Electromagnetism and Newton's 3$^{\rm rd}$ Law[1]

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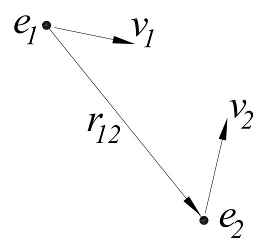
1 Introduction

From Newton's 3$^{\rm rd}$ law of action and reaction, $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ for any pair of masses i and j, we infer that the total momentum of an isolated system must be zero. And, if $\mathbf{F}_{ij}$ is along the line of centers between i and j, we infer that the total angular momentum of a isolated system is also zero.

A simple example of a violation of Newton's 3$^{\rm rd}$ law is sketched in the figure below. Charge e_1 moves with velocity $\mathbf{v}_1$ and charge e_2 moves with velocity $\mathbf{v}_2$ that is not parallel to $\mathbf{v}_1$. The sum of the Lorentz forces is,

$$\mathbf{F}_{12} + \mathbf{F}_{21} = e_1 e_1 \left(\frac{\mathbf{v}_1}{c} \times \mathbf{B}_2(\mathbf{r}_1) + \frac{\mathbf{v}_2}{c} \times \mathbf{B}_1(\mathbf{r}_2) \right), \tag{1}$$

in Gaussian units, with c as the speed of light in vacuum.



For low velocities where the Biot-Savart law is a good approximation, we have,

$$\mathbf{B}_i(\mathbf{r}_j) = \frac{e_i \mathbf{v}_i \times \hat{\mathbf{r}}_{ij}}{c\, r_{ij}^2}, \tag{2}$$

where $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$. Using this in eq. (1), we find,

$$\mathbf{F}_{12} + \mathbf{F}_{21} = e_1 e_1 \left[\frac{\mathbf{v}_1}{c} \times \left(\frac{\mathbf{v}_2 \times \hat{\mathbf{r}}_{21}}{r_{21}^2} \right) + \frac{\mathbf{v}_2}{c} \times \left(\frac{\mathbf{v}_1 \times \hat{\mathbf{r}}_{12}}{c\, r_{12}^2} \right) \right] = \frac{e_1 e_2}{c^2 r_{12}^2} [(\mathbf{v}_2 \cdot \hat{\mathbf{r}}_{12})\, \mathbf{v}_1 - (\mathbf{v}_1 \cdot \hat{\mathbf{r}}_{12})\, \mathbf{v}_2], \tag{3}$$

which is nonzero when $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel.[2]

[1] This note expands on pp. 286-290 of http://kirkmcd.princeton.edu/examples/ph501/ph501lecture24.pdf

[2] Ampère was aware of results like this, but insisted that magnetism obey Newton's 3$^{\rm rd}$ law, and hence rejected the Biot-Savart form, although he showed that the Biot-Savart force between closed, steady currents was the same as that for the magnetic force law that he favored. Ampére's insistence that magnetic forces obey Newton's 3$^{\rm rd}$ law earned him the sobriquet by Maxwell, in Art. 528 of [1], of the "Newton of electricity". See, for example, historical appendix A.10 of [2].

Ampére's authority held up acceptance of the "Lorentz" force law (stated obliquely by Maxwell in 1861 [3]) until efforts by Thomson [5] and Heaviside [7] in 1891 clarified that electromagnetic fields carry momentum as well as energy (following the first clear statement of the "Lorentz" force law by Heaviside in 1885 [10].

We wish to show that, although Newton's 3rd law is violated in electromagnetism, the total momentum of an isolated system is conserved once one considers electromagnetic momentum as well as mechanical momentum.

The concept of electromagnetic momentum was introduced by Maxwell in 1864, when he identified this with Faraday's "electronic state" in sec. 26 of [4], and clarified in sec. 57 that the electromagnetic momentum of charge e in an external vector potential $\mathbf{A}$ is $e\mathbf{A}/c$ (in the Coulomb gauge, as favored by Maxwell). For a collection of charges, we write,

$$\mathbf{P}_{\mathrm{EM}}^{(\mathrm{M})} = \sum_{i \neq j} \frac{e_i \mathbf{A}_j^{(\mathrm{C})}(\mathbf{r}_j)}{c}\,, \tag{4}$$

where superscript C is for Coulomb and M is for Maxwell. This formulation suggests that electromagnetic momentum is a property of the charges, rather than of the electromagnetic field.

