Hexagonal Pencil Rolling on an Inclined Plane

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Abstract

An everyday example of a nonholonomic mechanical system is a pencil rolling on an inclined plane. Aspects of the motion are discussed in various approximations, of which the most realistic assumes rolling without sliding and that a constant fraction of the pencil's kinetic energy is retained after each collision with the plane.

1 Introduction

The problem of the hexagonal pencil was drawn to the author's attention by Amin Rezaee Zadeh, who has performed experiments that demonstrate, for example, rolling motion of a hollow plastic "pencil" whose faces have a width of 4 mm on planes with angles of inclination as small as 2°, leading to a terminal velocity of about 12 cm/s at the latter inclination. He also observed that a "circular" pencil has a similar terminal velocity for rolling on a plane of small inclination.

The existence of a terminal velocity indicates that this problem involves energy dissipation, whether or not this is to be characterized as a kind of rolling friction. Also, the existence of steady rolling for very small angles of inclination proves to be noteworthy.

If the angle θ of the inclined plane to the horizontal is too large, the pencil will slide rather than roll. Indeed, if μ is the coefficient of static friction of the pencil on the plane, then pure rolling can occur only for,

$$\tan\theta < \mu. \tag{1}$$

In practice, a mixture of rolling and sliding can be observed for inclinations larger than the bound of eq. (1).

In the rest of this article we assume that pure rolling takes place, and that the pencil never loses contact with the plane.

During one full revolution of the pencil each of its N edges serves in turn as the instantaneous axis of rotation. Thus, the pencil rotates by $1/N$ of a turn while any given edge serves as the axis of rotation. For a sharp-edged pencil, the radius of curvature of an edge is very small (compared to the width a of a face of the pencil), and rolling friction should be negligible. However, at the end of each $1/N$ turn a face of the pencil collides with the inclined plane, and energy will be lost during that collision. The pencil has an asymptotic motion in which the potential energy gained as the pencil falls during each $1/N$ turn is lost in the collision at the end of that turn.

We will see that the existence of a terminal velocity for "circular" pencils on planes of small but finite inclination is not readily explained by supposing that a "circular" pencil is the large-N limit of a structure with N faces and N sharp edges. Rather, the energy dissipation in case of a rolling circular pencil is of a different character than that of a rolling hexagonal pencil.

The equation of motion of a sharp-edged pencil is simple, but it does not admit simple analytic solutions, nor even simple approximations based on initial conditions. Examples of the difficulties in integrating the equation of motion are given in the Appendix.

Section 2 gives a solution based on the assumption that energy dissipation in the collision of a face of the pencil with the inclined plane can be described by a coefficient of (in)elasticity, ϵ, which is the fraction of the pencil's kinetic energy that is retained after a collision with the plane. This approach introduces an unknown parameter, but appears to the author to be the most appropriate simple description of a situation in which the elastic properties of materials cannot be ignored. Section 3 discusses the rolling motion of circular pencils, in which it appears more appropriate to consider a frictional torque that acts throughout the rotation. Section 4 presents an elegant solution with no free parameters based on a problem from the 1995 Boston Area Undergraduate Physics Competition [1] and the 1998 International Physics Olympiad [2], which supposes (somewhat unrealistically) that the impulsive forces during a collision of the pencil with the plane act entirely along one edge of the pencil, so that angular momentum is conserved.

This problem is an extensive elaboration of considerations of impulsive motion given, for example, by Routh [3].

2 Analysis for $N = 6$ via a Coefficient of (In)elasticity

If the plane has angle of inclination $\theta < \pi/6$, the center of mass of a hexagonal pencil rises during the first part of any 1/6 turn. Hence, the pencil will not roll down the plane from rest unless $\theta > \pi/6$.

2.1 $\theta > \pi/6$

In this case a (sharp-edged) pencil rolls down the plane spontaneously from rest.[1]

The equation of motion of the rotating pencil is,

$$\tau = I\ddot{\alpha} = mga \sin\alpha, \tag{2}$$

where, as shown in Fig. 1, α is the angle to the vertical of line from the instantaneous axis of rotation of the pencil to its center of mass, a is the width of each of the six faces of the pencil, m is the mass of the pencil, I is the moment of inertia of the pencil about an edge, and g is the acceleration due to gravity. During each 1/6 turn,

$$\theta - \pi/6 \le \alpha \le \theta + \pi/6. \tag{3}$$

The equation of motion (2) is a so-called Mathieu equation, which does not lend itself to analytic solution. Instead, we use an energy analysis to estimate the asymptotic linear and angular velocity of the rolling pencil.

[1] Typical hexagonal pencils have rounded edges such that spontaneous rolling commences for $\theta \approx \pi/9$.

