Two-Dimensional Multipole Magnets

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1 Problem

Magnets used in the transport of charged-particle beams are often long compared to the size of their aperture. Consider the idealization of 2-dimensional magnetic fields inside a cylinder of radius a due to currents on the surface of the cylinder and parallel to its axis. Such surface currents can be well approximated by superconductors.

Deduce the vector potential, and the magnetic field, for a single current filament at azimuthal angle ϕ_0, and then for the case that the current varies with angle as $\cos m\phi_0$.

2 Solution[1]

2.1 Single Current Filament

For a steady current I in the filament at (a, ϕ_0, z) in a cylindrical coordinate system whose axis is that of the cylinder of radius a, the magnetic field $\mathbf{B}$ at (r, ϕ, z) is azimuthal about the filament, with magnitude $B = \mu_0 I/2\pi R$ (in SI units) as follows from Ampère's law $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$, where $\mathbf{J}$ is the current density, and,

$$R = \sqrt{a^2 + r^2 - 2ar\cos(\phi - \phi_0)}, \tag{1}$$

is the transverse distance between the filament and the observation point.

We briefly consider a cylindrical coordinate system (R, ϕ', z) whose axis is the current filament, such that the relation $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ implies that (in the Coulomb gauge) the vector potential $\mathbf{A}(\mathbf{x}) = \int \mu_0 \mathbf{J}(\mathbf{x}')\, d\mathbf{Vol}'/\left|\mathbf{x} - \mathbf{x}'\right|$ has only a z-component, which is azimuthally symmetric in this coordinate system. Hence,

$$B_{\phi'} = \frac{\mu_0 I}{2\pi R} = -\frac{\partial A_z}{\partial R}\,, \qquad A_z = -\frac{\mu_0 I}{2\pi}\ln R + \text{const.} \tag{2}$$

Returning to the coordinate system (r, ϕ, z), we define the vector potential to be zero on its axis, such that,

$$A_z = -\frac{\mu_0 I}{2\pi}\ln\frac{R}{a} = -\frac{\mu_0 I}{2\pi}\ln\left(1 + \frac{r^2}{a^2} - 2\frac{a}{r}\cos(\phi - \phi_0)\right). \tag{3}$$

For $r < a$ the vector potential has the Taylor-series expansion,

$$A_z(r < a) = \frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\frac{r^n}{a^n}\cos n(\phi - \phi_0), \tag{4}$$

[1]This solution follows [1].

as can be found by consulting, for example, https://www.wolframalpha.com about `taylor series log(sqrt(1 - 2 x cos(a) + x^2))`.

We can confirm eq. (4) by noting that,

$$\frac{R^2}{a^2} = \left(1 - \frac{r}{a}\, e^{i(\phi-\phi_0)}\right)\left(1 - \frac{r}{a}\, e^{-i(\phi-\phi_0)}\right), \tag{5}$$

$$\ln(1-\epsilon) = -\sum_{n=1}^{\infty} \frac{\epsilon^n}{n}, \tag{6}$$

$$\begin{aligned} \ln\frac{R}{a} &= \frac{1}{2}\ln\left(1 - \frac{r}{a}\, e^{i(\phi-\phi_0)}\right) + \frac{1}{2}\ln\left(1 - \frac{r}{a}\, e^{-i(\phi-\phi_0)}\right) = \sum_{n=1}^{\infty} \frac{1}{n}\frac{r^n}{a^n}\frac{e^{ni(\phi-\phi_0)} + e^{-ni(\phi-\phi_0)}}{2} \\ &= \sum_{n=1}^{\infty} \frac{1}{n}\frac{r^n}{a^n}\cos n(\phi-\phi_0). \end{aligned} \tag{7}$$

We could also expand the vector potential for $r > a$, but this is of less interest in that long, practical magnets have iron yokes outside their currents to shield nearby apparatus from their external magnetic field. If no such iron were present, the vector potential $A_z(r > a)$ is given, for example, in sec. 2.1 of [1].