As mentioned in footnote 2 above, in 1891 Thomson [5] and Heaviside [7] developed a concept of electromagnetic momentum based on the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$, relating this to the Poynting vector $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{B}$ [11]. We write this formulation as,

$$\mathbf{P}_{\mathrm{EM}}^{(\mathrm{P})} = \int \frac{\mathbf{S}}{c^2}\, d\mathrm{Vol} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\, d\mathrm{Vol}, \tag{5}$$

but note that for a collection of (moving) point charges it includes (unphysical) infinite self momenta. Hence, for point charges e_i we consider only the interaction field momentum,

$$\mathbf{P}_{\mathrm{EM}}^{(\mathrm{P})} = \sum_{i \neq j} \int \frac{\mathbf{E}_i \times \mathbf{B}_j}{4\pi c}\, d\mathrm{Vol}. \tag{6}$$

That Maxwell's electromagnetic momentum (4) is equivalent to the electromagnetic-field momentum (6) of Thomson and Heaviside in quasistatic examples was first demonstrated by Thomson [12]. See also [13].

In the rest of this note, we demonstrate that, for the example of two moving charges,

$$\frac{d\mathbf{P}_{\mathrm{mech}}}{dt} = \mathbf{F}_{12} + \mathbf{F}_{21} = -\frac{d\mathbf{P}_{\mathrm{EM}}}{dt}\,, \tag{7}$$

to order v^2/c^2, which latter is often called the Darwin approximation [14].

2 The Darwin Approximation

The Lagrangian for a charge e of mass m (with no magnetic moment) that moves with velocity $\mathbf{v}$ in an external electromagnetic field that is described by potentials V and $\mathbf{A}$ can be written as (see, for example, sec. 65 of [15] and sec. 12.6 of [16]),),

$$\mathcal{L} = -mc^2\sqrt{1 - v^2/c^2} - eV + e\frac{\mathbf{v}}{c} \cdot \mathbf{A}. \tag{8}$$

Darwin [14] worked in the Coulomb gauge, and kept term only to order v^2/c^2. Then, the scalar and vector potentials due to a charge e that has velocity $\mathbf{v}$ are (see sec. 65 of [15] or sec. 12.6 of [16]),

$$V = \frac{e}{R}, \qquad \mathbf{A} = \frac{e[\mathbf{v} + (\mathbf{v}\cdot\hat{\mathbf{R}})\,\hat{\mathbf{R}}]}{2cR}, \tag{9}$$

where $\hat{\mathbf{R}}$ is directed from the charge to the observer, whose (present) distance from the charge is $\mathbf{R}$.

The electric and magnetic fields of charge e at distance $\mathbf{R}$ from an observer follow in the Darwin approximation from the potentials (9),

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial ct} = \frac{e}{R^2}\hat{\mathbf{R}} - \frac{e}{2c^2R}\left(\mathbf{a} + (\mathbf{a}\cdot\hat{\mathbf{R}})\,\hat{\mathbf{R}} - \frac{v^2 - 3(\mathbf{v}\cdot\hat{\mathbf{R}})^2}{R}\hat{\mathbf{R}}\right), \tag{10}$$

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} = \frac{e\,\mathbf{v}\times\hat{\mathbf{R}}}{cR^2}, \tag{11}$$

where $\mathbf{a} = d\mathbf{v}/dt$ is the (present) acceleration of the charge.[3] As the magnetic field (11) varies as $1/R^2$, there is no radiation in the far zone in the Darwin approximation. However, the Poynting vector $\mathbf{S} = (c/4\pi)\,\mathbf{E}\times\mathbf{B}$ is nonzero, so there exists a flow of electromagnetic-field energy around the moving charge.

The Lorentz force on a charge e_1 with velocity $\mathbf{v}_1$ due to charge $\mathbf{e}_2$ with velocity $\mathbf{v}_2$ is, in the Darwin approximation,

$$\begin{aligned}\mathbf{F}_{12} &= e_1\left(\mathbf{E}_2 + \frac{\mathbf{v}_1}{c}\times\mathbf{B}_2\right)\\ &= \frac{e_1e_2}{r_{12}^3}\left[\left(1 + \frac{v_2^2}{2c^2} - \frac{3(\mathbf{r}_{12}\cdot\mathbf{v}_2)^2}{2c^2r_{12}^2}\right)\mathbf{r}_{12} - \frac{r_{12}^2\,\mathbf{a}_2 + (\mathbf{a}_2\cdot\mathbf{r}_{12})\,\mathbf{r}_{12}}{2c^2} + \frac{\mathbf{v}_1\times(\mathbf{v}_2\times\mathbf{r}_{12})}{c^2}\right].\end{aligned} \tag{12}$$

