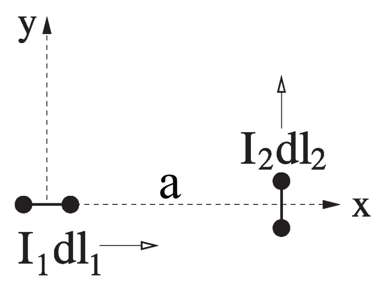
The Field Momentum of Two Time-Dependent Dipoles

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(February 4, 2022; updated February 23, 2022)

A recent article by Narayan [1] illustrated that magnetic forces do not obey Newton's 3rd law of action and reaction via the example of two "point" electric dipoles whose moments $\mathbf{p}_i = p_{0,i}\, e^{\lambda t}\, \hat{\mathbf{x}}_i$ grow exponentially with time. He correctly found that $\mathbf{F}_{12}+\mathbf{F}_{21}+d\mathbf{P}_{\rm EM}/dt = 0$ where $\mathbf{P}_{\rm EM} = \int \mathbf{E}\times\mathbf{B}\, d{\rm Vol}/4\pi c$ (in Gaussian units, with c as the speed of light in vacuum) is the electromagnetic-field momentum associated with the two dipoles.[1] This is an example of a general result by Page and Adams [2], published many years ago, in the so-called Darwin approximation [3][2] that keeps terms only to order v^2/c^2 (which ignores electromagnetic radiation), where $\mathbf{v}$ is velocity.[3] Narayan's demonstration is nice in that it holds to all orders of v/c.

However, Narayan mistakenly claimed that there are "no electromagnetic fields in the far field zone", and hence "no radiation is emitted" in his example, based on the appearance of his eqs. (2) and (3). While those forms are convenient for later computation, they do not well display the character of the electromagnetic fields of a time-dependent dipole. This was the topic of the Appendix of [7],[4] where the magnetic field of a time-dependent electric dipole $\mathbf{p}(t)$ (at the origin) was shown to be $\mathbf{B}(\mathbf{r},t) = [\ddot{\mathbf{p}}]\times\hat{\mathbf{r}}/c^2 r + [\dot{\mathbf{p}}]\times\hat{\mathbf{r}}/c\, r^2$ with $[\mathbf{p}] = \mathbf{p}(t-r/c)$. The term that varies as $1/r$ is a "radiation" field, nonzero except in the limit $t \to -\infty$ when the dipole moment is zero by construction. The radiation pattern of Narayan's two dipoles can be computed using eq. (66.6) of [4] and Fig. 1 of [1] (shown below) where $\hat{\mathbf{x}}_1 = \hat{\mathbf{x}}$ and where $\hat{\mathbf{x}}_2 = \hat{\mathbf{y}}$, noting also that $\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$,

$$\frac{dI}{d\Omega} = \frac{c\, r^2 [\mathbf{B}_{\rm rad}]^2}{4\pi} = \frac{[(\ddot{\mathbf{p}}_1 + \ddot{\mathbf{p}}_2)\times\hat{\mathbf{r}}]^2}{4\pi c^3} = \frac{\lambda^4[(p_1\,\hat{\mathbf{x}} + p_2\,\hat{\mathbf{y}})\times\hat{\mathbf{r}}]^2}{4\pi c^3}$$
$$= \frac{\lambda^4[(p_1^2+p_2^2)\cos^2\theta + (p_1\sin\phi - p_2\cos\phi)^2\sin^2\theta]}{4\pi c^3}\,. \qquad (1)$$



Then, the total intensity, integrated over solid angle, is $I = 2\lambda^4[p_1^2+p_2^2]/3c^3$.[5]

---

[1]We exclude the self-momentum of moving charges, and consider only the interaction field momentum.
[2]See also §65 of [4] and sec. 12.6 of [5].
[3]See also [6].
[4]See also the Appendix to [8].
[5]The electric-dipole radiation follows from eq. (67.8) of [4] as $I_{\rm E1} = 2[\ddot{\mathbf{p}}]^2/3c^3 = 2\lambda^4[\mathbf{p}_1+\mathbf{p}_2]^2/3c^3 = 2\lambda^4[p_1^2+p_2^2]/3c^3$, which is the same as the total intensity.