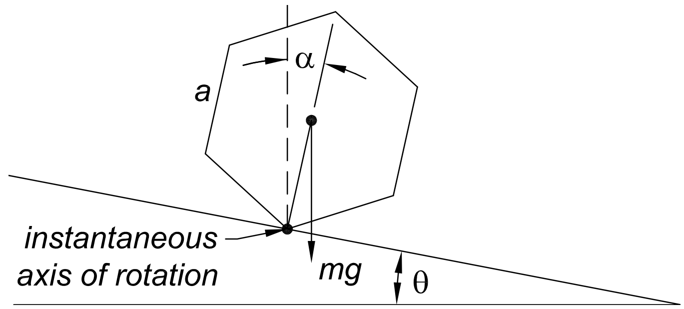


Figure 1: A hexagonal pencil whose edges have width a rolling about a temporarily fixed edge on a plane of inclination θ to the horizontal. The plane containing the axis of the pencil and the instantaneous axis of rotation makes angle α to the vertical.

During each 1/6 turn, the center of mass of the pencil falls by height $a \sin\theta$, as shown in Fig. 2. Thus, gravity adds energy,

$$\Delta E = mga \sin\theta \tag{4}$$

to the pencil each 1/6 turn.

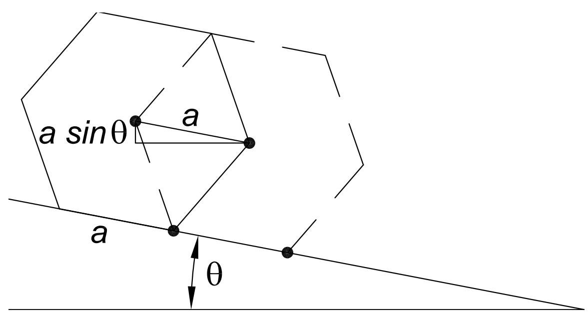


Figure 2: The center of mass of a hexagonal pencil falls through height $a \sin\theta$ during each 1/6 turn, where a is the width of a face of the pencil. For an N-sided pencil whose faces have width b, the center of mass falls by $b \sin\theta$ during each $1/N$ turn.

At the end of each 1/6 turn, the pencil collides with the surface of the inclined plane. Let ϵ $(0 \le \epsilon \le 1)$ be the coefficient of (in)elasticity for this collision, meaning that if the pencil has kinetic energy E just before a collision, it has kinetic energy ϵE just after the collision. The parameter ϵ is essentially the square of the coefficient of restitution defined for one-dimensional collisions.

Then, at the beginning of the second 1/6 turn, the pencil has kinetic energy $E_2 = \epsilon\Delta E$, assuming that the pencil starts from rest. By induction, the energy at the beginning of the third 1/6 turn is $E_3 = \epsilon(\epsilon+1)\Delta E = (\epsilon^2+\epsilon)\Delta E$, and the energy at the beginning of the nth 1/6 turn is,

$$E_n = \epsilon\Delta E(1+\epsilon+\epsilon^2+...+\epsilon^{n-2}) = \epsilon\Delta E\frac{1-\epsilon^{n-1}}{1-\epsilon}\,. \tag{5}$$

For any value of ϵ less than one, the kinetic energy of the pencil at the beginning of a 1/6 turn approaches the asymptotic value,

$$E_\infty = \frac{\epsilon}{1-\epsilon}\Delta E. \tag{6}$$

Hence, the kinetic energy at the end of each 1/6 turn approaches the value,

$$E'_\infty = E_\infty + \Delta E = \frac{1}{1-\epsilon}\Delta E. \tag{7}$$

The asymptotic time-average linear velocity $\langle v_\infty \rangle$ of the pencil is related to the asymptotic time-average angular velocity $\langle \omega_\infty \rangle$ by noting that the pencil advances distance $6a$ during one full turn whose period is $2\pi / \langle \omega_\infty \rangle$,

$$\langle v_\infty \rangle = \frac{3}{\pi} a \langle \omega_\infty \rangle . \tag{8}$$

The asymptotic time-average angular velocity is related to the asymptotic time-average kinetic energy of the pencil by,

$$\frac{1}{2} I \langle \omega_\infty \rangle^2 = \langle E_\infty \rangle \tag{9}$$

where $I = kma^2$ is the moment of inertia of a hexagonal pencil about an edge,

$$k = \frac{17}{12} \qquad \text{(solid hexagon)}, \qquad k = \frac{11}{6} \qquad \text{(hexagonal shell)}. \tag{10}$$

One way to estimate the time-average asymptotic kinetic energy is to replace the average over time by an average over angle α,

$$\langle E_\infty \rangle \approx E_\infty + \frac{3}{\pi} \int_{\theta-\pi/6}^{\theta+\pi/6} mga[\cos(\theta-\pi/6) - \cos\alpha]\, d\alpha = \frac{mga}{2}\left[\frac{1+\epsilon}{1-\epsilon}\sin\theta - \left(\frac{6}{\pi} - \sqrt{3}\right)\cos\theta\right], \tag{11}$$

recalling Fig. 1. This analysis suggests that there is a minimum angle $\theta_{\rm min}$ for steady rolling given by,

$$\tan\theta_{\rm min} = \frac{1-\epsilon}{1+\epsilon}\left(\frac{6}{\pi} - \sqrt{3}\right) = 0.178\frac{1-\epsilon}{1+\epsilon}\, . \tag{12}$$

The empirical evidence that steady rolling can exist for very small θ suggests that the coefficient of (in)elasticity ϵ is close to unity. However, we will find a slightly more restrictive limit on $\theta_{\rm min}$ in sec. 2.2.