This force depends on the acceleration $\mathbf{a}_2$ of the source charge e_2, but not on the acceleration $\mathbf{a}_1$ of the charge e_1, and has noncentral terms (not along $\hat{\mathbf{r}}_{12}$).[4]

The total force on the two charges is, noting that $\mathbf{r}_{21} = -\mathbf{r}_{12}$,

$$\begin{aligned}\mathbf{F}_{12} + \mathbf{F}_{21} = \frac{e_1e_2}{2c^2r_{12}^3}&\left[\left(v_2^2 - v_1^2 + \frac{3(\mathbf{r}_{12}\cdot\mathbf{v}_1)^2 - 3(\mathbf{r}_{12}\cdot\mathbf{v}_2)^2}{r_{12}^2}\right)\mathbf{r}_{12}\right.\\ &\left. -r_{12}^2\,(\mathbf{a}_1 + \mathbf{a}_2) - [(\mathbf{a}_1 + \mathbf{a}_2)\cdot\mathbf{r}_{12}]\,\mathbf{r}_{12} + 2(\mathbf{v}_1\times\mathbf{v}_2)\times\mathbf{r}_{12})\right],\end{aligned} \tag{15}$$

using the vector identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) - \mathbf{b}\times(\mathbf{a}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.

[3]Sec. 65 of [15] shows that in the Darwin approximation the Liénard-Wiechert potentials (Lorenz gauge) reduce to $V^{(\mathrm{L})} = e/R + (e/2c^2)\partial^2 R/\partial t^2$ and $\mathbf{A}^{(\mathrm{L})} = e\mathbf{v}/cR$, from which eqs. (10)-(11) also follow.

[4]For comparison, the (central) force law of Weber [18] (1846) is,

$$\mathbf{F}_{12}^{\mathrm{Weber}} = \frac{e_1e_2}{r_{12}^2}\hat{\mathbf{r}}_{12}\left[1 + \frac{1}{c^2}\left(v_1^2 + v_2^2 - 2\mathbf{v}_1\cdot\mathbf{v}_2 - \frac{3}{2}[\hat{\mathbf{r}}_{12}\cdot(\mathbf{v}_1 - \mathbf{v}_2)]^2\right)\right] + \frac{e_1e_2}{c^2r_{12}}\hat{\mathbf{r}}_{12}\,[\hat{\mathbf{r}}_{12}\cdot(\mathbf{a}_1 - \mathbf{a}_2)], \tag{13}$$

while that of Clausius [19] (1876) is,

$$\mathbf{F}_{12}^{\mathrm{Clausius}} = \frac{e_1e_2}{r_{12}^2}\hat{\mathbf{r}}_{12}\left(1 - \frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2}\right) - \frac{e_1e_2}{c^2r_{12}^2}[\hat{\mathbf{r}}_{12}\cdot(\mathbf{v}_1 - \mathbf{v}_2)](\mathbf{v}_1 - \mathbf{v}_2) + \frac{e_1e_2}{c^2r_{12}}(\mathbf{a}_1 - \mathbf{a}_2), \tag{14}$$

both of which depend on the acceleration $\mathbf{a}_1$ of the observing charge e_1.

3 Use of $\mathbf{P}_{\rm EM}^{\rm (M)}$

The electromagnetic momentum of the combined system of charges e_1 and e_2 is, according to Maxwell and recalling eq. (9),

$$\mathbf{P}_{\rm EM}^{\rm (M)} = \frac{e_1\,\mathbf{A}_2^{\rm (C)}(\mathbf{r}_1)}{c} + \frac{e_2\,\mathbf{A}_1^{\rm (C)}(\mathbf{r}_2)}{c} = \frac{e_1 e_2}{2c^2}\left[\frac{\mathbf{v}_1+\mathbf{v}_2}{r_{12}} + \frac{[(\mathbf{v}_1+\mathbf{v}_2)\cdot\mathbf{r}_{12}]\,\mathbf{r}_{12}}{r_{12}^3}\right]. \tag{16}$$