The magnetic field for $r < a$ follows from eq. (4),

$$B_r(r < a) = \frac{1}{r}\frac{\partial A_z}{\partial \phi} = \frac{\mu_0 I}{2\pi r}\sum_{n=1}^{\infty} \frac{r^n}{a^n}\sin n(\phi-\phi_0), \tag{8}$$

$$B_\phi(r < a) = -\frac{\partial A_z}{\partial r} = -\frac{\mu_0 I}{2\pi r}\sum_{n=1}^{\infty} \frac{r^n}{a^n}\cos n(\phi-\phi_0). \tag{9}$$

Equations (4) and (8)-(9) can be called multipole expansions.

2.2 The Surface Current Varies as $\cos m\phi_0$

For filamentary currents at $r = a$ of the form $I(\phi) = I_0 \cos m\phi$, the vector potential for $r < a$ has the Taylor-series/multipole expansion,

$$\begin{aligned} A_z^{(m)}(r < a) &= \frac{\mu_0 I_0}{2\pi}\sum_{n=1}^{\infty} \frac{1}{n}\frac{r^n}{a^n}\int_0^{2\pi} \cos n(\phi-\phi_0)\cos m\phi_0\, d\phi_0 \\ &= \frac{\mu_0 I_0}{2\pi}\sum_{n=1}^{\infty} \frac{1}{n}\frac{r^n}{a^n}\int_0^{2\pi} (\cos n\phi \cos n\phi_0 - \sin n\phi \sin n\phi_0)\cos m\phi_0\, d\phi_0 \\ &= \frac{\mu_0 I_0}{2m}\frac{r^m}{a^m}\cos m\phi, \end{aligned} \tag{10}$$

recalling that $\int_0^{2\pi} \cos m\phi \cos n\phi\, d\phi = \pi\delta_{mn}$ and $\int_0^{2\pi} \sin m\phi \cos n\phi\, d\phi = 0$.

The magnetic field for $r < a$ is then,

$$B_r^{(m)}(r < a) = -\frac{\mu_0 I_0 r^{m-1}}{2a^m}\sin m\phi, \qquad B_\phi^{(m)}(r < a) = -\frac{\mu_0 I_0 r^{m-1}}{2a^m}\cos m\phi. \tag{11}$$

The field components in rectangular coordinates are, for $r < a$,

$$\begin{aligned}
B_x^{(m)}(r<a) &= B_r^{(m)}\cos\phi - B_\phi^{(m)}\sin\phi = -\frac{\mu_0 I_0 r^{m-1}}{2a^m}(\sin m\phi\cos\phi - \cos m\phi\sin\phi) \\
&= -\frac{\mu_0 I_0 r^{m-1}}{2a^m}\sin(m-1)\phi, \qquad (12) \\
B_y^{(m)}(r<a) &= B_r^{(m)}\sin\phi + B_\phi^{(m)}\cos\phi = -\frac{\mu_0 I_0 r^{m-1}}{2a^m}(\sin m\phi\sin\phi + \cos m\phi\cos\phi) \\
&= -\frac{\mu_0 I_0 r^{m-1}}{2a^m}\cos(m-1)\phi. \qquad (13)
\end{aligned}$$

For $m = 1$ (dipole),

$$B_x^{(1)}(r<a) = 0, \qquad B_y^{(1)}(r<a) = -\frac{\mu_0 I}{2a} = B_0. \qquad (14)$$

