Princeton University

Ph205
Mechanics
Problem Set 12

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http://kirkmcd.princeton.edu/examples/

1. Recall Prob. 7, Set 11, http://kirkmcd.princeton.edu/examples/Ph205/ph205set12.pdf.

   A string of mass m and length l has both ends fixed, and mass M is attached at distance b from one end.

   Find the shift in the angular frequency of the $n^{\rm th}$ (transverse) normal mode using Rayleigh's perturbation method (pp. 235-237 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l22.pdf.)

2. A string is stretched with tension T between fixed points $x = 0$ and l. It has linear mass density,

$$\rho(x) = \begin{cases} \rho_0 + \epsilon & (0 < x < b), \\ \rho_0 - \epsilon & (b < x < l). \end{cases} \tag{1}$$

That is, it is made of two strings of different densities, joined at $x = b$.

(a) Solve the transverse wave equations for each substring separately, and match solutions at $x = b$ to show that the normal modes have angular frequencies Ω related by,

$$c_1 \tan \frac{\Omega b}{c_1} = -c_2 \tan \frac{\Omega(l-b)}{c_2}, \qquad \text{where} \qquad c_i = \sqrt{\frac{T}{\rho_0 \pm \epsilon}}, \tag{2}$$

is the wave velocity on string i.

This result is "exact", but not very transparent.

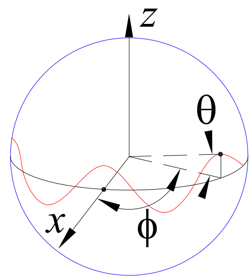
(b) Use Rayleigh's perturbation method to find the shift in angular frequencies relative to the case of $\epsilon = 0$.

Note that if $b = l/2$ there is no shift in frequencies — a simple result not readily apparent from part (a).

3. **Planetary String Theory.**

A string of linear mass density ρ is stretched with tension T around the equator of a sphere of radius a.

Consider transverse oscillations of the string, which slides without friction on the surface of the sphere, but somehow does not slip off it.



Let $\theta(\phi, t)$ be the angular displacement of the string. Use Lagrange's method to find the equation of motion. *Be careful about dimensions, and remember that the string always lies on the surface of the sphere.*

The normal modes have the form $\theta = \theta_n \cos(n\phi)\cos(\omega_n t)$. Show that,

$$\omega_n^2 = \frac{T}{\rho a^2}(n^2 - 1). \qquad (3)$$

If $n = 0$, the string would pop off the sphere; for $n = 1$ the string is not stretched and there is no oscillation.

4. **Transverse Waves on an Inelastic Vertical String**

What are the frequencies of small transverse oscillations in a vertical plane of an inelastic string of length l and linear mass density λ whose upper point is fixed at a point in a uniform gravitational field of strength g?

With y as the upward distance from the bottom of the string when at rest, deduce the equation of motion for (small) transverse displacements $s(y,t)$, and change variables to $x = \sqrt{y}$ to arrive at a version of Bessel's equation,

$$\frac{d^2 s}{dx^2} + \frac{1}{x}\frac{ds}{dx} + \frac{4\omega^2}{g}s = 0, \tag{4}$$

for oscillations with angular frequency ω.

Estimate the lowest oscillation frequency via Rayleigh's energy method using, say, a trial waveform $s(y) = l^p - y^p$ for y measured upwards from the lower end of the string, where p is to be optimized.

5. **Transverse Waves on an Inelastic Rotating String**

   What are the frequencies of small oscillations of an inelastic string of length l and linear mass density λ that is constrained to move on a plane which rotates at angular velocity Ω about a fixed axis in that plane, with one end of the string connected to a point on that axis?

   For x = outward distance from the axis for the string at equilibrium, and $z = x/l$, show that the equation of motion for oscillations of the form $s(x,t) = f(x)\cos\omega t$ is Legendre's equation,

$$\frac{d}{dz}\left[(1-z^2)\frac{df}{dz}\right] + \frac{2\omega^2}{\Omega^2}f = 0. \tag{5}$$

   Gravity can be neglected in this problem. Also, $\Omega l \ll c$, where c is the speed of light.

6. Approximate the lowest angular frequency ω of transverse vibrations of a bar that is clamped at one end, with the other end free, using Rayleigh's energy method.

   A brilliant guess of Rayleigh is that the shape $f(x)$ is very nearly that which is the solution to the statics problem of pushing on the bar at a point at distance b from the clamped end. Then, for $x > b$, the bar remains straight, which satisfies the boundary condition at the free end.

