The Fields of a Short, Linear Dipole Antenna If There Were No Displacement Current

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1 Problem

It is sometimes said that the main effect of Maxwell's "displacement current" is to produce radiation, which is a small effect in the near zone of a system. Consider the example of a short, center-fed, linear dipole antenna of length $2a$ and a specified current distribution to show that the fields its near zone calculated from Maxwell's equations without displacement current are the same as the nonradiation fields calculated using the full Maxwell's equations.

2 Solution

2.1 Fields with Neglect of the Displacement Current

If we neglect the displacement current, $(1/4\pi)\partial\mathbf{E}/\partial t$, Maxwell's equations for the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ are (in Gaussian units),

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\varrho, \quad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \quad \boldsymbol{\nabla}\cdot\mathbf{B} = 0, \quad \text{and} \quad \boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J}, \tag{1}$$

where ϱ is the charge density, $\mathbf{J}$ is the current density and c is the speed of light. These equations can by satisfied by,

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \qquad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}, \tag{2}$$

where the scalar potential Φ and the vector potential $\mathbf{A}$ are calculated using present quantities,

$$\Phi(\mathbf{x},t) = \int \frac{\varrho(\mathbf{x}',t)}{R}\, d\mathrm{Vol}', \qquad \mathbf{A}(\mathbf{x},t) = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{x}',t)}{R}\, d\mathrm{Vol}'. \tag{3}$$

Of course, this implies that changes in the charge or current distribution cause instantaneous changes in the potentials and fields.

As an example of eqs. (1)-(3), we consider a center-fed linear antenna of length $2a$ that is operated at angular frequency ω. We take the conductors to be along the z-axis, with the feed point at the origin. The current at the feed point is $I_0 e^{-i\omega t}$, but it must fall to zero at the tips of the antenna $z = \pm a$. When $a \ll \lambda$ the current distribution can only have a linear dependence on z, so its form must be,[1]

$$I(|z| < a, t) = I_0 e^{-i\omega t}\left(1 - \frac{|z|}{a}\right). \tag{4}$$

The charge distribution $\varrho(z,t)$ along the antenna can be deduced from the current distribution (4) using the equation of continuity (*i.e.*, charge conservation), which has the form $\partial I/\partial z = -\partial\varrho/\partial t$ Thus,

$$\varrho(|z| < a, t) = \pm\frac{iI_0 e^{-i\omega t}}{a\omega} = \pm\frac{iI_0 e^{-i\omega t}}{cka}\,, \tag{5}$$

where the $+(-)$ sign holds for $z > (<)\ 0$, and $k = \omega/c$.

Using eqs. (4) and (5) in eq. (3) we see that the scalar potential Φ is 90° out of phase with the vector potential $\mathbf{A}$. The time derivative $\partial\mathbf{A}/\partial t$ is 90° out of phase with $\mathbf{A}$, and hence is in phase with Φ. Then, eq. (2) indicates that the electric and magnetic fields are 90° out of phase throughout all space, IF the "displacement current" is neglected.

In detail, we find in cylindrical coordinates (ρ, ϕ, z) that,

$$\begin{aligned}
\Phi(\rho,\phi,z,t) &= Re\frac{iI_0 e^{-i\omega t}}{cka}\left(\int_0^a \frac{dz'}{\sqrt{\rho^2+(z-z')^2}} - \int_{-a}^0 \frac{dz'}{\sqrt{\rho^2+(z-z')^2}}\right) \\
&= \frac{I_0}{cka}\sin(\omega t)\left\{\ln\left[\sqrt{\rho^2+(z-a)^2}-(z-a)\right] + \ln\left[\sqrt{\rho^2+(z+a)^2}-(z+a)\right]\right. \\
&\qquad \left. -2\ln\left[\sqrt{\rho^2+z^2}-z\right]\right\},
\end{aligned} \tag{6}$$

and,

$$\begin{aligned}
A_z(\rho,\phi,z,t) &= Re\frac{I_0}{c}e^{-i\omega t}\left(\int_0^a \frac{(1-z'/a)\,dz'}{\sqrt{\rho^2+(z-z')^2}} + \int_{-a}^0 \frac{(1+z'/a)\,dz'}{\sqrt{\rho^2+(z-z')^2}}\right) \\
&= -\frac{I_0}{ca}\cos(\omega t)\left\{(z-a)\ln\left[\sqrt{\rho^2+(z-a)^2}-(z-a)\right]\right. \\
&\qquad +(z+a)\ln\left[\sqrt{\rho^2+(z+a)^2}-(z+a)\right] - 2z\ln\left[\sqrt{\rho^2+z^2}-z\right] \\
&\qquad \left. +\left(\sqrt{\rho^2+(z-a)^2}+\sqrt{\rho^2+(z+a)^2}-2\sqrt{\rho^2+z^2}\right)\right\},
\end{aligned} \tag{7}$$