Combining eqs. (8), (9) and (11) we estimate the asymptotic linear velocity to be,

$$\langle v_\infty \rangle = \frac{3}{\pi} a \langle \omega_\infty \rangle \approx \frac{3}{\pi}\sqrt{\frac{ag}{k}}\sqrt{\frac{1+\epsilon}{1-\epsilon}\sin\theta - 0.178\cos\theta}, \tag{13}$$

which vanishes for $\theta = \theta_{\rm min}$.

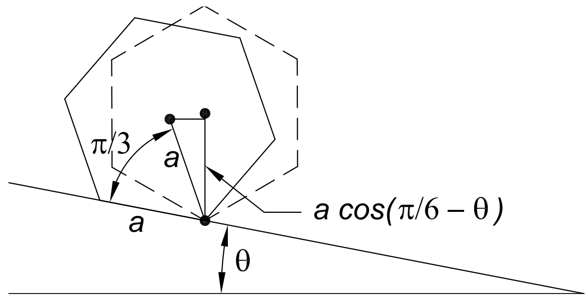


Figure 3: When a hexagonal pencil is at rest on a plane whose inclination is $\theta < \pi/6$, the center of mass of the pencil is at height $a\cos[\pi/2 - (\theta + \pi/3)] = a\cos(\pi/6 - \theta)$ above its lowest point.

2.2 $\theta < \pi/6$

In this case the pencil will not roll unless it is given an initial kinetic energy,

$$E_1 > mga[1 - \cos(\pi/6 - \theta)], \tag{14}$$

such that the center of mass of the pencil can rise to the vertical during the first 1/6 turn, as shown in Fig. 3.

The pencil will not continue to roll through a second 1/6 turn unless sufficient energy remains after its collision with the plane. That is, we need the energy at the beginning of the second 1/6 turn to satisfy,

$$E_2 = \epsilon(E_1 + \Delta E) > mga[1 - \cos(\pi/6 - \theta)]. \tag{15}$$

Similarly, the energy at the beginning of the nth 1/6 turn must satisfy,

$$E_n = \epsilon^{n-1}E_1 + \epsilon\Delta E\frac{1 - \epsilon^{n-1}}{1 - \epsilon} > mga[1 - \cos(\pi/6 - \theta)], \tag{16}$$

recalling the argument that led to eq. (5). The asymptotic condition is that,

$$\frac{\epsilon}{1 - \epsilon}\Delta E = \frac{\epsilon}{1 - \epsilon}mga\sin\theta > mga[1 - \cos(\pi/6 - \theta)], \tag{17}$$

or,

$$\epsilon > \frac{1}{1 + \frac{\sin\theta}{1 - \cos(\pi/6 - \theta)}}. \tag{18}$$

For small θ we must have,

$$\epsilon > \frac{1}{1 + \frac{\theta}{1 - \sqrt{3}/2}} \approx \frac{1}{1 + 7.46\,\theta}. \tag{19}$$

The value of ϵ can be determined from eq. (18), taken as an equality for the smallest angle of inclination $\theta_{\rm min}$ at which the pencil continues to roll after being given an initial velocity. This leads to the relation,

$$\tan\theta_{\rm min} = \frac{1 - \epsilon}{1 + \epsilon}\left(\frac{2}{\cos\theta_{\rm min}} - \sqrt{3}\right) \approx 0.268\frac{1 - \epsilon}{1 + \epsilon}, \tag{20}$$

where the approximation holds for small $\theta_{\rm min}$. This relation implies larger values of $\theta_{\rm min}$ than given by our previous estimate (12).

If conditions (14) and (19) are both satisfied, then the energetics of the asymptotic rolling motion of the pencil are again described by eqs. (4)-(9), and we can again estimate the asymptotic linear velocity of rolling by eq. (13). In particular, the smallest asymptotic velocity occurs for a plane of inclination $\theta_{\rm min}$, and we estimate,

$$\langle v_\infty \rangle_{\rm min} \approx \frac{1.1}{\pi}\sqrt{\frac{ag}{k}}\sqrt{\cos\theta_{\rm min}} \approx \frac{1.1}{\pi}\sqrt{\frac{ag}{k}}, \qquad (21)$$

which is essentially independent of the parameter ϵ.

For a hollow pencil with faces of length $a = 4$ mm, eq. (21) predicts an asymptotic rolling velocity of 5 cm/s.

2.3 Will the Pencil Lose Contact with the Plane?

If the normal force of the inclined plane on the pencil goes to zero, the pencil will lose contact with the plane.[2] The limiting case is that the component $mg\cos\theta$ of the force of gravity perpendicular to the plane is just sufficient to provide the acceleration of the center of mass of the pencil in this direction. Since the perpendicular distance of the center of mass from the plane is $a\cos(\alpha - \theta)$, as shown in Fig. 4, the acceleration of the center of mass towards the plane is,

$$a\ddot{\alpha}\cos(\alpha - \theta) = \frac{mga^2}{I}\sin\alpha\cos(\alpha - \theta) = \frac{g}{k}\sin\alpha\cos(\alpha - \theta), \qquad (22)$$

recalling eq. (2).