   To solve the statics problem, note that the transverse wave equation for the bar, p. 241 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l22.pdf, applies when $\ddot{s} = 0$ — the static limit. Show that in this case,

$$f(x) = \begin{cases} 3bx^3 - x^2 & (0 < x < b), \\ 3b^2x - b^3 & (b < x < l). \end{cases} \tag{6}$$

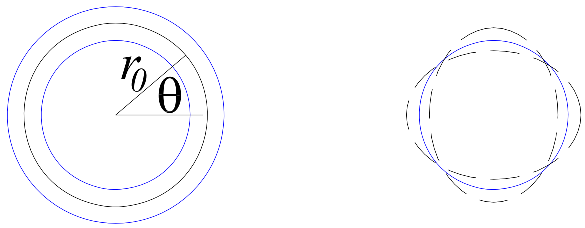
   Neglecting rotational kinetic energy in the wave equation, the resulting equation $1/\omega^2 = g(b)$ can be maximized to find lowest ω, and the corresponding best choice for b.

   This leads to a cubic equation (which can be solved using Wolfram Alpha).

   It turns out that $b = 4l/5$ is about right, and that $\omega \approx 3.512cd/l^2$, where $c^2 = AY/\rho$ and $d^2 = I/\rho A^2$ as on p. 241, Lecture 22 of the Notes.

   Hint: show that $\mathrm{KE} = (\rho A\,\omega^2 b^4/4)(3l^3 - 3bl^2 + b^2 l - 2b^3/35)$.

7. An elastic ring whose centerline has radius r_0 undergoes transverse vibrations in the plane of the ring, keeping the circumference, $2\pi r_0$, of the centerline constant.



The lowest mode is show on the right above.

During the vibration, a wedge-shaped element of the ring can move both radially and azimuthally, while deforming and rotating. Denoting the coordinates of the center of the deformed element as $r + \delta r$ and $\theta + \delta\theta$, show that the condition on the centerline implies that $\delta r = -r_0 d(\delta\theta)/d\theta$, by noting that the length $ds = r_0\, d\theta$ of a small segment of the centerline does not change as the ring deforms.

Construct the Lagrangian of the system, with θ as the independent variable, ignoring the (small) effects of kinetic energy of rotation, and of possible shearing motion. The potential energy associated with deformation of bar that is straight when undeformed, p. 239 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l22.pdf, is approximately correct for the ring, but with an approximation to the radius of curvature, namely,

$$\frac{1}{R} \approx \frac{1}{r} + \frac{d^2}{d\theta^2}\frac{1}{r}\,. \tag{7}$$

Use Hamilton's principle to deduce the equation of motion, which should contain a $6^{\rm th}$ derivative. For in-plane oscillatory modes of the form $\delta\theta = \epsilon \cos n\theta \cos \omega t$, show that the angular frequency obeys,

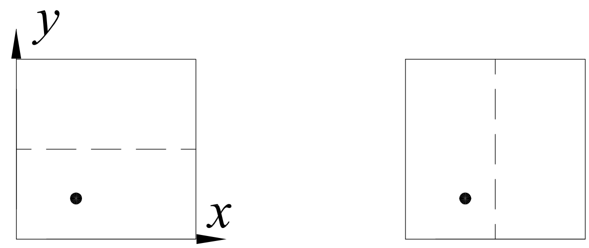
$$\omega^2 = \frac{YI}{\rho^2 A r_0^4}\frac{n^2(n^2-1)^2}{n^2+1}\,, \tag{8}$$

where Y is Young's modulus of elasticity, I is the moment of inertia of a cross section of the ring about its midline (perpendicular to the paper in the figure above),[1] ρ is the (volume) mass density, and A is the area of the cross section.

The modes with $n = 0$ and 1 are suppressed:
$n = 0 \Rightarrow$ rotation of the ring with no deformation;
$n = 1 \Rightarrow$ translation of the ring with no deformation.

[1]Many texts define their I to be our I/ρ.

8. Consider a square drum head of edge length l, mass density ρ_0 per unit area and surface tension T. A small mass m is attached at $(x, y) = (a, b)$ from one corner. What are the three lowest frequencies?



We can use Rayleigh's perturbation method if we know the form of the relevant unperturbed modes.

The form of the perturbed (1,1) mode is clear, but the (2,1) mode is degenerate with the (1,2). After the mass is added, the forms,

$$f_{1,2} = \sin\frac{\pi x}{l}\sin\frac{2\pi y}{l}, \qquad f_{2,1} = \sin\frac{2\pi x}{l}\sin\frac{\pi y}{l}, \tag{9}$$

are not normal modes any more.

