^2}\phi_n\right)\sin\frac{n\pi x}{l} = \frac{1}{\rho}F(x,t) = \frac{1}{\rho}\sum_n F_n(t)\sin\frac{n\pi x}{l}\,, \tag{6}$$

[4] See, for example, p. 145 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l13.pdf

where the Fourier coefficients $F_n(t)$ are related by,

$$\frac{2}{l}\int_0^l dx \sum_n F(x,t)\sin\frac{m\pi x}{l} = \frac{2}{l}\int_0^l dx \sum_n F_n(t)\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l} = F_m(t)$$
$$= \frac{2}{l}\int_0^l dx\, F(x,t)\sin\frac{m\pi x}{l} = \frac{2}{l}\begin{cases} F\,\sin\frac{2\pi t}{T}\sin\frac{m\pi b}{l} & (0 < t < T/2), \\ 0 & (\text{otherwise}). \end{cases} \qquad (7)$$

Hence, the coefficient $\phi_n(t)$ obeys the differential equation of a forced, undamped oscillator,

$$\ddot{\phi}_n + \frac{n^2\pi^2c^2}{l^2}\phi_n = \frac{2F}{\rho l}\begin{cases} \sin\frac{2\pi t}{T}\sin\frac{n\pi b}{l} & (0 < t < T/2), \\ 0 & (\text{otherwise}). \end{cases} \qquad (8)$$

Recalling the method of Green, discussed on p. 145 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l13.pdf, we have for ϕ_n at times $t > T/2$, noting that $\omega_1 = n\pi c/l$,

$$\phi_n(t > T/2) = \frac{l}{n\pi c}\frac{2F}{\rho l}\int_0^{T/2} dt'\,\sin\frac{2\pi t}{T}\sin\frac{n\pi b}{l}\sin\frac{n\pi c}{l}(t-t')$$
$$= \frac{2F}{n\pi c\rho}\sin\frac{n\pi b}{l}\int_0^{T/2} dt'\,\frac{1}{2}\left\{\cos\left[\left(\frac{2\pi}{T}+\frac{n\pi c}{l}\right)t' - \frac{n\pi}{l}t\right] - \cos\left[\left(\frac{2\pi}{T}-\frac{n\pi c}{l}\right)t' + \frac{n\pi}{l}t\right]\right\}$$
$$= \frac{F}{n\pi c\rho}\sin\frac{n\pi b}{l}\left\{\frac{\sin\left[\left(\frac{2\pi}{T}+\frac{n\pi c}{l}\right)\frac{T}{2} - \frac{n\pi}{l}t\right] + \sin\frac{n\pi ct}{l}}{\frac{2\pi}{T}+\frac{n\pi c}{l}} - \frac{\sin\left[\left(\frac{2\pi}{T}-\frac{n\pi c}{l}\right)\frac{T}{2} + \frac{n\pi}{l}t\right] - \sin\frac{n\pi ct}{l}}{\frac{2\pi}{T}-\frac{n\pi c}{l}}\right\}$$
$$= \frac{F}{n\pi c\rho}\sin\frac{n\pi b}{l}\left\{\frac{\sin\left[\pi - \frac{n\pi c}{l}\left(t-\frac{T}{2}\right)\right] + \sin\frac{n\pi ct}{l}}{\frac{2\pi}{T}+\frac{n\pi c}{l}} - \frac{\sin\left[\pi + \frac{n\pi c}{l}\left(t-\frac{T}{2}\right)\right] - \sin\frac{n\pi ct}{l}}{\frac{2\pi}{T}-\frac{n\pi c}{l}}\right\}$$
$$= \frac{F}{n\pi c\rho}\sin\frac{n\pi b}{l}\left\{\frac{\sin\frac{n\pi c}{l}\left(t-\frac{T}{2}\right) + \sin\frac{n\pi ct}{l}}{\frac{2\pi}{T}+\frac{n\pi c}{l}} + \frac{\sin\frac{n\pi c}{l}\left(t-\frac{T}{2}\right) + \sin\frac{n\pi ct}{l}}{\frac{2\pi}{T}-\frac{n\pi c}{l}}\right\}$$
$$= \frac{2F}{n\pi c\rho}\sin\frac{n\pi b}{l}\frac{\frac{4\pi}{T}\cos\frac{n\pi cT}{4l}\sin\frac{n\pi ct}{l}\left(t-\frac{T}{4}\right)}{\left(\frac{2\pi}{T}\right)^2 - \left(\frac{n\pi c}{l}\right)^2}\,. \qquad (9)$$

The displacement is now given by,

$$s(x,t) = \frac{2FT}{\pi^2 c\,\rho}\sum_n \frac{1}{n(1-(\frac{ncT}{2l})^2)}\sin\frac{n\pi b}{l}\cos\frac{n\pi cT}{4l}\sin\frac{n\pi x}{l}\sin\frac{n\pi c(t-T/4)}{l}\,. \qquad (10)$$

If we take $b = l/2$, the midpoint, and $T = 2l/c$, the fundamental period, then,

$$s(x,t) = \frac{Fl}{\pi^2 T}\sum_n \frac{\sin n\pi}{n(1-n^2)}\sin n\pi x\,\sin\frac{n\pi c(t-T/4)}{l}\,, \qquad (11)$$

so all harmonics vanish except $n = 1$, since $\lim_{n\to 1}\frac{\sin n\pi}{1-n^2} = \frac{\pi\cos n\pi}{-2n} = \frac{\pi}{2}$.

Even if $T = 2l/c$ can't be achieved exactly in practice, the series converges quickly as the terms go as $1/n^3$ for large n.

In contemporary pianos, $b = l/8$, in which case (for $T = 2l/c$),

$$s(x,t) = \frac{2Fl}{\pi^2 T} \sum_n \frac{1}{n(1-n^2)} \sin\frac{n\pi}{8} \cos\frac{n\pi}{2} \sin n\pi x \, \sin\frac{n\pi c(t - T/4)}{l}\,, \tag{12}$$

which includes harmonics $n = 1$, 2, 4, 6, 10, ...

Many harpsichords are built with $b = l/2$, which gives them a purer tone, although perhaps less interesting than that of a piano.