The limiting condition is that, at the maximum angle $\theta_{\rm max}$ of inclination for rolling without slipping, m times the acceleration (22) equals the normal component of mg, or,

$$k\cos\theta_{\rm max} = \sin\alpha\cos(\alpha - \theta_{\rm max}). \qquad (23)$$

Since angle α is never far from θ, the condition (23) is roughly that,

$$\tan\theta_{\rm max} \approx k, \qquad (24)$$

which is weaker than the condition (1), $\tan\theta < \mu$, for realistic values of the coefficient of static friction μ.

3 The Large-N Limit ("Circular" Pencil)

Suppose the pencil has N sides, where N is large. In this case, the effective radius of curvature of an edge becomes large compared to the width of a face, and rolling friction becomes more important that collisional energy loss.

Nonetheless, we first suppose that rolling friction can be neglected.

[2] In [1] it is argued that the pencil will lose contact with the plane if the force of gravity on the pencil is insufficient to provide the centripetal acceleration of the center of mass of the pencil about the instantaneous axis of rotation. However, the frictional force parallel to the plane that acts on this axis to keep the pencil from slipping also contributes to the centripetal force, so the argument of [1] does not hold.

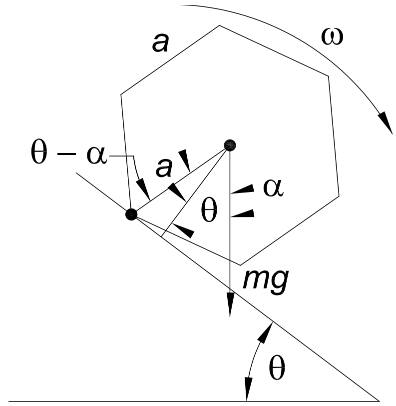


Figure 4: When the radius of length a of the pencil makes angle α to the vertical, the perpendicular distance from the center of mass to the inclined plane is $a\cos(\alpha - \theta)$.

3.1 Analysis Neglecting Rolling Friction

Then, so long as the angle of inclination obeys $\theta > \pi/N$, the pencil will roll spontaneously from rest. In the large-N limit, the pencil always rolls spontaneously from rest, so we use an analysis similar to that of sec. 2 to deduce the asymptotic rolling motion.

If the radius of the pencil is a, then the width of each of the N faces of the pencil is approximately $2\pi a/N$.

During each $1/N$ turn, the center of mass of the pencil falls by height $(2\pi/N)a\sin\theta$. Hence, the kinetic energy of the pencil increases by,

$$\Delta E = \frac{2\pi}{N} mga\sin\theta. \tag{25}$$

As in sec. 2.1, we find that the asymptotic energies at the beginning and end of each $1/N$ turn are,

$$E_\infty = \frac{\epsilon}{1-\epsilon}\Delta E, \qquad \text{and} \qquad E'_\infty = \frac{1}{1-\epsilon}\Delta E. \tag{26}$$

Again, we estimate the asymptotic time-average kinetic energy as the average with respect to angle α according to,

$$\frac{1}{2} I \langle\omega_\infty\rangle^2 = \langle E_\infty\rangle \approx E_\infty + \frac{N}{2\pi}\int_{\theta-\pi/N}^{\theta+\pi/N} mga[\cos(\theta - \pi/N) - \cos\alpha]\, d\alpha \approx \frac{2\pi}{N}\frac{mga}{2}\frac{1+\epsilon}{1-\epsilon}\sin\theta, \tag{27}$$

where $I = kma^2$ with $k \approx 3/2$ for a solid pencil and $k \approx 2$ for a pencil in the form of a hollow shell. The asymptotic linear velocity $\langle v_\infty\rangle$ of the pencil is,

$$\langle v_\infty\rangle = a\langle\omega_\infty\rangle = \sqrt{\frac{2a^2\langle E_\infty\rangle}{I}} \approx \sqrt{\frac{2\pi ag}{Nk}}\sqrt{\frac{1+\epsilon}{1-\epsilon}\sin\theta}. \tag{28}$$

If the coefficient of restitution ϵ has a value less than 1 independent of N (a doubtful assumption), our model predicts that the pencil rolls very slowly in the large-N limit. On the other hand, if $\epsilon = 1$, there is no finite asymptotic velocity to the rolling motion.

In practice, the rolling motion of a "circular" pencil on an inclined plane appears to have a finite asymptotic velocity. This suggests that our neglect of rolling friction is inappropriate for this case.