The new normal modes are linear combinations of $f_{1,2}$ and $f_{2,1}$ such that mass m lies on the nodal curve of one of the modes,

$$f_{2a} = \frac{Af_{1,2} + Bf_{2,1}}{\sqrt{A^2 + B^2}}, \qquad f_{2a} = \frac{Bf_{1,2} - Af_{2,1}}{\sqrt{A^2 + B^2}}, . \tag{10}$$

Show that the perturbed angular frequencies are,

$$\Omega_{1,1} \approx \omega_{1,1}\left(1 - \frac{2m}{\rho_0 l^2}\sin^2\frac{\pi a}{l}\sin^2\frac{\pi b}{l}\right), \tag{11}$$

$$\Omega_{2a} = \omega_{1,2} = \omega_{2,1}, \tag{12}$$

$$\Omega_{2b} \approx \omega_{2,1}\left[1 - \frac{8m}{\rho_0 l^2}\sin^2\frac{\pi a}{l}\sin^2\frac{\pi b}{l}\left(\cos^2\frac{\pi a}{l} + \cos^2\frac{\pi b}{l}\right)\right], \tag{13}$$

where the $\omega_{i,j}$ are the unperturbed frequencies.

That is, the degeneracy has been broken by the perturbation.

Rayleigh's method must be modified slightly to deal with degenerate modes.

9. Consider transverse vibrations of a circular membrane of radius a, mass density ρ per unit area, and surface tension T.

   Use $F = ma$ in polar coordinates for an area element $r\, dr\, d\theta$ to show that displacement $s(r, \theta, t)$ obeys the wave equation,

$$\frac{1}{c^2}\frac{\partial^2 s}{\partial t^2} = \frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{1}{r^2}\frac{\partial^2 s}{\partial \theta^2}, \qquad \text{with} \qquad c^2 = \frac{T}{\rho} \tag{14}$$

   (or use Lagrange's method).

   Try separation of variable, $s = f(r)g(\theta)h(t)$ to show that solutions are possible with $g = \cos n\theta$ or $\sin n\theta$, $h = \cos \omega t$ or $\sin \omega t$, and,

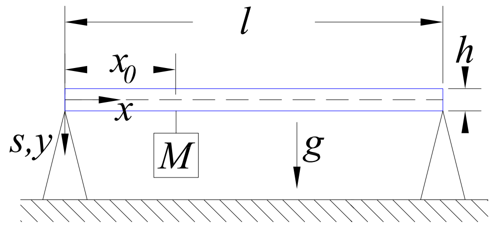
$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \left(\frac{\omega^2}{c^2} - \frac{n^2}{r^2}\right) f = 0, \tag{15}$$

   which is Bessel's equation of order n. The "boundary" equations for f are that $f(a) = 0 = f'(0)$.

   Apply Rayleigh's method to estimate the lowest normal frequency ($n = 0$), to show that $\omega_0 \approx 2.414c/a$ (compared to the "exact" value $2.405c/a$).

   See also Scientific American, p.172, Nov. 1982,
   http://kirkmcd.princeton.edu/examples/mechanics/rossing_sa_247-5_172_82.pdf.

10. A rectangular beam of length l, width w and height h is supported at the same height at both ends. The supports do not constrain the slope of the beam at its end (such that $s''(\text{end}) = 0$). Mass M is hung at distance x_0 from one end.



Give a Fourier-series expansion for the vertical displacement $s(x)$ of the beam, ignoring the deflection of the beam due to its own weight, and ignoring the variation in the deflection across the width of the beam.

Recall from p. 240, Lecture 22 of the Notes that the elastic potential energy of the displaced beam is,

$$V = \frac{II}{2\rho} \int (s'')^2 \, dx, \tag{16}$$

where Y is the Young's modulus of the beam, I is the moment of inertial per unit length of a cross section of the beam about its horizontal midline, and ρ is the mass density per unit length.

Show that,

$$s(x) = \frac{23Mgl^3}{\pi^4 Y w h^3} \sum_n \frac{1}{n^4} \sin \frac{n\pi x_0}{l} \sin \frac{n\pi x}{l} \, . \tag{17}$$

The deflection varies with the thickness h of the beam as $1/h^3$, so we say that the stiffness of the beam is proportional to $1/h^3$.

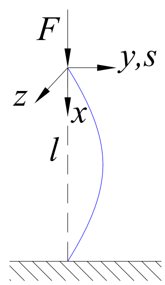
11. **Charlie Chaplin's Cane**

When Charlie leans on his cane, it pops into a bow shape

Consider a tall, slender beam (the cane) of length l, with a vertical force F applied to the top end, and the bottom end in no-slip contact with the ground, such that the ends of the beam are free to rotate. Show that the critical force, such that buckling/bowing in the x-y plane of an initially vertical beam occurs for any larger force, is given by,

$$F_{\rm crit} = \frac{\pi^2 Y I}{\rho l^2}\,, \tag{18}$$

where Y is Young's modulus of elasticity, $I = \int \rho y^2\, dx\, dy$ is the moment of inertia per unit length of a cross section of the beam,[2] and ρ is its (volume) mass density.



