Waves on a Mismatched Transmission Line

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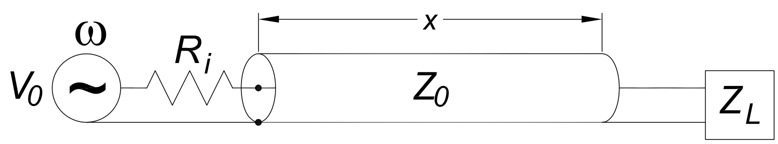
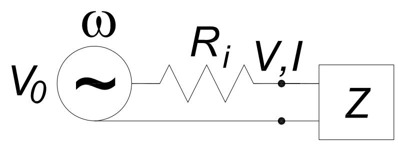
1 Problem

Discuss the waves on a transmission line of length x and characteristic (real) impedance Z_0 when it is connected to load of complex impedance Z_L and driven at angular frequency ω by a signal generator that can be modeled as an ideal voltage source V_0 in series with an internal resistance R_i.



2 Solution

The transmission line and load can be represented as a single complex impedance Z across the terminals of the signal generator, with equivalent circuit diagram as shown below.



The voltage V and the current I at the terminals are then related by,

$$V = IZ \tag{1}$$

The circuit equation $V_0 = I(R_i + Z)$ tells us that,

$$I = \frac{V_0}{R_i + Z}, \qquad V = V_0 \frac{Z}{R_i + Z}. \tag{2}$$

The (time-average) power delivered into the equivalent impedance Z is,

$$P = \frac{1}{2} Re V I^\star = \frac{V_0^2}{2} Re \frac{Z}{R_i + Z} = \frac{V_0^2}{2} \sqrt{\frac{Re^2(Z) + Im^2(Z)}{r_i^2 + 2R_i Re(Z) + Re^2(Z) + Im^2(Z)}}, \tag{3}$$

which is maximal for $Im(Z) = 0$ and $Re(Z) = R_i$, in which case the maximum power,[1] *is $P_{\max} = V_0^2/4R_i$. Sometimes it is said that the external impedance Z is* matched *to the signal generator if $Z = R_i$.*

It remains to deduce the equivalent terminal impedance Z in terms of the length x of the transmission line and the impedances Z_0 and Z_L.

[1]For comments on the related issue of maximum power in DC circuits, see, for example, [1].

The transmission line supports waves that propagate in both the $+x$ (forward) and $-x$ (reverse/return) directions. Taking $x = 0$ at the terminals of the generator, these waves can be written as,[2]

$$V_f(I_f)\, e^{i(\omega t - kx)}, \qquad V_f = Z_0 I_f, \qquad V_r(I_r)\, e^{i(\omega t + kx)}, \qquad V_f = -Z_0 I_r, \tag{4}$$

where V_f, I_f, V_r and I_r are complex constants, and $v = \omega/k$ is the (phase) velocity of the waves. Then, the terminal voltage V and terminal current I at $x = 0$ are related by,

$$V = V_f + V_r, \qquad I = I_f + I_r, \qquad V = IZ. \tag{5}$$

To relate V_f and V_r, we consider the load end of the transmission line, where the load voltage V_L and load current V_L are related by,

$$\begin{aligned} V_L &= V_f\, e^{i(\omega t - kx)} + V_r\, e^{i(\omega t + kx)} = Z_L I_L = Z_L \left(I_f\, e^{i(\omega t - kx)} + I_r\, e^{i(\omega t + kx)} \right) \\ &= \frac{Z_L}{Z_0} \left(V_f\, e^{i(\omega t - kx)} - V_r\, e^{i(\omega t + kx)} \right). \end{aligned} \tag{6}$$

Hence,

$$V_r = V_f\, e^{-2ikx} \frac{Z_0 - Z_L}{Z_0 + Z_L} = V_f \gamma\, e^{-2ikx}, \quad I_r = -\frac{V_r}{Z_0} = -I_f \gamma\, e^{-2ikx}, \quad \text{where} \quad \gamma = \frac{Z_0 - Z_L}{Z_0 + Z_L}. \tag{7}$$

The reflected wave vanishes only if $Z_L = Z_0$ (a pure resistance), in which case we say that the load is matched to the transmission line.[3] If in addition $Z_0 = R_i = Z_L$, then both the load and line are matched to the signal generator (and there is no reflected wave).