3.2 Analysis Including Rolling Friction and Neglecting Collisional Losses

While the pencil is rotating about one of its N edges, there is in general a frictional torque $\tau_{\rm friction} = -K\omega^p \approx -Kv^p/a^p \equiv -Cv^p$ that we have neglected thus far, where p is a constant in the range 1-2. Adding this to the equation of motion (1), and noting that in the large-N limit the angle α is essentially the same as angle θ at all times, we have,

$$I\ddot{\alpha} = mga\sin\alpha - K\omega^p \approx mga\sin\theta - Cv^p. \tag{29}$$

If we neglect collisional energy losses at the end of each $1/N$ turn, then the terminal velocity of the pencil according to eq. (29) is,

$$\langle v_\infty \rangle = \left(\frac{mga\sin\theta}{C} \right)^{1/p}. \tag{30}$$

This result is much more satisfactory than eq. (28), and indicates that rolling friction is important for a "circular" pencil, while collisional energy losses are important for a hexagonal pencil.

4 Analysis Assuming the Impulse Acts Only Along the Edge Newly in Contact with the Plane

This assumption was used in [1], but in an inconsistent manner. A more consistent analysis was given in [2].

4.1 Asymptotic Velocities at the Beginning and End of a $1/N$ Turn

If the pencil has N faces, then the angle between adjacent major radii is $\beta = 2\pi/N$, as shown in Fig. 5. If a major radius has length a, then the width of a face is $b = 2a\sin(\beta/2) = 2a\sin(\pi/N)$.

At the end of a $1/N$ turn the center of mass of the pencil has velocity vector $\mathbf{v}$ perpendicular to the major radius from the center of mass to the edge that is the (old) instantaneous axis of rotation, as shown in Fig. 6. As the face of the pencil collides with the plane the instantaneous axis shifts to the adjacent major radius, and an impulsive force is exerted over the colliding face.

An idealization is that this impulsive force is concentrated along the edge that becomes the new instantaneous axis. This assumption is not too plausible, but it leads to an analysis that has no free parameters (for given a and N of the pencil and angle of inclination θ).

Since the impulsive force acts by assumption only on the new axis of rotation, angular momentum L is conserved about this axis. If we label the velocity of the center of mass

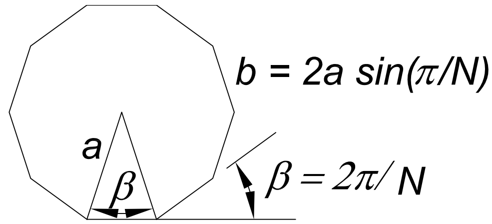


Figure 5: A pencil with N faces and major radius a has faces of width $b = 2a\sin(\pi/N)$.

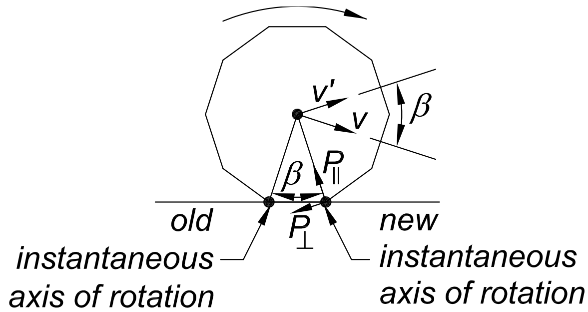


Figure 6: Just before a collision of the pencil with the plane, the center-of-mass velocity $\mathbf{v}$ is perpendicular to the radius to the old instantaneous axis of motion. Vector $\mathbf{v}_\perp$ is the component of $\mathbf{v}$ perpendicular to the new axis of rotation. The momentum impulse $\mathbf{P}$, assumed to occur at the new axis of rotation, has component $P_\parallel$ along the radius that intercepts this axis.

just after the collision as $v' = a\omega'$, then the angular momentum just after the collision is $L' = I\omega'$, where $I = kma^2$ is the moment of inertia of the pencil about an edge, and,

$$k_{\rm solid} = \frac{7}{6} + \frac{1}{3}\cos^2\frac{\pi}{N}\,, \qquad k_{\rm shell} = \frac{2}{3}\left(2 + \cos^2\frac{\pi}{N}\right). \tag{31}$$

For a hexagonal pencil, $k_{\rm solid} = 17/12$ and $k_{\rm shell} = 11/6$, while in the large-N limit, $k_{\rm solid} = 3/2$ and $k_{\rm shell} = 2$ as expected.

Just before the collision the angular momentum about the new axis is calculated as the sum of the angular momentum about the axis of the pencil plus the angular momentum of the motion of the center of mass about the new axis. If $v = a\omega$ is the velocity of the center of mass just before the collision, the component of this velocity transverse to the new axis is $v_\perp = v\cos\beta = a\omega\cos\beta$. The moment of inertia of the pencil about its axis is $I_{CM} = I - ma^2$ using the parallel-axis theorem. Hence, the angular momentum of the pencil about the new axis just before the collision is,

$$L = I_{CM}\omega + mav_\perp = (k - 1 + \cos\beta)ma^2\omega = L' = kma^2\omega', \tag{32}$$

invoking conservation of angular momentum about the new axis. Thus,

$$\frac{\omega'}{\omega} = \frac{v'}{v} = \frac{k - 1 + \cos\beta}{k} = 1 - \frac{1 - \cos\frac{2\pi}{N}}{k}\,. \tag{33}$$

For example, $\omega'/\omega = 11/17$ for a solid hexagonal pencil ($\cos\beta = 1/2$), $\omega'/\omega = 8/11$ for a hollow hexagonal pencil, and $\omega'/\omega = 1$ in the limit of a circular pencil whether solid or hollow.