You may ignore gravity, and the compression of length l of the beam.

For compressive force F, less than the critical force $F_{\rm crit}$ on the beam, it remains straight. At the critical force, any small transverse displacement in the x-y plane is also at static equilibrium. Therefore, consider the conditions for static equilibrium of such a displacement.

For this, both the total force and torque on any segment of the bar must be zero.

On p. 239, Lecture 22 of the Notes we found that the potential energy stored in a short section of length dl of a bent, elastic bar is,

$$dV = \frac{YI}{2\rho}\frac{dl}{R^2}\,, \tag{19}$$

where R is the local radius of curvature. Relate the derivation of this to the torque (often called the bending moment) on a cross section of the beam due to the internal forces on one side of it.....

[2]Many authors define I as our I/ρ.

Solutions

1. A string of mass m and length l has both ends fixed, and mass M is attached at distance b from one end.

   We can write the linear mass density of the string + mass M as $\rho_(x) = m/l + M\,\delta(x - b) \equiv \rho_0 + \rho_1$.. Then, from p. 236, Lecture 22 of the Notes,
   http://kirkmcd.princeton.edu/examples/Ph205/ph205l22.pdf,
   the angular frequency of the $n^{\rm th}$ (transverse) normal mode is given by,

$$\Omega_n = \omega_n \left(1 - \frac{1}{\rho_0 l}\int_0^l \rho_1(x)\sin^2\frac{n\pi x}{l}\,dx\right) = \omega_n\left(1 - \frac{M}{m}\sin^2\frac{n\pi b}{l}\right), \qquad (20)$$

   where $\omega_n = n\pi c/l$ is the angular frequency of the $n^{\rm th}$ normal mode of the unperturbed string.

   This is consistent with various results found in Prob. 7, Set 11,
   http://kirkmcd.princeton.edu/examples/Ph205/ph205set12.pdf.

   For example, if $\sin(n\pi b/l) = 0$, mass M is at a node of the $n^{\rm th}$ normal mode, which mode is unaffected by the presence of mass M (in the limit that its extent in x goes to zero).

2. A string is stretched with tension T between fixed points $x = 0$ and l. It has linear mass density,

$$\rho(x) = \begin{cases} \rho_0 + \epsilon & (0 < x < b), \\ \rho_0 - \epsilon & (b < x < l). \end{cases} \tag{21}$$

That is, it is made of two strings of different densities, joined at $x = b$.

(a) Each substring obeys the transverse wave equation,

$$s_i''(x,t) = \frac{T}{\rho_0 \pm \epsilon}\ddot{s}_i(x,t) = \frac{1}{c_i^2}\ddot{s}_i(x,t). \tag{22}$$

The boundary conditions are,

$$s_1(x,t) = 0 = s_2(l,t), \qquad s_1(b,t) = s_2(b,t), \qquad s_1'(b,t) = s_2'(b,t), \tag{23}$$

noting that the point at $x = b$ would have infinite acceleration if the slopes of the two substrings were different there.

Waves of angular frequency Ω that satisfy the boundary conditions at the $x = 0$ and l have the form,

$$s_1 = a_1 \sin k_1 x \cos \Omega t, \qquad s_2 = a_2 \sin k_2 x \cos \Omega t. \tag{24}$$

The wave equations (22) tell us that $k_i = \Omega/c_i$, and then the matching conditions at $x = b$ imply that,

$$a_1 \sin\frac{\omega b}{c_1} = a_2 \sin\frac{\Omega(l-b)}{c_2}, \tag{25}$$

$$a_1\frac{\Omega}{c_1}\cos\frac{\Omega b}{c_1} = -a_2\frac{\Omega}{c_2}\cos\frac{\Omega(l-b)}{c_2}, \tag{26}$$

$$c_1 \tan\frac{\Omega b}{c_1} = -c_2\tan\frac{\Omega(l-b)}{c_2}. \tag{27}$$

(b) For $\epsilon \ll \rho_0$ we can use Rayleigh's perturbation method, noting that when $\epsilon = 0$ the n^{th} (transverse) normal mode has angular frequency $\omega_n = n\pi c_0/l$ where $c_0 = \sqrt{T/\rho_0}$.