We can also consider the radiated momentum $d\mathbf{P}_{\rm rad}/dt = \int d\Omega\,(d^2\mathbf{P}_{\rm rad}/dt\,d\Omega)$,[6] where $d^2\mathbf{P}_{\rm rad}/dt\,d\Omega = (c\,\hat{\mathbf{r}}/4\pi)\,dI/d\Omega$. Recalling eq. (1), we find that although $d^2\mathbf{P}_{\rm rad}/dt\,d\Omega$ is nonzero, the total radiated momentum $d\mathbf{P}_{\rm rad}/dt$ is zero.

In a larger historical context, Ampére's insistence that magnetic forces obey Newton's 3rd law earned him the sobriquet by Maxwell [10] of the "Newton of electricity". Ampére's authority held up acceptance of the "Lorentz" force law (stated obliquely by Maxwell in 1861 [11]) until efforts by Thomson [13] and Heaviside [15] in 1891 clarified that electromagnetic fields carry momentum as well as energy (following the first clear statement of the "Lorentz" force law by Heaviside in 1885 [18][7]).

In 1864, Maxwell discussed "electromagnetic momentum", identifying this with Faraday's "electronic state" in sec. 26 of [23], and clarifying in sec. 57 that the "electromagnetic momentum" of charge q in an external vector potential $\mathbf{A}$ is $q\mathbf{A}(/c)$ (in the Coulomb gauge, as favored by Maxwell). This formulation suggests that "electromagnetic momentum" is a property of the charge, rather than of the electromagnetic field. That Maxwell's "electromagnetic momentum" is equivalent to the electromagnetic-field momentum of Thomson and Heaviside (the $\mathbf{P}_{\rm EM}$ of this note) in quasistatic examples was first demonstrated by Thomson [24]. See also [25, 26, 27].

Appendix: Onoochin's Variant

This Appendix is based on analysis by David Griffiths.

Vladimir Onoochin (private communication) has proposed a variant of Narayan's example, again with two time-dependent electric dipoles, for which the current density of the dipole at the origin can be written as,

$$\mathbf{J}_i(\mathbf{r},t) = I_i\,dl_i\,e^{\lambda t}\,\delta^3(\mathbf{r}-\mathbf{r}_i)\,\hat{\mathbf{x}}_i, \tag{2}$$

with $\mathbf{r}_1 = 0$ and $\mathbf{r}_2 = a\,\hat{\mathbf{x}}$. The continuity equation then says,

$$\frac{\partial\rho_i}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{J}_i = -I_i\,dl_i\,e^{\lambda t}\frac{\partial}{\partial x_i}\delta^3(\mathbf{R}_i), \tag{3}$$

where $\mathbf{R}_i = \mathbf{r}-\mathbf{r}_i$. This integrates to,

$$\rho_i(\mathbf{r},t) = -I_i\,dl_i\frac{\partial}{\partial x_i}\delta^3(\mathbf{R}_i)\int e^{\lambda t}\,dt = -I_i\,dl_i\frac{\partial}{\partial x_i}\delta^3(\mathbf{R}_i)\left[\frac{e^{\lambda t}}{\lambda}+k_i(\mathbf{r})\right], \tag{4}$$

for some $k_i(\mathbf{r})$ independent of time t. Narayan stipulated that "all sources" (including ρ_i) have the time dependence $e^{\lambda t}$, and hence tacitly assumed that $k_i = 0$,

$$\rho_{{\rm N},i}(\mathbf{r},t) = -I_i\,dl_i\frac{e^{\lambda t}}{\lambda}\frac{\partial}{\partial x}\delta^3(\mathbf{R}_i), \tag{5}$$

[6]See, for example, p. 3 of [9].

