

# LAB #7: COUPLED PENDULA AND NORMAL MODES

## BEFORE YOU COME TO LAB:

1. Read Knight's Chapter 14 (Oscillations), as assigned for lecture. Note that the equations describing Simple Harmonic Motion have applications in physics and engineering extending far beyond the simple systems described.
2. Read the lab writeup. Focus on the relationship between the physical phenomena and the equations as you carefully go through the Introduction and then through Appendix I of the writeup.
3. Consider the optional PreLab problem set attached.

## A. Introduction

In this week's Lecture and Lab, you will be studying the motion of simple harmonic oscillators. Small departures from equilibrium in almost any system result in a restoring force proportional to the departure, and consequently the motion is Simple Harmonic Motion (SHM). Electrical, acoustical, and optical systems oscillate with SHM, completely analogous to the mass-plus-spring and pendulum systems you'll be looking at during this Lab. Many physicists use their "physical intuition" about how systems with springs and masses behave to predict how many different systems will act. "Physical intuition" is not an innate human characteristic; a person develops physical intuition with experience. This Lab gives you the opportunity to develop such intuition about SHM in a "simple" mechanical system that exhibits surprising complex behavior.

The main apparatus in this Lab is two "identical" physical pendula (wooden  $2 \times 4$ s with an axle at one end), connected by a weak spring. In the absence of the spring, the two pendula would oscillate at (nearly) the same angular frequency  $\omega = 2\pi f$ , with angle  $\theta$  to the vertical obeying  $\theta_i = \theta_{i_0} \cos(\omega t + \phi_i)$ , with independent amplitudes  $\theta_{i_0}$  and phases  $\phi_i$ .

Coupling the two pendula by the spring produces two characteristic frequencies, which in turn lead to a complex motion, which is not a simple sinusoid. But the motion can be analyzed as a sum of two sinusoidal motions, each of which obeys the simple equations of SHM, and oscillates at its characteristic frequency. (See Appendix A.)

As you pursue your interest in science and engineering, you will find that a powerful approach to analyzing the time dependence of a complicated system is to look for special “characteristic” frequencies, and to determine the patterns of motion associated with simple sinusoidal oscillations at those frequencies. With these solutions in hand, then any general motion of a “linear” system can be described as a sum of these simple motions. This is the concept of **normal modes**, which refers to the set of patterns which each leads to simple sinusoidal variations in time.

In this Lab, you should observe various possible patterns of motion of the two-pendulum system, and record carefully descriptions of the patterns of motion that you see. Using a stopwatch, you can determine the frequencies associated with the various patterns which occur. Then, using a computer model which sums and graphs the combined effects of two sinusoids, you can simulate the patterns that you observe, and other patterns as well (for example, the first overtone in a musical instrument).

## B. The Apparatus and the Measurements

This apparatus consists of two pendula connected at their centers of mass with a spring, as sketched in Figure 1. *Slide the two pendula (2×4's) far enough apart that the spring is stretched when at rest. When you swing the pendula, the spring should never collapse completely. If it does, restart the pendula with smaller amplitudes, and/or slide the pendula farther apart.*

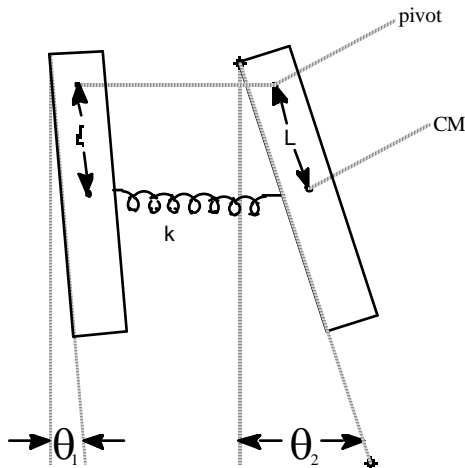


Figure 1: Coupled pendulums; each pendulum is made from a length of 2x4. A spring connects the two pendulums at their centers of mass (CMs).

Displace one pendulum while holding the other fixed, and then let both go free at the same time. The motion is complicated! At first one pendulum oscillates, but after a while its oscillations become small and the other pendulum oscillates. Later, the first pendulum oscillates again and the second does not. And then, this cycle of “transference of oscillation” repeats over and over (until friction damps all oscillations to zero).

Next, displace both pendula and let them go. You should find special combinations of the initial displacements such that there is no “transference of oscillation”, and both pendula oscillate with

the same frequency. How many different ways can you generate these “normal modes” of oscillation?