which satisfy the Lorenz gauge condition,

$$\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0. \tag{8}$$

Far from the antenna the potentials (6) and (7) simplify to the forms,

$$\Phi \approx \frac{I_0 a z}{ckr_0^3}\sin(\omega t), \qquad \text{and} \qquad A_z \approx \frac{I_0 a}{cr_0}\cos(\omega t), \tag{9}$$

[1] In an actual antenna with conducting arms there exists a small current that is 90° out of phase with the drive current, and which vanishes at $z = 0$ as well as $z = \pm a$. This current is needed to provide some additional electric field in the near zone such that the tangential component of the total electric field vanishes along the (good) conductors. Here, we assume the current somehow has the form (4), and that the antenna arms are not actually conductors.

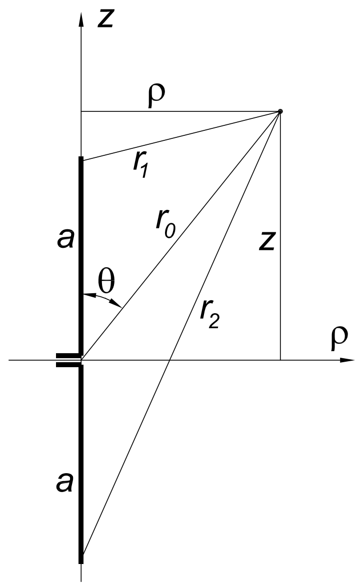
where $r_0 = \sqrt{\rho^2 + z^2}$. The far-zone scalar potential is that of a dipole consisting of charges $\pm i I_0/\omega$ separated by distance a, and the vector potential is that due to a length a of current I_0, with both potentials oscillating at frequency ω.

The magnetic field has only a ϕ component, and varies only as $\cos(\omega t)$,

$$\begin{aligned} B_\phi &= -\frac{\partial A_z}{\partial \rho} = \frac{\rho I_0}{ca}\cos(\omega t)\left[\frac{1}{r_1 - (z-a)} + \frac{1}{r_2 - (z+a)} - \frac{2}{r_0 - z}\right] \\ &= \frac{I_0}{c\rho a}(r_1 + r_2 - 2r_0)\cos(\omega t) \approx \frac{I_0 a \sin\theta}{c r_0^2}\cos(\omega t), \end{aligned} \tag{10}$$

where the approximation holds for $r_0 \gg a$, and the distances r_1 and r_2 are from the tips of the antenna to the observation point, as shown in the figure below, where,

$$r_0^2 = \rho^2 + a^2, \qquad r_1^2 = \rho^2 + (z-a)^2, \qquad r_2^2 = \rho^2 + (z+a)^2. \tag{11}$$



The ρ component of the electric field is,

$$\begin{aligned} E_\rho &= -\frac{\partial \Phi}{\partial \rho} = -\frac{\rho I_0}{cka}\sin(\omega t)\left[\frac{1}{r_1(r_1 - (z-a))} + \frac{1}{r_2(r_2 - (z+a))} - \frac{2}{r_0(r_0 - z)}\right] \\ &= -\frac{I_0}{c\rho k a}\left[\frac{z-a}{r_1} + \frac{z+a}{r_2} - \frac{2z}{r_0}\right]\sin(\omega t) \approx \frac{3 I_0 a \cos\theta \sin\theta}{c k r_0^3}\sin(\omega t), \end{aligned} \tag{12}$$

and the z component of the electric field is,

$$\begin{aligned} E_z &= -\frac{\partial \Phi}{\partial z} - \frac{1}{c}\frac{\partial A_z}{\partial t} = \frac{I_0}{cka}\sin(\omega t)\left\{\frac{1}{r_1} + \frac{1}{r_2} - \frac{2}{r_0}\right. \\ &\qquad \left. -k^2[r_1 + r_2 - 2r_0 + (z-a)\ln(r_1 - z + a) + (z+a)\ln(r_2 - z - a) - 2z\ln(r_0 - z)]\right\} \\ &\approx \frac{I_0 a}{ck}\left(\frac{3\cos^2\theta - 1}{r_0^3} + \frac{k^2}{r_0}\right)\sin(\omega t), \end{aligned} \tag{13}$$

where the approximation hold at large distances. The electric field varies only as $\sin(\omega t)$.