Then,

$$\Omega_n = \omega_n\left(1 - \frac{1}{\rho_0 l}\int_0^l \rho_1(x)\sin^2\frac{n\pi x}{l}\,dx\right)$$

$$= \omega_n\left[1 - \frac{\epsilon}{2\rho_0 l}\int_0^b\left(1-\cos\frac{2n\pi x}{l}\right)dx + \frac{\epsilon}{2\rho_0 l}\int_b^l\left(1-\cos\frac{2n\pi x}{l}\right]dx\right)$$

$$= \omega_n\left[1 - \frac{\epsilon}{2\rho_0 l}\left(b - \frac{l}{2n\pi}\sin\frac{2n\pi b}{l}\right) + \frac{\epsilon}{2\rho_0 l}\left(l - b + \frac{l}{2n\pi}\sin\frac{2n\pi(l-b)}{l}\right)\right]$$

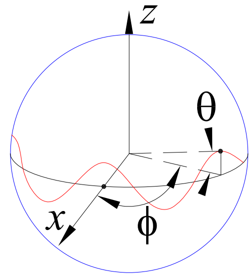
$$= \omega_n\left(1 + \frac{\epsilon(l-2b)}{2\rho_0 l} + \frac{\epsilon}{n\pi\rho_0}\sin\frac{2n\pi b}{l}\right). \tag{28}$$

Note that if $b = l/2$ there is no shift in frequencies — a simple result not readily apparent from part (a).

3. **Planetary String Theory.**

This problem is based on sec. 139, p. 213 of Lord Rayleigh, *Theory of Sound*, 2nd ed. (Macmillan, 1894), http://kirkmcd.princeton.edu/examples/mechanics/rayleigh_theory_of_sound_1.pdf

A string of linear mass density ρ is stretched with tension T around the equator of a sphere of radius a.



Denoting the latitude $\theta(\phi, t)$ as the angular displacement of the string, its kinetic energy is,

$$\mathrm{KE} = \int_0^{2\pi} \frac{\rho(a\dot{\theta})^2}{2}\, a\, d\phi = \frac{a^3\rho}{2}\int_0^{2\pi} \dot{\theta}^2\, d\phi. \tag{29}$$

The stored potential energy is the work done in stretching the string from its nominal equilibrium configuration along the equator,

$$V = \int T\, dl - 2\pi a T, \tag{30}$$

where for two points on the surface of the sphere, separated by $d\theta$ and $d\phi$,

$$dl^2 = a^2[(d\theta)^2 + \cos^2\theta(d\phi)^2] \approx a^2(d\phi)^2\left[1 - \theta^2 + \left(\frac{d\theta}{d\phi}\right)^2\right], \tag{31}$$

$$dl \approx a\, d\phi\left[1 - \frac{\theta^2}{2} + \frac{1}{2}\left(\frac{d\theta}{d\phi}\right)^2\right]. \tag{32}$$

Then,

$$V = \int_0^{2\pi} aT\left[-\frac{\theta^2}{2} + \frac{1}{2}\left(\frac{d\theta}{d\phi}\right)^2\right] d\phi, \tag{33}$$

and the Lagrangian of the system is,

$$L = \int_0^l \mathcal{L}\, d\phi = \int_0^{2\pi}(\mathrm{KE} - V)\, d\phi \approx \int_0^{2\pi}\left[\frac{a^3\rho\,\dot{\theta}^2}{2} + \frac{aT\,\theta^2}{2} - \frac{aT}{2}\left(\frac{d\theta}{d\phi}\right)^2\right] d\phi. \tag{34}$$

The equation of motion follows from the Lagrangian via Hamilton's principle as on p. 240, Lecture 22 of the Notes,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} + \frac{d}{d\phi}\frac{\partial\mathcal{L}}{\partial\, d\theta/d\phi} = \frac{\partial\mathcal{L}}{\partial\theta}, \qquad a^3\rho\,\ddot{\theta} - aT\frac{d^2\theta}{d\phi^2} = aT\,\theta. \tag{35}$$

For a normal mode of the form $\theta = \theta_n \cos(n\phi)\cos(\omega_n t)$, the equation of motion (35) implies that,

$$-a^3 \rho\, \omega_n^2 + aTn^2 = aT, \qquad \omega_n^2 = \frac{T}{\rho a^2}(n^2 - 1). \tag{36}$$

If $n = 0$, the string would pop off the sphere; for $n = 1$ the string is not stretched and there is no oscillation.

4. **Transverse Waves on an Inelastic Vertical String**

The equilibrium state of the string is, of course, that it hangs vertically, with its lower end at $y = 0$ and its upper end at $y = l$.