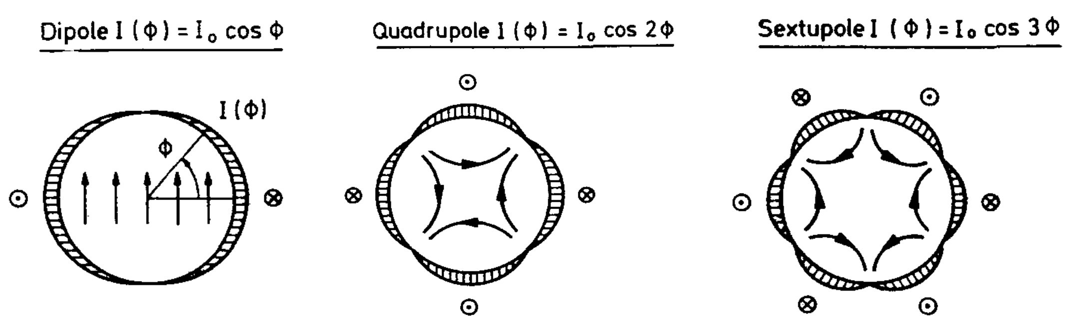
For $m = 2$ (quadrupole),

$$B_x^{(2)}(r<a) = -\frac{\mu_0 I_0}{2a^2} r\sin\phi = -\frac{\mu_0 I_0 y}{2a^2}, \qquad B_y^{(2)}(r<a) = -\frac{\mu_0 I_0}{2a^2} r\cos\phi = -\frac{\mu_0 I_0 x}{2a^2}. \qquad (15)$$

For $m = 3$ (sextupole),[2]

$$\begin{aligned}
B_x^{(3)}(r<a) &= -\frac{\mu_0 I_0}{2a^3} r^2\sin 2\phi = -\frac{\mu_0 I_0}{a^3} r^2 \sin\phi\cos\phi = -\frac{\mu_0 I_0 xy}{a^3}, \qquad (16) \\
B_y^{(3)}(r<a) &= -\frac{\mu_0 I_0}{2a^3} r^2\cos 2\phi = -\frac{\mu_0 I_0}{2a^3} r^2(\cos^2\phi - \sin^2\phi) = -\frac{\mu_0 I_0}{2a^3}(x^2 - y^2). \qquad (17)
\end{aligned}$$

The field patterns of the three lowest nontrivial multipoles are shown below (from [1]).[3] The field of multipole m has the approximate character of that associated with $2m$ filaments (poles) on the cylinder of radius a, with adjacent filaments having opposite signs of their currents.



2.3 Nomenclature for Two- *vs.* Three-Dimensional Multipoles

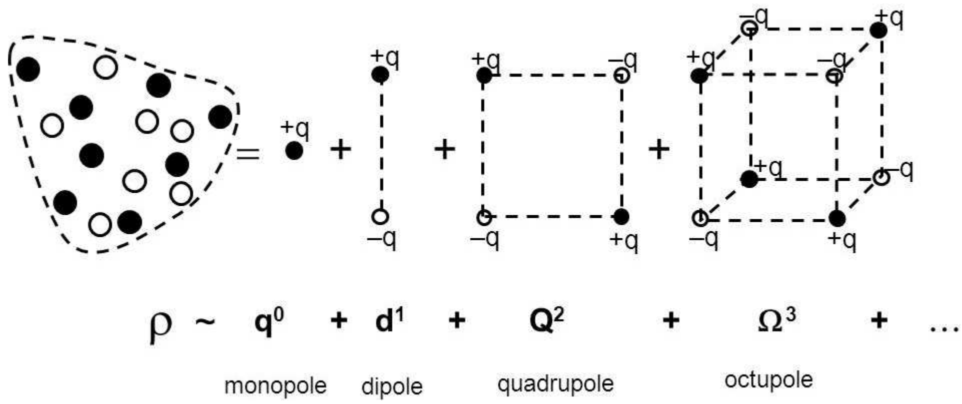
The multipole expansion considered here arose in a 2-dimensional context, where a series expansion of the form $\sum_m a_m \cos m\phi$ is useful. In contrast, in 3-dimensional cases one

[2] An early discussion of sextupole magnets is in [2].