The components of the electric field in spherical coordinates for $r_0 \gg a$ are,

$$E_r \approx \frac{I_0 a}{ck}\left(\frac{2}{r_0^3} - \frac{k^2}{r_0}\right)\cos\theta\sin(\omega t), \qquad E_\theta \approx \frac{I_0 a}{ck}\left(\frac{1}{r_0^3} + \frac{k^2}{r_0}\right)\sin\theta\sin(\omega t). \tag{14}$$

For large r_0 the electric field, but not the magnetic field, has a term that falls off as $1/r_0$. Since $\mathbf{E}$ and $\mathbf{B}$ are out of phase the time-average Poynting vector is zero, and there is no (time-average) transport of energy away from the source when displacement current is neglected.

In the near zone, where $kr_j \ll 1$ for $j = 0, 1, 2$ the nonzero field components can be written as,

$$B_\phi(kr_j \ll 1) = \frac{I_0}{c\rho a}[r_1 + r_2 - 2r_0]\cos(\omega t), \tag{15}$$

$$E_\rho(kr_j \ll 1) = -\frac{I_0}{c\rho ka}\left[\frac{z-a}{r_1} + \frac{z+a}{r_2} - \frac{2z}{r_0}\right]\sin(\omega t), \tag{16}$$

$$E_z(kr_j \ll 1) \approx \frac{I_0}{cka}\left[\frac{1}{r_1} + \frac{1}{r_2} - \frac{2}{r_0}\right]\sin(\omega t). \tag{17}$$

2.2 Near Fields When Displacement Current Is Included

This section follows sec. 2.3 of [1] as to the near fields of a linear dipole antenna of length $2a$, with an assumed current distribution,

$$I(z,t) = I_0 \frac{\sin[k(a - |z|)]\cos\omega t}{\sin ka}, \tag{18}$$

which is normalized such that $I(z = 0) = I_0 \cos\omega t$.

The electric and magnetic fields can be calculated from the retarded vector potential, which has only a z-component in this example,

$$A_z(\mathbf{x}, t) = \frac{1}{c}\int_{-a}^{a} dz' \, \frac{I(z', t' = t - R/c)}{R} = \frac{I_0 e^{-i\omega t}}{c}\int_{-a}^{a} dz' \, \sin\sin[k(a - |z'|)]\frac{e^{ikR}}{R}, \tag{19}$$

where $k = 2\pi/\lambda = \omega/c$ and $R = |\mathbf{x} - \mathbf{x}'|$. Then, the fields $\mathbf{E}$ and $\mathbf{B}$ are related by[2]

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}, \qquad \text{and} \qquad \left[\frac{i}{kc}\frac{\partial\mathbf{E}}{\partial t} =\right]\mathbf{E} = \frac{i}{k}\boldsymbol{\nabla}\times\mathbf{B}. \tag{20}$$

We evaluate the field components in a cylindrical coordinate system (ρ, ϕ, z) to find for a small antenna with $ka \ll 1$,

$$B_\rho = 0, \tag{21}$$

$$B_\phi = -Re\left\{\frac{iI_0 e^{-i\omega t}}{c\rho ka}\left[e^{ikr_1} + e^{ikr_2} - 2e^{ikr_0}\left(1 - \frac{k^2a^2}{2}\right)\right]\right\} \tag{22}$$

[2] We note that the sequence of calculations in eqs. (19) and (20) could be interpreted as implying that the conduction current I leads to the vector potential and the magnetic field, and then the curl of the magnetic field leads to the "displacement current" $(1/4\pi)\partial E/\partial t$.