The tension in the string is,

$$T(y) = \lambda g y. \tag{37}$$

The equation of motion for a transverse displacement $s(y,t)$ in a vertical plane of a segment dy of the string is

$$\lambda\, dx\, \ddot{s} = T(y+dy)s'(y+dy) - T(y)s'(y) = \frac{\partial T s'}{\partial y}\, dy = \lambda g \frac{\partial (y s')]}{\partial y}\, dy \tag{38}$$

For oscillations at angular frequency ω of the form $s(y,t) = s(y)e^{i\omega t}$, eq. (38) reduces to,

$$\frac{d(ys')}{dy} + \frac{\omega^2}{g}s = y\frac{d^2 s}{dy^2} + \frac{ds}{dy} + \frac{\omega^2}{g}s = 0. \tag{39}$$

This is a form of Bessel's equation of order zero, as can be seen using the substitution $x = \sqrt{y}$, with which eq. (39) becomes,

$$x^2\frac{d^2 s}{dx^2} + x\frac{ds}{dx} + \frac{4\omega^2}{g}x^2 s = 0, \tag{40}$$

whose solutions are,

$$s(y) = s_0 J_0(2\omega\sqrt{y/g}). \tag{41}$$

The condition that $s(y = l) = 0$ determine a series of frequencies of small oscillation,

$$2\omega\sqrt{\frac{l}{g}} = 2.405,\ \ 5.520,\ \ 8.654, \ldots, \tag{42}$$

or,

$$\omega = 1.202\sqrt{\frac{g}{l}},\ \ 2.760\sqrt{\frac{g}{l}},\ \ 4.318\sqrt{\frac{g}{l}}, \ldots \tag{43}$$

Rayleigh noted that for a springlike system, $\langle\text{KE}\rangle = \langle\text{PE}\rangle$ (virial theorem), so that a trial waveform with parameter p can be used to estimate the frequency $\omega(p)$ using this constraint. Then the lowest frequency is obtained by minimizing $\omega(p)$ with respect to the parameter p.

We consider the form,

$$s(y,t) = (l^p - y^p)e^{i\omega t}, \tag{44}$$

for which the time-average kinetic energy is,

$$\begin{aligned}\langle\text{KE}\rangle &= \left\langle \int_0^l \frac{\lambda \dot{s}^2}{2}\, dy \right\rangle = \frac{\lambda\omega^2}{4}\int_0^l (l^p - y^p)^2\, dy = \frac{\lambda\omega^2}{4}l^{2p+1}\left(1 - \frac{2}{p+1} + \frac{1}{2p+1}\right) \\ &= \frac{\lambda\omega^2}{4}l^{2p+1}\frac{2p^2}{(p+1)(2p+1)}, \end{aligned}\tag{45}$$

and the time-average potential energy (= work done in stretching the string) is,

$$\langle \mathrm{PE} \rangle = \left\langle \int_0^l T(\sqrt{1+s'^2} - 1)\, dy \right\rangle \approx \left\langle \int_0^l \frac{Ts'^2}{2}\, dy \right\rangle = \frac{\lambda g}{4} \int_0^l y(-py^{p-1})^2\, dy = \frac{\lambda g}{4} l^{2p} \frac{p}{2}\,. \tag{46}$$

Equating the kinetic and potential energies, we have that,

$$\omega^2(p) = \frac{g}{l} \frac{(p+1)(2p+1)}{4p}\,. \tag{47}$$

The minimum frequency occurs for $p = 1/\sqrt{2}$, which implies that its value is,

$$\omega \approx \sqrt{\frac{g}{l}} \sqrt{\frac{1.707 \cdot 2.414}{2.828}} = 1.207 \sqrt{\frac{g}{l}}\,, \tag{48}$$

which compares well with the "exact" value of $1.202\sqrt{g/l}$.

For additional discussion, see A.B. Western, Demonstration for observing $J_0(x)$ *on a resonant rotating vertical chain, Am. J. Phys.* **48**, *54 (1980),*
http://kirkmcd.princeton.edu/examples/mechanics/western_ajp_48_54_80.pdf
An early paper on this topic is by J.H. Rohrs, Oscillations of a Suspension Chain, Trans. Camb. Phil. Soc. **9**, *Part III, 49 (1851),*
http://kirkmcd.princeton.edu/examples/mechanics/rohrs_tcps_9(3)_49_51.pdf

5. **Transverse Waves on an Inelastic Rotating String**

This problem was suggested by Sam Treiman.

The equilibrium state of the string is that is lies along the line in the rotating plane that passes through the point of connecting and is perpendicular to the axis. Let x measure the distance along this line, with $x = 0$ at the axis. Let $s(x, t)$ be the (small) transverse displacement (in the rotating plane) of the string from its equilibrium state.

Then, in the rotating frame, an segment dx of the string about point x experiences a "fictitious" outward force $\lambda\ dx\ \Omega^2 x$, which is balanced by the x-component of the tension $T(x)$ in the string. For small oscillations the x-component of T is well approximated as T, so,

$$T(x + dx) - T(x) = T'\,dx = \lambda\Omega^2 x\,dx, \qquad T' = \lambda\Omega^2 x, \tag{49}$$

and,

$$T(x) = \frac{\lambda\Omega^2}{2}(l^2 - x^2), \tag{50}$$

noting that the tension vanishes at the free end, $T(l) = 0$.