[3] The "trivial" case of $m = 0$ has uniform current on the cylinder of radius a, and no magnetic field inside that cylinder.

often considers expansions of the form $\sum_l P_l(\cos\theta)$ or $\sum_{l,m} c_{l,m} Y_l^m(\theta,\phi)$ where $P_l(\cos\theta)$ is a Legendre polynomial and Y_l^m is a spherical harmonic. In 3-dimensional expansions, one describes terms of order l as monopole, dipole, quadrupole, octupole, hexadecupole, ..., for l = 0,1,2,3,4, ..., whereas in 2-dimensional expansions the names dipole, quadrupole, sextupole, octupole, decupole, are associated with terms of order m = 1,2,3,4,5 ...[4]

The figure below, illustrating a multipole expansion of a three-dimensional charge distribution, indicates how the lowest multipole with an intrinsically three-dimensional character is the octupole. Hence, the lowest two-dimensional multipole to have a different character than the three-dimensional multipole of the same order is the sextupole (of order 3).



While the 2-dimensional multipole of order m has the character of $2m$ filaments/poles in a circular array, the 3-dimensional multiple of order l has the character of 2^l point charges/poles on a cubical lattice.

A Appendix: Two-Dimensional Electric Multipoles

For completeness, we note that line charges on the surface of a cylinder lead to two-dimensional electric-field multipoles very similar to the magnetic multipoles found above.

A.1 Single Line Charge

For a fixed charge λ per unit length along a filament at (a, ϕ_0, z) in a cylindrical coordinate system whose axis is that of the cylinder of radius a, the electric field $\mathbf{E}$ at (r, ϕ, z) is radial with respect to the filament, with magnitude $E = \lambda/2\pi\epsilon_0 R$ as follows from $\boldsymbol{\nabla}\cdot\mathbf{E} = \rho/\epsilon_0$, where ρ is the electric current density, and R of eq. (2) is the transverse distance between the filament and the observation point.

We again briefly consider a cylindrical coordinate system (R, ϕ', z) whose axis is the current filament, such that the relation $\mathbf{E} = -\boldsymbol{\nabla} V$ implies that (in the Coulomb gauge) the scalar potential V is related by, Hence,

$$B_{r'} = \frac{\lambda}{2\pi\epsilon_0 R} = -\frac{\partial V}{\partial R}\,, \qquad V = -\frac{\lambda}{2\pi\epsilon_0}\ln R + \text{const.} \tag{18}$$

[4]In two dimensions, the vector potential of a single current filament not at the origin is given by eq. (4), which contains multipoles of every nonzero order. Hence, one does not speak of a monopole ($m = 0$) in the 2-dimensional multipole expansion.

Returning to the coordinate system (r, ϕ, z), we define the scalar potential to be zero on its axis, such that,

$$V = -\frac{\lambda}{2\pi\epsilon_0} \ln\frac{R}{a} = -\frac{\lambda}{2\pi\epsilon_0} \ln\left(1 + \frac{r^2}{a^2} - 2\frac{a}{r}\cos(\phi - \phi_0)\right). \tag{19}$$

For $r < a$ the scalar potential has the Taylor-series expansion,

$$V(r < a) = \frac{\lambda}{2\pi\epsilon_0} \sum_{n=1}^{\infty} \frac{1}{n}\frac{r^n}{a^n} \cos n(\phi - \phi_0). \tag{20}$$

The electric field for $r < a$ follows from eq. (20),

$$E_r(r < a) = -\frac{\partial V}{\partial r} = -\frac{\lambda}{2\pi\epsilon_0 r} \sum_{n=1}^{\infty} \frac{r^n}{a^n} \cos n(\phi - \phi_0), \tag{21}$$

$$E_\phi(r < a) = -\frac{1}{r}\frac{\partial V}{\partial \phi} = \frac{\lambda}{2\pi\epsilon_0 r} \sum_{n=1}^{\infty} \frac{r^n}{a^n} \sin n(\phi - \phi_0). \tag{22}$$

Equations (20) and (21)-(22) can be called electric multipole expansions.