$$= \frac{I_0}{c\rho ka}\left[\sin(kr_1 - \omega t) + \sin(kr_2 - \omega t) - 2\left(1 - \frac{k^2a^2}{2}\right)\sin(kr_0 - \omega t)\right], \tag{23}$$

$$B_z = 0, \tag{24}$$

$$E_\rho = -Re\left\{\frac{iI_0e^{-i\omega t}}{c\rho ka}\left[\frac{(z-a)e^{ikr_1}}{r_1} + \frac{(z+a)e^{ikr_2}}{r_2} - 2\frac{ze^{ikr_0}}{r_0}\left(1 - \frac{k^2a^2}{2}\right)\right]\right\} \tag{25}$$

$$= \frac{I_0}{c\rho ka}\left[(z-a)\frac{\sin(kr_1 - \omega t)}{r_1} + (z+a)\frac{\sin(kr_2 - \omega t)}{r_2} - 2z\left(1 - \frac{k^2a^2}{2}\right)\frac{\sin(kr_0 - \omega t)}{r_0}\right], \tag{26}$$

$$E_\phi = 0, \tag{27}$$

$$E_z = Re\left\{\frac{iI_0e^{-i\omega t}}{cka}\left[\frac{e^{ikr_1}}{r_1} + \frac{e^{ikr_2}}{r_2} - 2\frac{e^{ikr_0}}{r_0}\left(1 - \frac{k^2a^2}{2}\right)\right]\right\} \tag{28}$$

$$= -\frac{I_0}{cka}\left[\frac{\sin(kr_1 - \omega t)}{r_1} + \frac{\sin(kr_2 - \omega t)}{r_2} - 2\left(1 - \frac{k^2a^2}{2}\right)\frac{\sin(kr_0 - \omega t)}{r_0}\right]. \tag{29}$$

For completeness, we also display the field components in a spherical coordinate system (r, θ, ϕ), noting that $\rho = r_0 \sin\theta$,

$$B_r = 0, \tag{30}$$

$$B_\theta = 0, \tag{31}$$

$$B_\phi = -Re\left\{\frac{iI_0e^{-i\omega t}}{cr_0ka}\left[e^{ikr_1} + e^{ikr_2} - 2e^{ikr_0}\left(1 - \frac{k^2a^2}{2}\right)\right]\sin\theta\right\} \tag{32}$$

$$= \frac{I_0}{cr_0ka}\left[\sin(kr_1 - \omega t) + \sin(kr_2 - \omega t) - 2\left(1 - \frac{k^2a^2}{2}\right)\sin(kr_0 - \omega t)\right]\sin\theta, \tag{33}$$

$$E_r = Re\left\{\frac{iaI_0e^{-i\omega t}}{cr_0ka}\left[\frac{e^{ikr_1}}{r_1} - \frac{e^{ikr_2}}{r_2}\right]\right\} \tag{34}$$

$$= -\frac{I_0a}{cr_0ka}\left[\frac{\sin(kr_1 - \omega t)}{r_1} - \frac{\sin(kr_2 - \omega t)}{r_2}\right], \tag{35}$$

$$E_\theta = -Re\left\{\frac{iI_0e^{-i\omega t}}{cr_0^2ka\sin\theta}\left[\frac{(r_0^2 - az)e^{ikr_1}}{r_1} + \frac{(r_0^2 + az)e^{ikr_2}}{r_2} - 2r_0e^{ikr_0}\left(1 - \frac{k^2a^2}{2}\right)\right]\right\} \tag{36}$$

$$= \frac{I_0}{cr_0^2ka\sin\theta}\left[(r_0^2 - az)\frac{\sin(kr_1 - \omega t)}{r_1} + (r_0^2 + az)\frac{\sin(kr_2 - \omega t)}{r_2} - 2r_0\left(1 - \frac{k^2a^2}{2}\right)\sin(kr_0 - \omega t)\right]. \tag{37}$$

$$E_\phi = 0, \tag{38}$$

The radiation fields are most prominent in the far zone, where $r = r_0 \approx r_1 \approx r_2$. In spherical coordinates the only nonzero components to the radiation fields are,[3]

$$B_\phi = E_\theta = -Re\left\{\frac{iI_0ka}{c}\frac{e^{i(kr_0 - \omega t)}}{r_0}\sin\theta\right\} = \frac{I_0a}{c}\frac{\sin(kr_0 - \omega t)}{r_0}\sin\theta. \tag{39}$$

The radiation fields depend on the small quantity ka, which permits us to identify the radiation fields in the near zone, where they are only a small part of the total fields.

[3]Verification that eqs. (36) and (37) become eq. (39) in the far zone is a bit subtle. See [1].