We can confirm the result (33) by an analysis that includes the impulse on the edge newly in contact with the inclined plane. As shown in Fig. 6, we let $P_\parallel$ be the component of the impulse along the line from the edge to the axis of the pencil, and $P_\perp$ be the component transverse to this line. The changes in linear momentum of the pencil caused by these impulses are,

$$P_\parallel = mv\sin\beta = ma\omega\sin\beta, \tag{34}$$

$$P_\perp = mv\cos\beta - mv' = ma(\omega\cos\beta - \omega'), \tag{35}$$

while the change in angular momentum about the axis of the pencil is given by

$$(k-1)ma^2(\omega - \omega') = I_{CM}(\omega - \omega') = -aP_\perp = ma^2(\omega' - \omega\cos\beta), \tag{36}$$

which leads again to eq. (33).

The solution given in [1] seems to be based on the assumption that $P_\perp = 0$, in which case eq. (35) implies that $\omega' = \omega\cos\beta$, as claimed there. However, this assumption is inconsistent with conservation of angular momentum about the edge newly in contact with the plane (eqs. (32)-(33)), and also inconsistent with the torque analysis (36) about the axis of the pencil.

The kinetic energy lost in the collision is given by,

$$\Delta E = \frac{1}{2}I(\omega^2 - \omega'^2) = \frac{kmv^2}{2}\left(1 - \frac{\omega'^2}{\omega^2}\right), \tag{37}$$

where v is the velocity of the center of mass of the pencil just before the collision. In the large-N limit where $\omega' = \omega$, no energy is lost during the collision.

During each $1/N$ turn the center of mass of the pencil falls by height $h = b\sin\theta$, recalling Fig. 2, and gravitational potential energy mgh is converted into kinetic energy. The asymptotic condition is that the potential energy gained during each $1/N$ turn equals the kinetic energy lost during the collision at the end of that turn.

The asymptotic velocity v_e of the center of mass at the end of a $1/N$ turn is given by,

$$mgb\sin\theta = \Delta E = \frac{kmv_e^2}{2}\left(1 - \frac{\omega'^2}{\omega^2}\right). \tag{38}$$

or,

$$v_e^2 = \frac{4ag}{k}\frac{\sin\frac{\pi}{N}}{1 - \omega'^2/\omega^2}\sin\theta. \tag{39}$$

Since $\Delta E \to 0$ in the large-N limit, there is no finite asymptotic velocity for any nonzero value of the inclination θ in this limit.

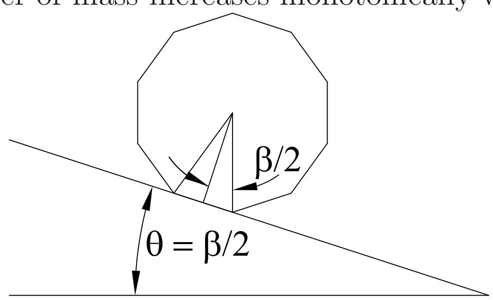
The asymptotic velocity v_b of the center of mass at the beginning of a $1/N$ turn is the same as the asymptotic velocity just after a collision,

$$v_b^2 = v_e^2\frac{\omega'^2}{\omega^2} = \frac{4ag}{k}\frac{\frac{\omega'^2}{\omega^2}\sin\frac{\pi}{N}}{1 - \omega'^2/\omega^2}\sin\theta. \tag{40}$$

4.2 Asymptotic Average Linear Velocity

We now estimate the asymptotic average linear velocity $\langle v_\infty \rangle$ of the pencil parallel to the inclined plane.

If the angle of inclination obeys $\theta < \beta/2 = \pi/N$, then the center of mass of the pencil first rises and then falls during a $1/N$ turn, and hence the center-of-mass velocity first falls below v_b and then rises to v_e. In contrast, if $\theta > \beta/2$, then the center of mass always falls and the velocity of the center of mass increases monotonically with time from v_b to v_e.



4.2.1 $\theta > \pi/N$

In this case the center of mass velocity increases monotonically during a $1/N$ turn, and we estimate the asymptotic average velocity $\langle v_\infty \rangle$ parallel to the inclined plane as,

$$\left\langle v_\infty^2 \right\rangle \approx \frac{v_b^2 + v_e^2}{2} = \frac{2ag}{k} \frac{1 + \omega'^2/\omega^2}{1 - \omega'^2/\omega^2} \sin\frac{\pi}{N} \sin\theta, \tag{41}$$

recalling eqs. (39) and (40).

In the large-N limit no energy is lost during collisions according to the model of this section, so the asymptotic average velocity diverges, in contrast to the predictions of the models of secs. 2 and 3.