Using a stopwatch, measure the frequencies of the pendula without the spring, and then the frequencies of the normal modes that you have discovered when the spring is attached.

After you have analyzed the simple patterns of motion, again displace one pendulum while holding the other one fixed. Then release them simultaneously. Describe the resulting motion. Using the stopwatch, measure any relevant frequencies, and relate these to the frequencies of the normal modes they you previously observed.

Make a VideoPoint movie of the complex motion, using a capture rate of 5 frames per second. (Since you will be interested mainly in frequencies, there is no need to include a meter stick in the field of view.) Make sure that the video covers at least two (2) complete cycles of the “transference of oscillation” (and so includes 5 times at which the left pendulum has minimal amplitude of oscillation). Digitize the positions of some identifiable point near the bottom of each of the pendula. Make plots of horizontal position of these points *versus* time.

The challenge now is to extract the frequencies and amplitudes of the normal modes from your plots. You will do this by varying the parameters in a computer model of your experiment until you obtain a good match between your plots of position *vs.* time and the computer simulation.

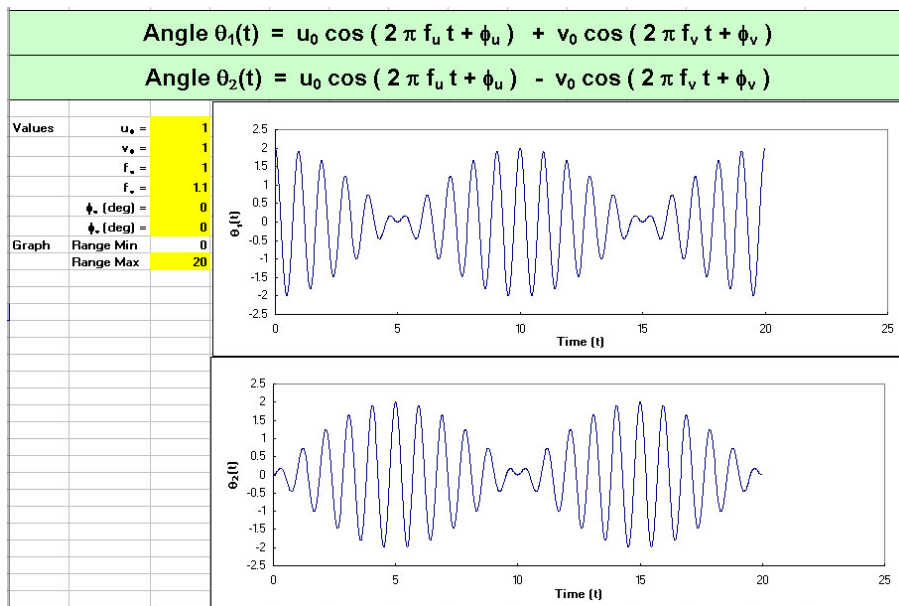
### C. Computer Modeling of Your Results

An Excel program, Normal\_Modes\_Modeling, can be found in your Physics 103 folder (and on the Blackboard). It provides a way of confirming that two sinusoidal normal modes can generate the complex behavior that you observed with the two pendulums.

The program calculates the functions

$$\theta_1(t) = u_0 \cos(\omega_u t + \phi_u) + v_0 \cos(\omega_v t + \phi_v), \quad \theta_2(t) = u_0 \cos(\omega_u t + \phi_u) - v_0 \cos(\omega_v t + \phi_v),$$

found in eqs. (7) and (8) of the Appendix.



The figure on p. 49 shows an example Excel screen. Amplitudes, frequencies and phases can be changed at will. Note also that you may need to change the maximum value of the range of time plotted in order to see the entire behavior.

Your plots of horizontal position  $x$  vs. time of points on your two pendula should have the same form as the computer plots of  $\theta_1(t)$  and  $\theta_2(t)$  since  $x = x_0 + D \sin \theta \approx x_0 + D\theta$ , where  $D$  is the vertical distance from the pivot to the point you digitized. Adjust the input parameters (in yellow cells) in the Excel program until the computer plots match your data plots. Since the vertical scale of the plots is arbitrary, you can leave parameter  $u_0$  (or  $v_0$ ) fixed to speed up the process of “fitting” the parameters. Another trick is to leave  $\phi_u$  fixed at first, while adjusting parameters  $v_0$  and  $\phi_v$  until the computer pattern matches that of your data, except for an offset in time. You can determine the frequencies  $f_u$  and  $f_v$  without determining the phase  $\phi_u$ , but if you wish to obtain the best match between the computer plots and your data, make the final adjustment of  $\phi_u$  while keeping the difference  $\phi_u - \phi_v$  constant

What are the uncertainties on the frequencies  $f_u$  and  $f_v$  that you determine by this “fitting” procedure? That is, how much can you change these parameters in the computer model such that the match between the model and your data is “good enough?”