We ignore effects of the Coriolis force in the approximation that the motion is purely transverse.

The equation of transverse motion for a segment of the string is,

$$\lambda\,dx\,\ddot{s} = T(x + dx)s'(x + dx) - T(x)s'(x) = \frac{\partial(Ts')}{\partial x}\,dx = \frac{\lambda\Omega^2}{2}\frac{\partial[(l^2 - x^2)s']}{\partial x}\,dx. \tag{51}$$

For oscillations at angular frequency ω of the form $s(x, t) = s(x)e^{i\omega t}$, eq. (51) reduces to,

$$\frac{d[(l^2 - x^2)s']}{dx} + \frac{2\omega^2}{\Omega^2}s = 0. \tag{52}$$

Changing to the dimensionless variable $z = x/l$, this becomes,

$$\frac{d}{dz}\left[(1 - z^2)\frac{ds}{dz}\right] + \frac{2\omega^2}{\Omega^2}s = 0. \tag{53}$$

We recognize this as Legrendre's equation, whose solutions are the Legendre polynomials $P_m(z)$ where,

$$\frac{2\omega^2}{\Omega^2} = m(m + 1), \tag{54}$$

for non-negative integers m.

The string obeys the boundary condition that $s(0) = 0$, which is satisfied only by Legendre polynomials of odd m. Hence the frequencies of small oscillation are,

$$\omega = \Omega\sqrt{\frac{m(m + 1)}{2}} \qquad (m \text{ odd}), \tag{55}$$

$$\omega = \Omega, \sqrt{6}\Omega, \sqrt{15}\Omega, \ldots \tag{56}$$

The corresponding waveforms are,

$$s(z) = s_0 z, \frac{s_0}{2}(5z^3 - 3z), \frac{s_0}{8}(63z^5 - 70z^3 + 15z), ... \tag{57}$$

where s_0 is the amplitude of the oscillation at $z = 1$ ($x = l$).

We have obtained this solution to a second-order differential equation using only a single boundary condition. Note that by expanding eq. (53), and setting $z = 1$, we obtain,

$$\frac{ds}{dz} = \frac{\omega^2}{\Omega^2} s \qquad (z = 1), \tag{58}$$

which is a kind of "automatic" boundary condition that can be satisfied by a real string.

6. *This problem is based on sec. 182, p. 287 of* Lord Rayleigh, *Theory of Sound*, 2nd ed. (Macmillan, 1894), http://kirkmcd.princeton.edu/examples/mechanics/rayleigh_theory_of_sound_1.pdf

We seek an estimate of the lowest angular frequency ω of transverse vibrations of a bar that is clamped at one end, with the other end free. For this we follow Rayleigh in noting that for such vibrations $\langle \text{KE} \rangle = \langle \text{PE} \rangle$.

The wave equation for transverse vibrations of a bar of cross sectional area A, mass density ρ, moment of inertial I per unit length (for rotations about the midline of a cross sectional slice), and Young's modulus Y is, p. 241, Lecture 22 of the Notes,

$$\rho A \ddot{s} - I\ddot{s}'' + \frac{YI}{\rho} s'''' = 0. \tag{59}$$

In this problem we neglect the term $I\ddot{s}''$ associated with the rotational kinetic energy, and consider the approximate equation,

$$\ddot{s} + \frac{YI}{A\rho^2} s'''' = 0 = \ddot{s} + (cd)^2 s'''', \qquad \text{where} \qquad c^2 = \frac{AY}{\rho} \qquad d^2 = \frac{I}{\rho A^2}. \tag{60}$$

We also follow Rayleigh in supposing that the shape $f(x)$ in $s(x,t) = f(x)\cos\omega t$ is very nearly that which is the solution to the statics problem of pushing on the bar at a point at distance b from the clamped end. Then, for $x > b$, the bar remains straight, which satisfies the boundary condition at the free end.

In the static limit, the wave equation (60) reduces to $f'''' = 0$, such that,

$$f = a_0 + a_1 x + a_2 x^2 + a_3 x^3, \tag{61}$$

for constants a_i. For a bar clamped at $x = 0$, we have that $f(0) = 0 = f'(0)$, *i.e.*, $a_0 = 0 = a_1$, while for $b < x < l$ the bar is straight, with $f'(x > b) =$ constant and $f''(x > b) = 0$.

In particular, $f''(b) = 2a_2 + 6a_3 b = 0$, *i.e.*, $a_2 = -3a_3 b$. Hence, $f(x > b) = a_4 + a_5 x$, and,

$$f(x < b) = a_3(x^3 - 3bx^2), \tag{62}$$

$$f'(b) = -3a_3 b^2 = a_5, \qquad f(b) = -2a_3 b^3 = a_4 + a_5 b = a_4 - 3a_3 b^3, \tag{63}$$

$$f(x > b) = a_3(b^3 - 3b^2 x). \tag{64}$$

We now suppose that the static forms, eqs. (62) and (64), apply approximately for standing waves as well, and evaluate the corresponding time-average kinetic and potential energies, which are equal for springlike oscillations.