Close to the antenna, where $kr_j \ll 1$ we have $\cos(kr_j) \approx 1$ and $\sin(kr_j) \approx kr_j$ for $j = 0, 1, 2$. The nonzero field components (23), (26) and (29) in cylindrical coordinates simplify to the forms close to the antenna,

$$B_\phi(kr_j \ll 1) \approx \frac{I_0}{c\rho}\left[\frac{r_1 + r_2 - 2r_0}{a}\cos(\omega t) - ka\sin(\omega t)\right], \tag{40}$$

$$E_\rho(kr_j \ll 1) \approx -\frac{I_0}{c\rho ka}\left[\frac{z-a}{r_1} + \frac{z+a}{r_2} - \frac{2z}{r_0}\left(1 - \frac{k^2a^2}{2}\right)\right]\sin(\omega t), \tag{41}$$

$$E_z(kr_j \ll 1) \approx \frac{I_0}{cka}\left[\frac{1}{r_1} + \frac{1}{r_2} - \frac{2}{r_0}\left(1 - \frac{k^2a^2}{2}\right)\right]\sin(\omega t). \tag{42}$$

Close to the antenna all of the electric field varies as $\sin(\omega t)$, and so is 90° out of phase with the drive current. The largest part of the magnetic field is in phase with the current, but the radiation part of the magnetic field (which varies as ka) is 90° out of phase with the current, and is therefore in phase with the electric field. Furthermore, the radiation parts of the electric and magnetic field have very similar magnitudes close to the antenna, even though the total electric field is much larger than the total magnetic field here.

Thus, the assumed current distribution (4) generates radiation fields in its near zone that are similar in character to the radiation fields in the far zone: $E_{\rm rad} \approx B_{\rm rad}$ in magnitude an phase, and directed at right angles to one another.

The nonradiation ("reactive") parts of the fields in the near zone are,

$$B_\phi(kr_j \ll 1, \text{non rad}) \approx \frac{I_0}{c\rho a}[r_1 + r_2 - 2r_0]\cos(\omega t), \tag{43}$$

$$E_\rho(kr_j \ll 1, \text{non rad}) \approx -\frac{I_0}{c\rho ka}\left[\frac{z-a}{r_1} + \frac{z+a}{r_2} - \frac{2z}{r_0}\right]\sin(\omega t), \tag{44}$$

$$E_z(kr_j \ll 1, \text{non rad}) \approx \frac{I_0}{cka}\left[\frac{1}{r_1} + \frac{1}{r_2} - \frac{2}{r_0}\right]\sin(\omega t), \tag{45}$$

which are the same as those found in eqs. (15)-(17) by ignoring the displacement-current in Maxwell's equations.

A Appendix: The Electric and Magnetic Fields are Not Just Retarded Static Fields

It is well known that the electric and magnetic fields described by Maxwell's equations can be deduced from the retarded potentials [2],

$$\Phi(\mathbf{x}, t) = \int \frac{\varrho(\mathbf{x}', t')}{R}\, d\text{Vol}', \qquad \mathbf{A}(\mathbf{x}, t) = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{x}', t')}{R}\, d\text{Vol}'. \tag{46}$$

which have the form of the static potentials and current distributions evaluated at the retarded time $t' = t - R/c$, where $R = |\mathbf{x} - \mathbf{x}'|$, rather than at the present time. However, it does not follow that the electric and magnetic fields have the form of the static fields with the charge and current distributions evaluated at the retarded time. Instead, the fields can be calculated from the charge and current distributions according to,[4]

$$\mathbf{E} = \int \frac{[\varrho]\,\hat{\mathbf{R}}}{R^2}\, d\mathrm{Vol}' + \frac{1}{c}\int \frac{([\mathbf{J}]\cdot\hat{\mathbf{R}})\hat{\mathbf{R}} + ([\mathbf{J}]\times\hat{\mathbf{R}})\times\hat{\mathbf{R}}}{R^2}\, d\mathrm{Vol}' + \frac{1}{c^2}\int \frac{([\dot{\mathbf{J}}]\times\hat{\mathbf{R}})\times\hat{\mathbf{R}}}{R}\, d\mathrm{Vol}', \tag{47}$$

and,

$$\mathbf{B} = \frac{1}{c}\int \frac{[\mathbf{J}]\times\hat{\mathbf{R}}}{R^2}\, d\mathrm{Vol}' + \frac{1}{c^2}\int \frac{[\dot{\mathbf{J}}]\times\hat{\mathbf{R}}}{R}\, d\mathrm{Vol}', \tag{48}$$

where $\hat{\mathbf{R}} = \mathbf{R}/R = (\mathbf{x}-\mathbf{x}')/\left|\mathbf{x}-\mathbf{x}'\right|$, and quantities inside brackets, [...], are evaluated at the retarded time $t' = t - R/c$.