## D. Optional Computer Modeling

You can download the Excel program from the Blackboard and use it to model other systems that involve the sum (or difference) between two simple harmonic oscillations.

In the following, you may want to make the amplitude of the higher-frequency term smaller than that of the fundamental.

(a) Look at two frequencies which are exactly a factor of two apart. These are most relevant to such instruments as the flute and the organ, in which the overtones tend to be almost exact multiples of the fundamental note being played.

(b) Look at two frequencies which are almost, but not quite, a factor of two apart. (Say, with a frequency ratio of 2.2 to 1.) Such waveforms are perhaps relevant to the sound produced by a saxophone, in which the tapered bore causes the overtones to not be exact multiples of the fundamental note.

(c) What about non-integer ratios of frequencies? Think about what period of time makes such a system return to its initial state. (Use simple integer ratios first, such as 3:2.)

## Appendix: Coupled Oscillations

Consider the case of the two pendula coupled with a spring at their centers of mass, as shown in Fig. 1. We suppose that the distances  $L$  from the pivots to the centers of mass are the same, and that the masses  $m$  and the moments of inertia  $I$  about the pivots are the same for the two pendula. Positive angles  $\theta$  correspond to counterclockwise displacements from the vertical.

There are two torques on each pendulum. The weight of a pendulum gives a torque, which tries to restore the pendulum to vertical,

$$\tau_{1,w} = -m_1 g l \sin \theta \approx -m_1 g l \theta,$$

using the small-angle approximation. The spring provides a restoring torque on the *left* pendulum which acts to bring the separation  $L_s$  between the centers of mass of the pendula to the unstretched length of the spring,  $L_0$ . Assuming that  $L_0$  equals the distance between the centers of mass of the pendula when vertical, the force on the *left* pendulum due to the spring is

$$F_s = k(L_s - L_0) \approx kL(\theta_2 - \theta_1),$$

and the torque on the left pendulum due to the spring is

$$\tau_s \approx LF_s \approx -kL(\theta_1 - \theta_2).$$

The restoring torque on the *right* pendulum is equal in magnitude, but in the opposite direction.

Newton's law in angular variables applied to each pendulum gives the following equations of motion:

$$I\alpha_1 = I \frac{d^2\theta_1}{dt^2} = I\ddot{\theta}_1 = -mgL\theta_1 - kL(\theta_1 - \theta_2), \quad (1)$$

$$I\alpha_2 = I \frac{d^2\theta_2}{dt^2} = I\ddot{\theta}_2 = -mgL\theta_2 + kL(\theta_1 - \theta_2). \quad (2)$$

Neither of these equations describes simple harmonic motion because of the spring coupling terms. However, then by adding and subtracting eqs. (1) and (2) we obtain two new equations each of which describes simple harmonic motion.

$$I(\ddot{\theta}_1 + \ddot{\theta}_2) = -mgL(\theta_1 + \theta_2), \quad (3)$$

and

$$I(\ddot{\theta}_1 - \ddot{\theta}_2) = -(mgL + 2kL^2)(\theta_1 - \theta_2). \quad (4)$$

Equation (3) is an equation for simple harmonic motion in the variable

$$u = \theta_1 + \theta_2, \quad \text{such that}^1 \quad u(t) = 2u_0 \cos(\omega_u t + \phi_u), \quad \text{where} \quad \omega_u^2 = \frac{mgL}{I}. \quad (5)$$

Similarly, equation (4) describes simple harmonic motion in the variable

$$v = \theta_1 - \theta_2, \quad \text{such that} \quad v(t) = 2v_0 \cos(\omega_v t + \phi_v), \quad \text{where} \quad \omega_v^2 = \frac{mgL + 2kL^2}{I}. \quad (6)$$

The solutions  $u(t)$  and  $v(t)$  are called **normal modes** = motions in which both pendula move at the same frequency. The normal mode  $u(t)$  is excited when both pendula are displaced equally in the same direction. The spring then has no effect. The normal mode  $v(t)$  is excited by starting the pendula with equal and opposite displacements. The frequency for this mode is higher

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<sup>1</sup> The factor of 2 in  $u(t)$  and  $v(t)$  is inserted to make eqs. (7) and (8) look simpler.

because this motion excites the spring, which increases the forces and torques on the pendula, which then move faster..