To take the time derivative of eq. (16), we note that $\dot{\mathbf{r}}_i = \mathbf{v}_i$, $\dot{\mathbf{r}}_{12} = \mathbf{v}_2 - \mathbf{v}_1$, $\dot{\mathbf{v}}_i = \mathbf{a}_i$ and $\dot{r}_{12} = \hat{\mathbf{r}}_{12}\cdot(\mathbf{v}_2-\mathbf{v}_1)$ via $r_{12}^2 = \mathbf{r}_{12}^2$. Then,

$$\begin{aligned}
\frac{d\mathbf{P}_{\rm EM}^{\rm (M)}}{dt} &= \frac{e_1 e_2}{2c^2}\left[\frac{\mathbf{a}_1+\mathbf{a}_2}{r_{12}}\right. \\
&+\frac{[(\mathbf{a}_1+\mathbf{a}_2)\cdot\mathbf{r}_{12}]\,\mathbf{r}_{12} + [(\mathbf{v}_1+\mathbf{v}_2)\cdot(\mathbf{v}_2-\mathbf{v}_1)]\,\mathbf{r}_{12} + [(\mathbf{v}_1+\mathbf{v}_2)\cdot\mathbf{r}_{12}]\,(\mathbf{v}_2-\mathbf{v}_1)}{r_{12}^3} \\
&\left.-\frac{(\mathbf{v}_1+\mathbf{v}_2)[\hat{\mathbf{r}}_{12}\cdot(\mathbf{v}_2-\mathbf{v}_1)]}{r_{12}^2} - \frac{3[\hat{\mathbf{r}}_{12}\cdot(\mathbf{v}_2-\mathbf{v}_1)][(\mathbf{v}_1+\mathbf{v}_2)\cdot\hat{\mathbf{r}}_{12}]\,\hat{\mathbf{r}}_{12}}{r_{12}^3}\right]. \\
&= \frac{e_1 e_2}{r_{12}^3}\left[\frac{r_{12}^2\mathbf{a}_1 + \mathbf{a}_2 + [(\mathbf{a}_1+\mathbf{a}_2)\cdot\mathbf{r}_{12}]\,\mathbf{r}_{12} + (v_2^2 - v_1^2)\,\mathbf{r}_{12}}{2c^2}\right. \\
&\left.+\frac{\mathbf{r}_{12}\times(\mathbf{v}_1\times\mathbf{v}_2)}{c^2} - \frac{3(\mathbf{r}_{12}\cdot\mathbf{v}_2)^2 - 3(\mathbf{r}_{12}\cdot\mathbf{v}_1)^2}{2c^2 r_{12}^2}\,\mathbf{r}_{12}\right] = -\mathbf{F}_{12} - \mathbf{F}_{21},
\end{aligned} \tag{17}$$

which provides the first confirmation of eq. (7).

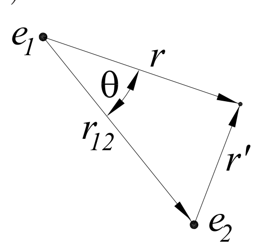
4 Use of $\mathbf{P}_{\rm EM}^{\rm (P)}$

The demonstration of eq. (7) using the Poynting form (6) of electromagnetic momentum is more intricate than that for $\mathbf{P}_{\rm EM}^{\rm (M)}$, although we expect that $\mathbf{P}_{\rm EM}^{\rm (P)} = \mathbf{P}_{\rm EM}^{\rm (M)}$, as the present example is "quasistatic" in the Darwin approximation.[5]

Our computation is to be accurate to order v^2/c^2, and since the magnetic field of eq. (11) is of order v/c we need only consider the leading term in the electric field of eq. (10),

$$\begin{aligned}
\mathbf{P}_{\rm EM}^{\rm (P)} &= \int \frac{\mathbf{E}_1\times\mathbf{B}_2 + \mathbf{E}_2\times\mathbf{B}_1}{4\pi c}\,d{\rm Vol} = \frac{e_1 e_2}{4\pi c^2}\int \frac{\mathbf{r}\times(\mathbf{v}_2\times\mathbf{r}') + \mathbf{r}'\times(\mathbf{v}_1\times\mathbf{r})}{r^3 r'^3}\,d{\rm Vol} \\
&= \frac{e_1 e_2}{4\pi c^2}\int \frac{(\mathbf{r}\cdot\mathbf{r}')\,(\mathbf{v}_1+\mathbf{v}_2) - (\mathbf{r}'\cdot\mathbf{v}_1)\,\mathbf{r} - (\mathbf{r}\cdot\mathbf{v}_2)\,\mathbf{r}'}{r^3 r'^3}\,d{\rm Vol},
\end{aligned} \tag{18}$$

where vector $\mathbf{r}(\mathbf{r}')$ is from charge 1(2) to the observation point.