4.2.2 $\theta < \pi/N$

In this case the pencil has minimum angular velocity when angle $\alpha = 0$ and the center of mass is directly above the point of contact.

The dependence of angular velocity ω on angle α can be found from conservation of energy. Noting that the angle α_e at the end of a $1/N$ turn is,

$$\alpha_e = \theta + \frac{\beta}{2} = \theta + \frac{\pi}{N}, \tag{42}$$

we have,

$$\frac{1}{2} I\omega^2(\alpha) = \frac{1}{2} I\omega_e^2 - mga(\cos\alpha - \cos\alpha_e), \tag{43}$$

or,

$$\omega^2(\alpha) = \omega_e^2 - \frac{2g}{ak}(\cos\alpha - \cos\alpha_e). \tag{44}$$

Thus, the minimum angular velocity follows from eq. (44) as,

$$\omega_{\min}^2 = \omega_e^2 - \frac{2g}{ak}(1 - \cos\alpha_e). \tag{45}$$

The minimum angle $\theta_{\rm min}$ of inclination such that asymptotic rolling motion can exist corresponds to $\omega_{\rm min} = 0$ and follows from eqs. (39), (42) and (45) as,

$$\sin\theta_{\rm min} = \frac{1 - \omega'^2/\omega^2}{2\sin\frac{\pi}{N}}\left[1 - \cos\left(\theta_{\rm min} + \frac{\pi}{N}\right)\right]. \tag{46}$$

In the large-N limit $\theta_{\rm min} \approx (\pi/N)^3$, which is negligible. For a hexagonal pencil ($N = 6$), eq. (46) leads to the relation,

$$1 - \left(\frac{4k^2}{4k-1} - \frac{1}{2}\right)\sin\theta_{\rm min} = \frac{\sqrt{3}}{2}\cos\theta_{\rm min}, \tag{47}$$

or,

$$\theta_{\rm min} = 6.6^\circ \qquad \text{(solid hexagon)}, \qquad \theta_{\rm min} = 4.8^\circ \qquad \text{(hollow hexagon)}, \tag{48}$$

The result (48) tells us that the present analysis is not a good approximation to the motion of real pencils, which exhibit asymptotic rolling motion even for angles of inclination as little as 2°.

Nonetheless, we complete the analysis by supposing that $\theta_{\rm min} < \theta < \pi/N$, for which we estimate the average asymptotic angular velocity $\langle\omega_\infty\rangle$ as,

$$\langle\omega_\infty^2\rangle \approx \frac{\omega_{\rm min}^2 + \omega_e^2 + \omega_b^2}{3} = \frac{2g}{3ak}\left[2\frac{2 + \omega'^2/\omega^2}{1 - \omega'^2/\omega^2}\sin\frac{\pi}{N}\sin\theta + \cos\left(\theta + \frac{\pi}{N}\right) - 1\right]. \tag{49}$$

During one full turn the pencil advances distance Nb along the plane, so,

$$\langle v_\infty\rangle = \frac{Nb}{T_\infty} = \frac{Nb\,\langle\omega_\infty\rangle}{2\pi} = \frac{N}{\pi}\sin\frac{\pi}{N}\; a\,\langle\omega_\infty\rangle, \tag{50}$$

and our estimate of the asymptotic average linear velocity is,

$$\langle v_\infty\rangle \approx \frac{N}{\pi}\sin\frac{\pi}{N}\sqrt{\frac{2ag}{3k}\left[2\frac{2 + \omega'^2/\omega^2}{1 - \omega'^2/\omega^2}\sin\frac{\pi}{N}\sin\theta + \cos\left(\theta + \frac{\pi}{N}\right) - 1\right]}. \tag{51}$$

For a hexagonal pencil, we find,

$$\langle v_\infty\rangle \approx \frac{3}{\pi}\sqrt{\frac{2ag}{3k}\left[\frac{12k^2 - 4k + 1}{4k-1}\sin\theta + \cos\left(\theta + \frac{\pi}{6}\right) - 1\right]} \qquad \text{(hexagonal pencil)}. \tag{52}$$

5 Summary

Observations of asymptotic rolling of a hexagonal pencil on inclined planes of angles as small at 2° to the horizontal are not consistent with the model (sec. 4) that angular momentum is conserved in the collisions of the pencil with the plane. A better model (sec. 2) is that a constant fraction ϵ of the pencil's kinetic energy is retained after each collision. This model indicates that $\epsilon \approx 0.8$ for the pencil that could still roll down a 2° plane. However, the model of nearly elastic collision of the pencil with the plane does not predict a finite asymptotic velocity for rolling of a "circular" pencil; rather rolling friction limits the velocity in this case (sec. 3).

6 Appendix: Approximate Solution to the Equation of Motion of a Hexagonal Pencil for Small θ

If we knew the period T_∞ of a 1/6 turn during the asymptotic rolling motion, we could calculate the asymptotic linear velocity according to,

$$\langle v_\infty \rangle = \frac{a}{T_\infty}\,, \tag{53}$$

since the pencil advances distance a along the incline during each 1/6 turn. Here, we estimate the period T_∞ via various approximate solutions to the equation of motion (2). However, these estimates are not better than eq. (13) which was obtained by the energy method.