[7]In 1846, the first force law between moving charges was given by Weber as $\mathbf{F} = q_1q_2(1-\dot{r}^2/2c^2 + r\ddot{r}/c^2)\,\hat{\mathbf{r}}/r^2$, p. 375 of [19], which obeys Newton's 3rd law. In 1881, Thomson wrote the magnetic force on charge q as $q\mathbf{v}\times\mathbf{B}/2c$ in eq. (5) of [20]. In 1895, Lorentz wrote "his" force law as $\mathbf{F} = q(\mathbf{D}+\mathbf{v}/c\times\mathbf{H})$ in eq. (V), p. 21, of [21], although he seems mainly to have considered its use in vacuum where $\mathbf{D} = \mathbf{E}$ and $\mathbf{H} = \mathbf{B}$. See also eq. (23), p. 14, of [22].

as in eq. (5) of [1]. In contrast, Onoochin chose $k_i = -1/\lambda$, which leads to,

$$\rho_{\mathrm{O},i}(\mathbf{r},t) = -I_i\,dl_i \left[\frac{e^{\lambda t}-1}{\lambda}\right] \frac{\partial}{\partial x_i}\delta^3(\mathbf{R}_i), \tag{6}$$

and considered the limit $\lambda \to 0$,

$$\rho_{\mathrm{O},i}(\mathbf{r},t) = -I_i\,dl_i\,t\frac{\partial}{\partial x_i}\delta^3(\mathbf{R}_i), \qquad \mathbf{J}_{\mathrm{O},i}(\mathbf{r},t) = I_i\,dl_i\,\delta^3(\mathbf{R}_i)\,\hat{\mathbf{x}}_i. \tag{7}$$

In Onoochin's variant, the electric charge density $\rho_{\mathrm{O},i}$ depends linearly on time while the current density $\mathbf{J}_{\mathrm{O},i}$ is independent of time. These conditions were called "semistatic" in sec. III of [28], for which case the electric and magnetic fields can be computed using the instantaneous Coulomb and Biot-Savart laws (without retardation).

The electric-dipole moment in Onoochin's variant is, omitting the subscript O from now on,

$$\begin{aligned}\mathbf{p}_i(t) = \int \rho_i(\mathbf{r},t)\,\mathbf{r}\,d^3\mathbf{r} &= -I_i\,dl_i\,t\int\left[\frac{\partial}{\partial x_i}\delta^3(\mathbf{R}_i)\right]\mathbf{r}\,d^3\mathbf{r} \\ &= I_i\,dl_i\,t\int \delta^3(\mathbf{R}_i)\frac{\partial\,\mathbf{r}}{\partial x_i}\,d^3\mathbf{r} = I_i\,dl_i\,t\,\hat{\mathbf{x}}_i.\end{aligned} \tag{8}$$

The electromagnetic fields follow[8] as the instantaneous electric-dipole field,

$$\mathbf{E}_i(\mathbf{r},t) = \frac{3(\mathbf{p}_i\cdot\hat{\mathbf{R}}_i)\,\hat{\mathbf{R}}_i - \mathbf{p}_i}{R_i^3} - \frac{4\pi}{3}\mathbf{p}_i\,\delta^3(\mathbf{R}_i), \tag{9}$$

and the Biot-Savart magnetic field,

$$\mathbf{B}_i(\mathbf{r},t) = \int \frac{\mathbf{J}_i(\mathbf{r}',t)\times\hat{\mathbf{R}}'_i}{cR_i'^2}\,d^3\mathbf{r}' = \frac{I_i\,dl_i\,\hat{\mathbf{x}}_i\times\hat{\mathbf{R}}_i}{cR_i^2}\,, \tag{10}$$

where $\mathbf{R}' = \mathbf{r}' - \mathbf{r}_i$. There is no radiation associated with Onoochin's "semistatic" variant.