If the charge and current distributions are oscillatory with a single angular frequency ω, we can write,

$$\varrho(\mathbf{x},t) = \varrho_0(\mathbf{x})e^{-i\omega t}, \qquad \text{and} \qquad \mathbf{J}(\mathbf{x},t) = \mathbf{J}_0(\mathbf{x})e^{-i\omega t}. \tag{49}$$

The oscillatory factor $e^{-i\omega t}$ when evaluated at the retarded time $t' = t - R/c$ becomes the waveform $e^{-i\omega(t'-R/c)} = e^{i(kR-\omega t)}$, where $k = \omega/c = 2\pi/\lambda$. In this case, the electric and magnetic fields can be written as,

$$\begin{aligned}\mathbf{E} \;=\; & \int \frac{\varrho_0\hat{\mathbf{R}}}{R^2}e^{i(kR-\omega t)}\, d\mathrm{Vol}' + \frac{1}{c}\int \frac{(\mathbf{J}_0\cdot\hat{\mathbf{R}})\hat{\mathbf{R}} + (\mathbf{J}_0\times\hat{\mathbf{R}})\times\hat{\mathbf{R}}}{R^2}e^{i(kR-\omega t)}\, d\mathrm{Vol}' \\ & -\frac{ik}{c}\int \frac{(\mathbf{J}_0\times\hat{\mathbf{R}})\times\hat{\mathbf{R}}}{R}e^{i(kR-\omega t)}\, d\mathrm{Vol}', \end{aligned} \tag{50}$$

and,

$$\mathbf{B} = \frac{1}{c}\int \frac{\mathbf{J}_0\times\hat{\mathbf{R}}}{R^2}e^{i(kR-\omega t)}\, d\mathrm{Vol}' - \frac{ik}{c}\int \frac{\mathbf{J}_0\times\hat{\mathbf{R}}}{R}e^{i(kR-\omega t)}\, d\mathrm{Vol}', \tag{51}$$

The first term of eqs. (47) and (50) could be called the retarded Coulomb field, and the first term of eqs. (48) and (51) could be called the retarded Biot-Savart field. Both of these terms vary as the inverse square of the distance between the source and observer, and so they are important in the near zone and negligible in the far zone.

It is perhaps surprising that the electric field has a second term that varies inversely with the square of the distance, and which is due to the current distribution rather than the charge distribution.[5] This term is an indirect effect of Maxwell's "displacement current", and in examples such as the present it makes a significant contribution to the difference between the actual near-zone fields and those approximated by neglect of the "displacement current".

The last terms of eqs. (47)-(48) and (50)-(51) vary inversely with the distance between the source and observer. These terms are the radiation fields, which are the most significant additions to the fields when the "displacement current" is included in Maxwell's equations.

[4]Equations (47) and (48) first appeared in [3].

[5]The second term of the electric field vanishes for steady currents. See sec. 3 of [4]. While this term is expressed as a function only of the conduction currents, it would be absent if the "displacement current" were not present in Maxwell's equations.

The form of these terms shows that each current element whose time derivative is nonzero creates electric and magnetic radiation fields that are 90° out of phase with respect to the current, and which are equal in magnitude and at right angles to one another.

References

[1] K.T. McDonald, *Radiation in the Near Zone of a Center-Fed Linear Antenna* (June 21, 2003), http://kirkmcd.princeton.edu/examples/linearantenna.pdf

[2] L. Lorenz, *On the Identity of the Vibrations of Light with Electrical Currents*, Phil. Mag. **34**, 287 (1867), http://kirkmcd.princeton.edu/examples/EM/lorenz_pm_34_287_67.pdf

[3] W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd ed. (Addison-Wesley, 1962), http://kirkmcd.princeton.edu/examples/EM/panofsky-phillips.pdf

[4] K.T. McDonald, *The Relation Between Expressions for Time-Dependent Electromagnetic Fields Given by Jefimenko and by Panofsky and Phillips* (Dec. 6, 1996), http://kirkmcd.princeton.edu/examples/jefimenko.pdf