A.2 The Surface Charge Varies as $\cos m\phi_0$

For surface charge density at $r = a$ of the form $\lambda(\phi) = \lambda_0 \cos m\phi$, the scalar potential for $r < a$ has the Taylor-series expansion,

$$\begin{aligned} V^{(m)}(r < a) &= \frac{\lambda}{2\pi\epsilon_0} \sum_{n=1}^{\infty} \frac{1}{n}\frac{r^n}{a^n} \int_0^{2\pi} \cos n(\phi - \phi_0) \cos m\phi_0 \, d\phi_0 \\ &= \frac{\lambda}{2\pi\epsilon_0} \sum_{n=1}^{\infty} \frac{1}{n}\frac{r^n}{a^n} \int_0^{2\pi} (\cos n\phi \cos n\phi_0 - \sin n\phi \sin n\phi_0) \cos m\phi_0 \, d\phi_0 \\ &= \frac{\lambda}{2\epsilon_0 m}\frac{r^m}{a^m} \cos m\phi. \end{aligned} \tag{23}$$

The electric field for $r < a$ is then,

$$E_r^{(m)}(r < a) = -\frac{\lambda_0 r^{m-1}}{2\epsilon_0 a^m} \cos m\phi, \qquad E_\phi^{(m)}(r < a) = \frac{\lambda_0 r^{m-1}}{2\epsilon_0 a^m} \sin m\phi. \tag{24}$$

The field components in rectangular coordinates are, for $r < a$,

$$\begin{aligned} E_x^{(m)}(r < a) &= E_r^{(m)} \cos\phi - E_\phi^{(m)} \sin\phi = -\frac{\lambda_0 r^{m-1}}{2\epsilon_0 a^m}(\cos m\phi \cos\phi + \sin m\phi \sin\phi) \\ &= -\frac{\lambda_0 r^{m-1}}{2\epsilon_0 a^m} \cos(m-1)\phi, \end{aligned} \tag{25}$$

$$\begin{aligned} E_y^{(m)}(r < a) &= E_r^{(m)} \sin\phi + E_\phi^{(m)} \cos\phi = \frac{\lambda_0 r^{m-1}}{2\epsilon_0 a^m}(-\cos m\phi \sin\phi + \sin m\phi \cos\phi) \\ &= \frac{\lambda_0 r^{m-1}}{2\epsilon_0 a^m} \sin(m-1)\phi. \end{aligned} \tag{26}$$

For $m = 1$ (dipole),

$$E_x^{(1)}(r < a) = -\frac{\lambda_0}{2\epsilon_0 a} = E_0, \qquad E_y^{(1)}(r < a) = 0. \tag{27}$$

For $m = 2$ (quadrupole),

$$E_x^{(2)}(r < a) = -\frac{\lambda_0}{2\epsilon_0 a^2}\, r\cos\phi = -\frac{\lambda_0}{2\epsilon_0 a^2}\, x, \qquad E_y^{(2)}(r < a) = \frac{\lambda_0}{2\epsilon_0 a^2}\, r\sin\phi = \frac{\lambda_0}{2\epsilon_0 a^2}\, y. \tag{28}$$

For $m = 3$ (sextupole),

$$E_x^{(3)}(r < a) = -\frac{\lambda_0}{2\epsilon_0 a^3}\, r^2\cos 2\phi = -\frac{\lambda_0}{2\epsilon_0 a^3}\, r^2(\cos^2\phi - \sin^2\phi) = -\frac{\lambda_0}{\epsilon_0 a^3}\,\frac{x^2 - y^2}{2}\,, \tag{29}$$

$$E_y^{(3)}(r < a) = \frac{\lambda_0}{2\epsilon_0 a^3}\, r^2\sin 2\phi = \frac{\lambda_0}{\epsilon_0 a^3}\, r^2\sin\phi\cos\phi = \frac{\lambda_0}{\epsilon_0 a^3}\, xy. \tag{30}$$

The electric field pattern of order m is rotated counterclockwise by $\pi/2m$ with respect to the magnetic pattern.