Turning to the full configuration with two electric dipoles according to Onoochin's variant, as in the figure on p. 1 above, with $\mathbf{p}_1 = I_1\,dl_1\,t\,\hat{\mathbf{x}}$ at $\mathbf{r}_1 = 0$ and $\mathbf{p}_2 = I_2\,dl_2\,t\,\hat{\mathbf{y}}$ at $\mathbf{r}_2 = a\,\hat{\mathbf{x}}$, the total electric force on the two dipoles is zero, since Coulomb's law obeys Newton's 3$^{\mathrm{rd}}$ law. However, the total magnetic force is nonzero,

$$\begin{aligned}\mathbf{F}_{\mathrm{tot}} = \mathbf{F}_{12} + \mathbf{F}_{21} &= \int\frac{\mathbf{J}_1}{c}\times\mathbf{B}_2\,d\mathrm{Vol} + \int\frac{\mathbf{J}_2}{c}\times\mathbf{B}_1\,d\mathrm{Vol} \\ = \frac{I_1\,dl_1\,\hat{\mathbf{x}}}{c}\times\left(\frac{I_2\,dl_2\,\hat{\mathbf{y}}\times(-\hat{\mathbf{x}})}{ca^2}\right) &+ \frac{I_2\,dl_2\,\hat{\mathbf{y}}}{c}\times\left(\frac{I_2\,dl_1\,\hat{\mathbf{x}}\times\hat{\mathbf{x}}}{ca^2}\right) = -\frac{(I_1\,dl_1)(I_2\,dl_2)}{a^2c^2}\,\hat{\mathbf{y}}.\end{aligned} \tag{11}$$

The interaction field momentum is, recalling that $\mathbf{R}_1 = \mathbf{r}$ and $\mathbf{R}_2 = \mathbf{r} - a\,\hat{\mathbf{x}} \equiv \mathbf{r}'$,

$$\begin{aligned}\mathbf{P}_{\mathrm{EM}} &= \int\frac{\mathbf{E}_1\times\mathbf{B}_2 + \mathbf{E}_2\times\mathbf{B}_1}{4\pi c}\,d\mathrm{Vol} \\ = \frac{(I_1\,dl_1)(I_2\,dl_2)t}{4\pi c^2}&\int\left[\left(\frac{3(\hat{\mathbf{x}}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \hat{\mathbf{x}}}{r^3} - \frac{4\pi}{3}\,\hat{\mathbf{x}}\,\delta^3(\mathbf{r})\right)\times\frac{\hat{\mathbf{y}}\times\mathbf{r}'}{r'^3}\right. \\ &\left.+\left(\frac{3(\hat{\mathbf{y}}\cdot\hat{\mathbf{r}}')\,\hat{\mathbf{r}}' - \hat{\mathbf{y}}}{r'^3} - \frac{4\pi}{3}\,\hat{\mathbf{y}}\,\delta^3(\mathbf{r} - a\,\hat{\mathbf{x}})\right)\times\frac{\hat{\mathbf{x}}\times\mathbf{r}}{r^3}\right]d\mathrm{Vol}.\end{aligned} \tag{12}$$

[8]These fields also follow from the Appendix of [7], and eqs. (46)-(47) of [29].

To evaluate the integral, we adopt a spherical coordinate system (r, θ, ϕ) with the x-axis as the polar axis, and,

$$\mathbf{r} \cdot \hat{\mathbf{x}} = r\cos\theta, \qquad \mathbf{r} \cdot \hat{\mathbf{y}} = r\sin\theta\cos\phi, \qquad \mathbf{r} \cdot \hat{\mathbf{z}} = r\sin\theta\sin\phi. \tag{13}$$

Then, $r' = \sqrt{r^2 - 2ar\cos\theta + a^2}$,

$$\mathbf{r}' \cdot \hat{\mathbf{x}} = r\cos\theta - a, \qquad \mathbf{r}' \cdot \hat{\mathbf{y}} = r\sin\theta\cos\phi, \qquad \mathbf{r}' \cdot \mathbf{r} = r^2 - ar\cos\theta, \tag{14}$$