[5] See, for example, [13].

The first term of eq. (18) is independent of angle ϕ, and involves the factor $\mathbf{r}\cdot\mathbf{r}'$ which is related by,

$$\mathbf{r}_{12} = \mathbf{r} - \mathbf{r}', \qquad r_{12}^2 = r^2 + r'^2 - 2\mathbf{r}\cdot\mathbf{r}', \qquad \mathbf{r}\cdot\mathbf{r}' = \frac{r'^2 + r^2 - r_{12}^2}{2}. \tag{19}$$

We adopt a spherical coordinate system (r, θ, ϕ) with origin at charge 1 and z-axis along $\mathbf{r}_{12}$, for which the volume element is $d\mathrm{Vol} = r^2\, dr\, \sin\theta\, d\theta\, d\phi$. The distance r' from the observation point to charge 2 is related by $r'^2 = r^2 - 2r\, r_{12}\cos\theta + r_{12}^2$, where r_{12} is constant during the integration. A clever trick from [20] is to note that on a sphere of constant radius r we have $r'\, dr' = r\, r_{12}\sin\theta\, d\theta$, so we can write the volume element as $d\mathrm{Vol} = r\, r'\, dr\, dr'\, d\phi / r_{12}$.

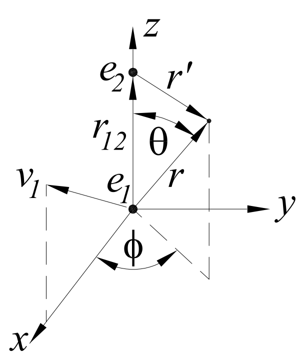
The limits of the r' integration depend on whether r is greater or less than r_{12},

$$\int d\mathrm{Vol} = \frac{1}{r_{12}}\int_0^{r_{12}} r\, dr \int_{r_{12}-r}^{r_{12}+r} r'\, dr' \int_0^{2\pi} d\phi + \frac{1}{r_{12}}\int_{r_{12}}^{\infty} r\, dr \int_{r-r_{12}}^{r+r_{12}} r'\, dr' \int_0^{2\pi} d\phi. \tag{20}$$

Then, the first integral in eq. (18) is,

$$\begin{aligned}
\frac{e_1 e_2(\mathbf{v}_1+\mathbf{v}_2)}{4\pi c^2}\, 2\pi(\mathbf{v}_1+\mathbf{v}_2)\int\!\!\int dr\, dr' \frac{r r'}{r_{12}} \frac{r'^2 + r^2 - r_{12}^2}{2r^3 r'^3} &= \frac{e_1 e_2(\mathbf{v}_1+\mathbf{v}_2)}{4c^2 r_{12}}\int \frac{dr}{r^2}\left[r' - \frac{r^2 - r_{12}^2}{r'}\right]_{r'_{\mathrm{min}}}^{r'_{\mathrm{max}}} \\
&= \frac{e_1 e_2}{4c^2 r_{12}}\left(\int_0^{r_{12}} \frac{dr}{r^2}\left[r' - \frac{r^2 - r_{12}^2}{r'}\right]_{r_{12}-r}^{r_{12}+r} + \int_{r_{12}}^{\infty}\frac{dr}{r^2}\left[r' - \frac{r^2 - r_{12}^2}{r'}\right]_{r-r_{12}}^{r+r_{12}}\right) \\
&= \frac{e_1 e_2(\mathbf{v}_1+\mathbf{v}_2)}{4c^2 r_{12}}\left(\int_0^{r_{12}}\frac{dr}{r^2}(2r - 2r) + \int_{r_{12}}^{\infty}\frac{dr}{r^2}(2r_{12} + 2r_{12})\right) = \frac{e_1 e_2(\mathbf{v}_1+\mathbf{v}_2)}{c^2 r_{12}}.
\end{aligned} \tag{21}$$

To evaluate the second integral in eq. (18) we take velocity $\mathbf{v}_1$ to be in the x-z plane.