B Appendix: Magnetic Scalar Potential

In regions where the electric current density $\mathbf{J}$ is zero, and the electromagnetic fields are steady, such as $r < a$ in the present example, we have that $\boldsymbol{\nabla}\times\mathbf{B} = 0$, with the implication that the magnetic field can be related to a scalar potential V_M according to $\mathbf{B} = -\boldsymbol{\nabla} V_M$.

For filamentary currents on the cylinder $r = a$ that vary as $\cos m\phi_0$, we then can write eq. (11) as,

$$B_r^{(m)}(r < a) = -\frac{\mu_0 I_0 r^{m-1}}{2a^m}\sin m\phi = -\frac{\partial V_M^{(m)}}{\partial r}\,, \tag{31}$$

$$B_\phi^{(m)}(r < a) = -\frac{\mu_0 I_0 r^{m-1}}{2a^m}\cos m\phi = -\frac{1}{r}\frac{\partial V_M^{(m)}}{\partial \phi}\,, \tag{32}$$

where the magnetic scalar potential is,

$$V_M^{(m)}(r < a) = \frac{\mu_0 I_0 r^m}{2m a^m}\sin m\phi. \tag{33}$$

Is seems, however, that there is no simple method to arrive at eq. (33) without knowing the magnetic field $\mathbf{B}^{(m)}(r < a)$.[5]

[5] We could return to the case of a single filament, as in sec. 2.1, and note that the scalar potential,

$$V_M(r < a) = \frac{\mu_0 I}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\frac{r^n}{a^n}\sin n(\phi - \phi_0), \tag{34}$$

C Appendix: Two-Dimensional Multipoles via Conjugate Functions

As noted by Maxwell in Art. 183 of [3], any analytic function $f(z) = U + i\,V$ of the complex variable $z = x + i\,y$ obeys,

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\,, \tag{36}$$

and hence both $U(x, y)$ and $V(x, y)$ are solutions to Laplace's equation in two dimensions. Thus, both U and V are possible electrostatic and magnetostatic potentials in regions free of electric charges and currents.

The simple case of $f = z^m = (r\,e^{i\phi})^m = r^m \cos m\phi + i\,r^m \sin m\phi$ was discussed in Art. 317 of [4]. We recognize that $V^{(m)} \propto r^m \cos m\phi$ (or $r^m \sin m\phi$) constitutes a set of possible potentials, which for positive integers m correspond to the two-dimensional multipoles found above.[6]

References

[1] K.H. Meβ and P. Schmüser, *Superconducting Accelerator Magnets*, CERN-89-04, p. 87, http://kirkmcd.princeton.edu/examples/magnets/mess_cern-89-04_87.pdf

[2] H. Friedburg, *Optische Abbildung mit neutralen Atomen*, Z. Phys. **130**, 493 (1951), http://kirkmcd.princeton.edu/examples/magnets/friedburg_zp_130_493_51.pdf

[3] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 1 (Clarendon Press, 1873), http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v1_73.pdf

[4] J.H. Jeans, *The Mathematical Theory of Electricity and Magnetism* (Cambridge U. Press, 1908), http://kirkmcd.princeton.edu/examples/EM/jeans_electricity.pdf

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would lead to the form (33) upon integration (as in eq. (10)) over the current distribution $I(a, \phi_0) = I_0 \cos m\phi_0$. For a single filament, we could again consider a cylindrical coordinate system (R, ϕ', z) whose axis is that of the filament,

$$B_{\phi'} = \frac{\mu_0 I}{2\pi R} = -\frac{1}{R}\frac{\partial V_M}{\partial \phi'}\,, \qquad V_M = -\frac{\mu_0 I}{2\pi}\phi'. \tag{35}$$

Then, in the coordinate system (r, ϕ, z) the sine law tells us that $\sin\phi' = (r/R)\sin(\phi - \phi_0)$, but the Taylor series for $\sin^{-1}\phi'$ is not readily seen to have the form of eq. (34).

[6]This method was used to deduce the sextupole fields in [2].