The variables that we observe directly are  $\theta_1(t)$  and  $\theta_2(t)$ ,

$$\theta_1(t) = \frac{u+v}{2} = u_0 \cos(\omega_u t + \phi_u) + v_0 \cos(\omega_v t + \phi_v), \quad (7)$$

$$\theta_2(t) = \frac{u-v}{2} = u_0 \cos(\omega_u t + \phi_u) - v_0 \cos(\omega_v t + \phi_v). \quad (8)$$

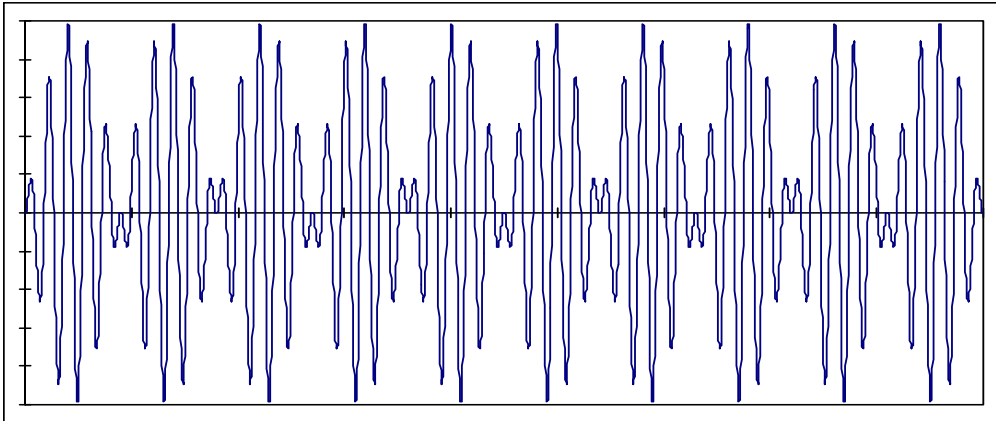
Using some trig identities, eqs. (7) and (8) can be rewritten as

$$\theta_1(t) = 2(u_0 + v_0) \cos(\omega_+ t + \phi_+) \cos(\omega_- t + \phi_-) - 2(u_0 - v_0) \sin(\omega_+ t + \phi_+) \sin(\omega_- t + \phi_-), \quad (9)$$

$$\theta_2(t) = -2(u_0 + v_0) \sin(\omega_+ t + \phi_+) \sin(\omega_- t + \phi_-) + 2(u_0 - v_0) \cos(\omega_+ t + \phi_+) \cos(\omega_- t + \phi_-), \quad (10)$$

where  $\omega_+ = \frac{\omega_u + \omega_v}{2}$ ,  $\phi_+ = \frac{\phi_u + \phi_v}{2}$ ,  $\omega_- = \frac{\omega_v - \omega_u}{2}$ ,  $\phi_- = \frac{\phi_v - \phi_u}{2}$ . (11)

These complicated forms can be thought of (or heard as!) an oscillation at the average frequency  $\omega_+$  that is modulated by an oscillation at the much lower frequency  $\omega_-$ . As example of this is shown in the figure below, for the case that  $u_0 = v_0$ .



In acoustics, when the frequency difference  $\omega_-$  is very small, the sound you hear seems to come in pulses or “beat”, and the difference  $f_v - f_u = \omega_v / 2\pi - \omega_u / 2\pi$  is call the “beat frequency.”

## **PRELAB Problems for Lab #7: Coupled Pendulums and Normal Modes**

1. Consider a simple pendulum consisting of a mass  $m$  suspended on a string of length  $L$ .
  - (a) What is the pendulum's natural frequency if  $L = 1$  m?
  - (b) What length  $L$  for a clock pendulum would make the clock “tick” once every second, if the tick occurs every time the clock's pendulum passes through its equilibrium position?
  
2. Two identical carts are placed on a frictionless air track, so that they are free to move along a horizontal line. Weak springs of negligible mass are used to join the two cars and to connect each of the cars with a fixed point its end of the air track. The three springs have identical spring constants  $k$ . Each car has a mass  $M$ .
  - (a) What will be the frequency of motion if each cart is moved some distance  $D$  to the right of its equilibrium position, and they are simultaneously released from that position? (Hint: Think of the springs, and use the similarity of the two cart's situations.)
  - (b) What will be the frequency of motion if, instead of the above, one cart is moved a distance  $D$  to the right of its equilibrium position, and the other cart is moved a distance  $D$  to the left?