Then,

$$\mathbf{r} = r(\sin\theta\cos\phi, \sin\theta\cos\phi, \cos\theta), \qquad \mathbf{r}' = (r\sin\theta\cos\phi, r\sin\theta\cos\phi, r\cos\theta - r_{12}), \tag{22}$$

$$(\mathbf{r}'\cdot\mathbf{v}_1)\,\mathbf{r} = r(r v_{1x}\sin\theta\cos\phi + r v_{1z}\cos\theta - r_{12}v_{1z})(\sin\theta\cos\phi, \sin\theta\cos\phi, \cos\theta). \tag{23}$$

$$r'^2 = r^2 + r_{12}^2 - 2r r_{12}\cos\theta, \qquad \cos\theta = \frac{r^2 + r_{12}^2 - r'^2}{2r r_{12}}. \tag{24}$$

The integral of $(\mathbf{r}'\cdot\mathbf{v}_1)\,\mathbf{r}$ over ϕ has (x, y, z) components
$\pi r(r v_{1x}\sin^2\theta, 0, 2r v_{1z}\cos^2\theta - 2r_{12}v_{1z}\cos\theta)$, and its volume integral is,

$$-\frac{e_1 e_2}{4\pi c^2}\frac{\pi}{r_{12}}\int_0^{\infty} r\, dr \int_{r'_{\mathrm{min}}}^{r'_{\mathrm{max}}} r'\, dr'\, \frac{r(r v_{1x}(1-\cos^2\theta), 0, 2r v_{1z}\cos^2\theta - 2r_{12}v_{1z}\cos\theta)}{r^3 r'^3}$$

$$= -\frac{e_1 e_2}{4c^2 r_{12}} \int_0^{r_{12}} dr \int_{r_{12}+r}^{r_{12}-r} dr' \left[\frac{v_{1x}}{r'^2} - \frac{v_{1x}(r^2 + r_{12}^2 - r'^2)^2}{4r^2 r'^2 r_{12}^2}, 0, \right.$$
$$\left. \frac{v_{1z}(r^2 + r_{12}^2 - r'^2)^2}{2r^2 r'^2 r_{12}^2} - \frac{v_{1z}(r^2 + r_{12}^2 - r'^2)}{r^2 r'^2} \right]$$
$$-\frac{e_1 e_2}{4c^2 r_{12}} \int_{r_{12}}^{\infty} dr \int_{r+r_{12}}^{r-r_{12}} dr' \left[\frac{v_{1x}}{r'^2} - \frac{v_{1x}(r^2 + r_{12}^2 - r'^2)^2}{4r^2 r'^2 r_{12}^2}, 0, \right.$$
$$\left. \frac{v_{1z}(r^2 + r_{12}^2 - r'^2)^2}{2r^2 r'^2 r_{12}^2} - \frac{v_{1z}(r^2 + r_{12}^2 - r'^2)}{r^2 r'^2} \right]. \tag{25}$$

The third integral in eq. (18) is the same as the second but with index $1 \to 2$.

The remaining integrals are "elementary", but very tedious to evaluate. So, we accept without detailed confirmation that the result is $\mathbf{P}_{\rm EM}^{\rm (P)} = \mathbf{P}_{\rm EM}^{\rm (M)}$.[6,7]

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[5] J.J. Thomson, *On the Illustration of the Properties of the Electric Field by Means of Tubes of Electrostatic Induction*, Phil. Mag. **31**, 149 (1891), eqs. (2) and (6), http://kirkmcd.princeton.edu/examples/EM/thomson_pm_31_149_91.pdf See also [6].

[6] K.T. McDonald, *J.J. Thomson and "Hidden" Momentum* (Apr. 30, 2014), http://kirkmcd.princeton.edu/examples/thomson.pdf

[7] O. Heaviside, *On the Forces, Stresses and Fluxes of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **183**, 423 (1892), sec. 26, http://kirkmcd.princeton.edu/examples/EM/heaviside_ptrsla_183_423_92.pdf This was a clarification of his discussion in 1886, eq. (7a) of [8].

[6]Page and Adams seemed not to have been aware in [20] that this result is to be expected, or that the computation of $\mathbf{P}_{\rm EM}^{\rm (M)}$ is much simpler than that of $\mathbf{P}_{\rm EM}^{\rm (P)}$. Hence, that they arrived at the form of our eq. (16) in their eq. (8) suggests that the actually did complete the integration in our eq. (25).

[7]That $\mathbf{P}_{\rm mech} = -\mathbf{P}_{\rm EM}^{\rm (P)}$ was verified to order $1/c^2$ for a special case of two moving charges in [22].

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