and,

$$\begin{aligned}
\mathbf{P}_{\rm EM} &= \frac{(I_1\, dl_1)(I_2\, dl_2)\, t}{4\pi c^2} \left[\frac{8\pi}{3a^2}\, \hat{\mathbf{y}} + \int d\mathrm{Vol}\, \frac{[2r\cos\theta + a(1 - 3\cos^2\theta)]\, \hat{\mathbf{y}} - 3\sin\theta\cos\theta\cos\phi\, (\mathbf{r} - a\, \hat{\mathbf{x}}))}{r^3 r'^3} \right. \\
&\qquad \left. + \int d\mathrm{Vol}\, \frac{r\sin\theta\cos\phi\, [3(r^2 - ar\cos\theta)/r'^2 - 1]\, \hat{\mathbf{x}} - 3r\sin\theta\cos\phi(r\cos\theta - a)\, \mathbf{r}/r'^2}{r^3 r'^3} \right] \\
&= \frac{(I_1\, dl_1)(I_2\, dl_2)\, t}{4\pi c^2}\, \hat{\mathbf{y}} \left[\frac{4\pi}{3a^2} + 2\pi \int_0^\infty dr \int_{-1}^1 d\cos\theta\, \frac{2r\cos\theta + a(1 - 3\cos^2\theta) - (3/2) r\sin\theta\cos\theta}{r'^3} \right. \\
&\qquad \left. - 3\pi \int_0^\infty dr \int_{-1}^1 d\cos\theta\, \frac{r\sin\theta\cos\theta\, (r\cos\theta - a)}{r'^5} \right].
\end{aligned} \tag{15}$$

These integrals are difficult to evaluate, but of possible interest is that the delta function in the electric field at the center of "point" dipole 1 contributes to the total field momentum, but the delta function of dipole 2 does not.[9]

As Onoochin's variant is "semistatic", we can also compute the field momentum via Maxwell's formulation [27], $\mathbf{P}_{\rm EM} = \sum q_j \mathbf{A}(\mathbf{r}_j)/c$. We see that the (Coulomb-gauge) vector potential of dipole 1 is the same at the position of both charges of dipole 2, and so does not contribute to the momentum. The vector potential of dipole 2 along the x-axis is $\mathbf{A}_2(x) = I_2\, dl_2\, \hat{\mathbf{y}}/c\, |a - x|$, so the electromagnetic momentum is,

$$\mathbf{P}_{\rm EM} = \frac{q_1 dl_1}{c} \frac{d\mathbf{A}_2(0)}{dx} = \frac{(I_1\, dl_1)(I_2\, dl_2)\, t\, \hat{\mathbf{y}}}{a^2 c^2}, \tag{19}$$

[9] For completeness, we consider the field momentum of Onoochin's dipole 1, for which $\mathbf{p}_1 = I_1\, dl_1\, t\, \hat{\mathbf{x}}$, and its electromagnetic fields are, recalling eqs. (9)-(10) with $\mathbf{r}_1 = 0$,

$$\mathbf{E}_1(\mathbf{r}, t) = \frac{3(\mathbf{p}_1 \cdot \hat{\mathbf{r}})\, \hat{\mathbf{r}} - \mathbf{p}_1}{r^3} - \frac{4\pi}{3} \mathbf{p}_1\, \delta^3(\mathbf{r}), \qquad \mathbf{B}_1(\mathbf{r}, t) = \frac{I_1\, dl_1\, \hat{\mathbf{x}} \times \hat{\mathbf{r}}}{c\, r^2}. \tag{16}$$

The field momentum of this dipole is, recalling eq. (13),

$$\begin{aligned}
\mathbf{P}_{\rm EM,1} = \int \frac{\mathbf{E}_1 \times \mathbf{B}_1}{4\pi c}\, d\mathrm{Vol} &= \frac{(I_1\, dl_1)^2\, t}{4\pi c^2} \int \frac{[3(\hat{\mathbf{x}} \cdot \hat{\mathbf{r}})\, \hat{\mathbf{r}} - \hat{\mathbf{x}}] \times (\hat{\mathbf{x}} \times \hat{\mathbf{r}})}{r^5}\, d\mathrm{Vol} \\
&= \frac{(I_1\, dl_1)^2\, t}{4\pi c^2} \int \frac{[3(\hat{\mathbf{x}} \cdot \hat{\mathbf{r}})\, \hat{\mathbf{r}} - \hat{\mathbf{x}}] \times (\hat{\mathbf{x}} \times \hat{\mathbf{r}})}{r^5}\, d\mathrm{Vol} \\
= \frac{(I_1\, dl_1)^2\, t}{4\pi c^2} \int \frac{3(\hat{\mathbf{x}} \cdot \hat{\mathbf{r}})[\hat{\mathbf{x}} - (\hat{\mathbf{x}} \cdot \hat{\mathbf{r}})\, \hat{\mathbf{r}}] - (\hat{\mathbf{x}} \cdot \hat{\mathbf{r}})\, \hat{\mathbf{x}} + \hat{\mathbf{r}}}{r^5}\, d\mathrm{Vol} &= 0.
\end{aligned} \tag{17}$$