The largest value of angle α during a 1/6 turn is $\pi/6+\theta$, so the approximation $\sin\alpha \approx \alpha$ is still reasonably valid for small θ ($< \pi/6$). Hence, the equation of motion (2) is approximately,

$$I\ddot{\alpha} = kma^2\ddot{\alpha} = mga\sin\alpha \approx mga\alpha, \qquad (\theta \ll 1) \tag{54}$$

and we obtain the approximate solution,

$$\alpha \approx Ae^{\sqrt{g/ka}\,t} + Be^{-\sqrt{g/ka}\,t} = (A+B)\cosh\sqrt{\frac{g}{ka}}t + (A-B)\sinh\sqrt{\frac{g}{ka}}t. \tag{55}$$

The constants A and B can be determined from the initial conditions. We define time $t = 0$ to be the beginning of a 1/6 turn, so,

$$\alpha(0) = A + B = \theta - \frac{\pi}{6}\,. \tag{56}$$

The angular velocity at the beginning of a 1/6 turn is,

$$\omega(0) = \dot{\alpha}(0) = \sqrt{\frac{g}{ka}}(A - B). \tag{57}$$

This varies from turn to turn, so the integration of the equation of motion must be supplemented by a model of the collision as the end of each 1/6 turn.

In the rest of this Appendix we restrict our attention to the asymptotic rolling motion, for which in the model of sec. 2,

$$\omega_\infty(0) = \sqrt{\frac{2E_\infty}{I}} = \sqrt{\frac{g}{ka}}\sqrt{\frac{2\epsilon\sin\theta}{1-\epsilon}}. \tag{58}$$

recalling eqs. (4) and (6). Thus,

$$A - B = \sqrt{\frac{2\epsilon\sin\theta}{1-\epsilon}}, \tag{59}$$

and,

$$\alpha \approx \left(\theta - \frac{\pi}{6}\right)\cosh\sqrt{\frac{g}{ka}}t + \sqrt{\frac{2\epsilon\sin\theta}{1-\epsilon}}\sinh\sqrt{\frac{g}{ka}}t \tag{60}$$

The asymptotic period T_∞ of a 1/6 turn, which ends with $\alpha = \theta + \pi/6$, is related by eq. (60) as,

$$\sqrt{\frac{2\epsilon\sin\theta}{1-\epsilon}}\sinh\sqrt{\frac{g}{ka}}T_\infty \approx \frac{\pi}{6}\left(1 + \cosh\sqrt{\frac{g}{ka}}T_\infty\right), \tag{61}$$

or,

$$\tanh\sqrt{\frac{g}{ka}}\frac{T_\infty}{2} \approx \frac{\pi}{6}\sqrt{\frac{1-\epsilon}{2\epsilon\sin\theta}}, \tag{62}$$

where we suppose that $\theta \ll \pi/6$.

If we also use the approximation of eq. (20) that $2\epsilon\sin\theta_{\rm min}/(1-\epsilon) \approx 0.27$, then we have,

$$\tanh\sqrt{\frac{g}{ka}}\frac{T_{\infty,\rm max}}{2} \approx 1, \tag{63}$$

which provides no meaningful estimate of $T_{\infty,\rm max}$ for rolling motion on a plane of minimal inclination $\theta_{\rm min}$.

We actually do better to use only the first-order approximation that $\tanh x = x$, for which eq. (62) implies,

$$T_\infty \approx \frac{\pi}{3}\sqrt{\frac{ka}{g}}\sqrt{\frac{1-\epsilon}{2\epsilon\sin\theta}}, \tag{64}$$

and hence,

$$\langle v_\infty\rangle = \frac{a}{T_\infty} = \frac{3}{\pi}\sqrt{\frac{ag}{k}}\sqrt{\frac{2\epsilon\sin\theta}{1-\epsilon}}. \tag{65}$$

In sum, it appears that estimates of the asymptotic rolling velocity of a hexagonal pencil based on integration of its equation of motion are not superior to those given in sec. 2.

References

[1] Problem 6 of the 1995 Boston Area Undergraduate Physics Competition,
http://liquids.seas.harvard.edu/oleg/competition/images/exam95.pdf
http://liquids.seas.harvard.edu/oleg/competition/images/solut95.pdf
http://kirkmcd.princeton.edu/examples/mechanics/baupc95_exam.pdf
http://kirkmcd.princeton.edu/examples/mechanics/baupc95_sol.pdf

[2] Problem 1 of the 29th International Physics Olympiad (July 4, 1998),
http://www.hi.is/pub/ipho/
http://kirkmcd.princeton.edu/examples/mechanics/ipo29_98.pdf

[3] E.J. Routh, *The Elementary Part of a Treatise on the Dynamics of a System of Rigid Bodies*, 7$^{\rm th}$ ed., (Macmillan, 1905), sec. 171,
http://kirkmcd.princeton.edu/examples/mechanics/routh_elementary_rigid_dynamics.pdf