In general, the terminal voltage V and current I are related to the voltage and current of the waves on the transmission line by,

$$V = V_f + V_r = V_f (1 + \gamma\, e^{-2ikx}) \qquad I = I_f + I_r = I_f (1 - \gamma\, e^{-2ikx}) = \frac{V_f}{Z_0} 1 - \gamma\, e^{-2ikx}), \tag{8}$$

and the terminal impedance $Z = V/I$ is,

$$Z = Z_0 \frac{1 + \gamma\, e^{-2ikx}}{1 - \gamma\, e^{2ikx}}. \tag{9}$$

Finally, recalling eq. (2), the voltage of the forward wave can be written as,

$$\begin{aligned} V_f &= \frac{V}{1 + \gamma\, e^{-2ikx}} = \frac{V_0}{1 + \gamma\, e^{-2ikx}} \frac{Z}{R_i + Z} = \frac{V_0}{1 - \gamma\, e^{-2ikx}} \frac{Z_0}{R_i + Z_0 \frac{1 + \gamma\, e^{-2ikx}}{1 - \gamma\, e^{2ikx}}} \\ &= V_0 \frac{Z_0}{Z_0 + R_i + (Z_0 - R_i) \gamma\, e^{-2ikx}} \\ &= V_0 \frac{Z_0 (Z_0 + Z_L)}{(Z_0 + R_i)(Z_0 + Z_L) + (Z_0 - R_i)(Z_0 - Z_L)\, e^{-2ikx}}, \end{aligned} \tag{10}$$

[2]A derivation of the transmission-line equations is given in, for example, [2].

[3]Other aspects of impedance matching of transmission lines are considered in [3].

and the voltage of the reverse wave is,

$$V_r = V_f\, e^{-2ikx} \frac{Z_0 - Z_L}{Z_0 + Z_L} = V_0 \frac{Z_0(Z_0 - Z_L)}{(Z_0 + R_i)(Z_0 + Z_L) + (Z_0 - R_i)(Z_0 - Z_L)\, e^{-2ikx}}\,. \tag{11}$$

In the general case where $Z_0 \neq R_i$ and $Z_L \neq Z_0$, the voltage of the forward wave depends on the load impedance (and according to the first form of eq. (11) we could also say that the forward voltage depends on the reverse voltage).

However, if the line is matched to the generator (or the generator is matched to the line), $Z_0 = R_i$, and then,

$$V_f = V_0 \frac{Z_0}{Z_0 + R_i} = \frac{V_0}{2}\,, \qquad V_r = V_f\, e^{-2ikx} \frac{Z_0 - Z_L}{Z_0 + Z_L} = \frac{V_0}{2}\, e^{-2ikx}, \tag{12}$$

so the forward wave amplitude V_f is independent of the load whether or not the load impedance is matched to that of the generator/line. **Hence, it is best to use a transmission line that is matched to the generator, whether or not the load is matched to the line.**[4]

If the load is matched to the line, $Z_0 = Z_L$, but the line is not matched to the generator, $Z_0 \neq R_i$, then $Z = Z_0$ and $V_f = V_0 Z_0/(Z_0 + R_i)$. The forward voltage does not depend on the load impedance in this case either.

In the general case that the line is not matched to the generator and the load is not matched to the line, can/should we say that the forward voltage depends on the reverse voltage (or that the forward wave depends on the reverse wave? We can rewrite eq. (7) as,

$$V_f = V_r\, e^{2ikx} \frac{Z_0 + Z_L}{Z_0 - Z_L}\,, \tag{13}$$

which can be considered to show how the forward voltage "depends" of the reverse voltage, with an implication that the forward wave "depends" on the reverse wave.

This view can be reinforced by considerations of the transient behavior of V_f in case of a very long transmission line. After the signal generator is turned on, but before the forward wave arrives at the distant load, the terminal impedance appears to be just $Z_0 \neq R_i$, and the voltage of the forward wave is $V_f = V_0 Z_0/(Z_0 + R_i)$. But, after the forward wave encounters the mismatched load, and if $Z_L \neq Z_0$, a reverse wave returns to the generator, which leads to changes in the forward wave, until eventually it settles down to the steady-state value of eq. (10). Hence, it is reasonable to consider that the forward wave has been affected by/depends on the reverse wave in the general, mismatched case that $R_i \neq Z_0 \neq Z_L$.

References

[1] K.T. McDonald, *Maximum Power from DC Current and Voltage Sources* (Feb. 18, 2013), http://kirkmcd.princeton.edu/examples/maxpow.pdf

[4]In DC circuits with a battery as the power source, it is typically considered preferable if the internal resistance R_i of the battery is as small as possible. But, in AC circuits where the power source/signal generator is connected to a transmission line, it is preferable that the internal resistance R_i of the generator be nonzero, and equal to the (real) impedance Z_0 of the transmission line.

[2] K.T. McDonald, *Distortionless Transmission Line* (Nov. 11, 1996),
http://kirkmcd.princeton.edu/examples/distortionless.pdf

[3] K.T. McDonald, *Impedance Matching of Transmission Lines* (July 20, 2005),
http://kirkmcd.princeton.edu/examples/impedance_matching.pdf