Alternatively, we can compute the field momentum using Maxwell's form,

$$\mathbf{P}_{\rm EM,1} = \frac{q_1 \mathbf{A}(x = dl_1/2) - q_1 \mathbf{A}(x = -dl_1/2)}{c} = \frac{(q_1 - q_1) I_1\, dl_1\, \hat{\mathbf{x}}}{c^2\, dl_1/2} = 0. \tag{18}$$

Similarly, the field momentum of dipole 2 is zero.

noting that $q_1 = I_1\, t$.

Recalling eq. (11), we now have that $\mathbf{F}_{12} + \mathbf{F}_{21} + d\mathbf{P}_{\rm EM}/dt = 0$, as expected. With the use of Maxwell's form for electromagnetic momentum, this demonstration for Onoochin's variant is much quicker than for Narayan's version of two time-dependent dipoles.

References

[1] O. Narayan, *Momentum conservation in the Biot-Savart law*, Am. J. Phys. **89**, 1033 (2021), http://kirkmcd.princeton.edu/examples/EM/narayan_ajp_89_1033_21.pdf

[2] L. Page and N.I. Adams, Jr., *Action and Reaction Between Moving Charges*, Am. J. Phys. **13**, 141 (1945), http://kirkmcd.princeton.edu/examples/EM/page_ajp_13_141_45.pdf

[3] C.G. Darwin, *The Dynamical Motions of Charged Particles*, Phil. Mag. **39**, 537 (1920), http://kirkmcd.princeton.edu/examples/EM/darwin_pm_39_537_20.pdf
See also §65 of [4] and sec. 12.6 of [5].

[4] L.D. Landau and E.M. Lifshitz, *Classical Theory of Fields*, 4th ed. (Pergamon, 1975), http://kirkmcd.princeton.edu/examples/EM/landau_ctf_75.pdf

[5] J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, 1999), http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf

[6] E. Breitberger, *Magnetic Intereactions between Charged Particles*, Am. J. Phys. **36**, 505 (1968), http://kirkmcd.princeton.edu/examples/EM/breitenberger_ajp_36_505_68.pdf

[7] P.R. Berman, *Dynamic creation of electrostatic fields*, Am. J. Phys. **76**, 48 (2008), http://kirkmcd.princeton.edu/examples/EM/berman_ajp_76_48_08.pdf

[8] K.T. McDonald, *The Fields of a Pulsed, Small Dipole Antenna* (Mar. 16, 2007), http://kirkmcd.princeton.edu/examples/pulsed_dipole.pdf

[9] K.T. McDonald, *The Force on an Antenna Array* (May 1, 1979), http://kirkmcd.princeton.edu/examples/antenna_force.pdf

[10] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 2 (Clarendon Press, 1873), Art. 528, http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v2_73.pdf

[11] J.C. Maxwell, *On Physical Lines of Force. Part II.—The Theory of Molecular Vortices applied to Electric Currents*, Phil. Mag. **21**, 281, 338 (1861), eq. (77), http://kirkmcd.princeton.edu/examples/EM/maxwell_pm_21_281_61.pdf
http://kirkmcd.princeton.edu/examples/EM/maxwell_pm_21_338_61.pdf
See also Appendices A.28.1.7, A.28.2.6, A.28.3.7 and A.28.4.7 of [12].

[12] K.T. McDonald, *Is Faraday's Disk Dynamo a Flux-Rule Exception?* (July 27, 2019), http://kirkmcd.princeton.edu/examples/faradaydisk.pdf

[13] J.J. Thomson, *On the Illustration of the Properties of the Electric Field by Means of Tubes of Electrostatic Induction*, Phil. Mag. **31**, 149 (1891), eqs. (2) and (6),
http://kirkmcd.princeton.edu/examples/EM/thomson_pm_31_149_91.pdf
See also [14].

[14] K.T. McDonald, *J.J. Thomson and "Hidden" Momentum* (Apr. 30, 2014),
http://kirkmcd.princeton.edu/examples/thomson.pdf

[15] O. Heaviside, *On the Forces, Stresses and Fluxes of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **183**, 423 (1892), sec. 26,
http://kirkmcd.princeton.edu/examples/EM/heaviside_ptrsla_183_423_92.pdf
This was a clarification of his discussion in 1886, eq. (7a) of [16].

[16] O. Heaviside, *Electromagnetic Induction and Its Propagation*, Electrician **16**, 186 (1886), http://kirkmcd.princeton.edu/examples/EM/heaviside_electrician_16_186_86.pdf
See also p. 545-548 of [17].

[17] O. Heaviside, *Electrical Papers*, Vol. 1 (Macmillan, 1892),
http://kirkmcd.princeton.edu/examples/EM/heaviside_electrical_papers_1.pdf

[18] O. Heaviside, *Electromagnetic Induction and Its Propagation*, Electrician **14**, 306 (1885), http://kirkmcd.princeton.edu/examples/EM/heaviside_electrician_14_178_85.pdf
See also p. 448 of [17].

[19] W. Weber, *Elektrodynamische Maassbestimmungen*, Abh. König. Sächs. Gesell. Wiss. 209 (1846), kirkmcd.princeton.edu/examples/EM/weber_aksgw_209_46.pdf
kirkmcd.princeton.edu/examples/EM/weber_aksgw_209_46_english.pdf

[20] J.J. Thomson, *On the Electric and Magnetic Effects produced by the Motion of Electrified Bodies*, Phil. Mag. **11**, 229 (1881),
http://kirkmcd.princeton.edu/examples/EM/thomson_pm_11_229_81.pdf

[21] H.A. Lorentz, *Versuch einer Theorie des Electrischen und Optischen Erscheinungen in bewegten Körpern*, (Brill, Leiden, 1895),
http://kirkmcd.princeton.edu/examples/EM/lorentz_electrical_theory_95.pdf
http://kirkmcd.princeton.edu/examples/EM/lorentz_electrical_theory_95_english.pdf

[22] H.A. Lorentz, *The Theory of Electrons* (B.G. Teubner, 1909),
http://kirkmcd.princeton.edu/examples/EM/lorentz_theory_of_electrons_09.pdf

[23] J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459 (1865), kirkmcd.princeton.edu/examples/EM/maxwell_ptrsl_155_459_65.pdf

[24] J.J. Thomson, *On Momentum in the Electric Field*, Phil. Mag. **8**, 331 (1904), p. 348,
http://kirkmcd.princeton.edu/examples/EM/thomson_pm_8_331_04.pdf

[25] G.R. Trammel, *Aharonov-Bohm Paradox*, Phys. Rev. **134**, B1183 (1964),
http://kirkmcd.princeton.edu/examples/EM/trammel_pr_134_B1183_64.pdf

[26] M.G. Calkin, *Linear Momentum of Quasistatic Electromagnetic Fields*, Am. J. Phys. **34**, 921 (1966), http://kirkmcd.princeton.edu/examples/EM/calkin_ajp_34_921_66.pdf

[27] K.T. McDonald, *Four Expressions for Electromagnetic Field Momentum* (April 10, 2006), http://kirkmcd.princeton.edu/examples/pem_forms.pdf

[28] D.J. Griffths and M.A. Heald, *Time-dependent generalization of the Biot-Savart and Coulomb laws*, Am. J. Phys. **59**, 111 (1991), http://kirkmcd.princeton.edu/examples/EM/berman_ajp_59_111_91.pdf

[29] D.J. Griffths, *Dynamic dipoles*, Am. J. Phys. **79**, 867 (2011), http://kirkmcd.princeton.edu/examples/EM/berman_ajp_79_867_11.pdf