Noting that the linear mass density of the bar is ρA, the time-average kinetic energy is,

$$\langle \text{KE} \rangle = \int_0^l \frac{\rho A \langle \dot{s}^2 \rangle}{2}\, dl = \frac{\rho A \omega^2}{4} \int_0^l f^2(x)\, dx$$

$$= \frac{\rho a_3^2 A\omega^2}{4}\int_0^b (x^6 - 6bx^5 + 9b^2x^4)\,dx + \frac{\rho a_3^2 A\omega^2}{4}\int_b^l (b^6 - 6b^5x + 9b^4x^2)\,dx$$

$$= \frac{\rho a_3^2 A\omega^2}{4}\left[\frac{b^7}{7} - b^7 + \frac{9b^7}{5} + b^6(l-b) - 3b^5(l^2-b^2) + 3b^4(l^3-b^3)\right]$$

$$= \frac{\rho a_3^2 b^4 A\omega^2}{4}\left(3l^3 - 3bl^2 + b^2 l - \frac{2b^3}{35}\right). \qquad (65)$$

From p. 240 of the Notes, the potential energy is PE $= (YI/2\rho)\int_0^l (s'')^2\,dx$, so the time-average potential energy is,

$$\langle \text{PE} \rangle = \frac{IY}{2\rho}\int_0^l \left\langle s''^2 \right\rangle dl = \frac{IYa_3^2}{4\rho}\int_0^l f''^2(x)\,dx = \frac{IYa_3^2}{4\rho}\int_0^b 36(x^2 - 2bx + b^2)\,dx$$

$$= \frac{IYa_3^2}{4\rho}(12b^3 - 36b^3 + 36b^3) = \frac{3IYa_3^2 b^3}{4\rho}. \qquad (66)$$

Equating the time-average kinetic and potential energies, we find,

$$\frac{1}{\omega^2} = \frac{\rho^2 bA}{3IY}\left(3l^3 - 3bl^2 + b^2 l - \frac{2b^3}{35}\right) = \frac{l^4}{12c^2d^2}\left(\frac{3b}{l} - \frac{3b^2}{l^2} + \frac{b^3}{l^3} - \frac{2b^4}{35l^4}\right), \qquad (67)$$

with c and d as in eq. (60).

To find the lowest angular frequency ω, we maximize eq. (67) with respect to b, which leads to the cubic equation,

$$3 - 6\frac{b}{l} + \frac{3b^2}{l^2} - \frac{8b^3}{35l^3} = 0. \qquad (68)$$

Using Wolfram Alpha, we find that $b/l \approx 0.802$, and $\omega \approx 3.512cd/l^2$.
There are 2 other roots of eq. (68) with $b/l > 1$.

The problem is briefly discussed in prob. 6, §25, p. 117 of L.D. Landau and E.M. Lifshitz, *Theory of Elasticity*, 2nd ed. (Pergamon, 1970),
http://kirkmcd.princeton.edu/examples/mechanics/landau_e_70.pdf,
who find that $\omega = a^2\,cd/l^2$ where $\cos a \cosh a = -1$ *i.e.*, $a^2 = 3.516$ (in close agreement with the above result).

7. We consider the in-plane vibrations of an elastic ring of central radius r_0 in which the centerline does not stretch or compress. We also neglect possible shearing motions, as well as the kinetic energy of rotations of volume elements of the ring.

   Then, a wedge-shaped volume element, centered on polar coordinates r_0 and θ has displaced coordinates $r_0 + \delta r$ and $\theta + \delta\theta$.

   A short segment of length $r_0\, d\theta$ on the undisplaced centerline has displaced length related by

$$ds^2 = d(\delta r)^2 + (r_0 + \delta r)^2[d\theta + d(\delta\theta)]^2 \approx r_0^2 d\theta^2 + 2r_0^2\, d\theta\, d(\delta\theta) + 2r_0\, \delta r\, d\theta, \qquad (69)$$

   to first order in the very small quantities δr and $d(\delta\theta) \approx \delta(d\theta)$. The condition that the centerline does not stretch or compress implies that ds^2 remains $r_0^2\, d\theta^2$, and hence,

$$r_0\, d(\delta\theta) + \delta r\, d\theta = 0, \qquad \delta r = -r_0 \frac{d(\delta\theta)}{d\theta} \equiv -r_0\, (\delta\theta)'. \qquad (70)$$

   The kinetic energy of the vibrating ring, of mass density ρ and cross-sectional area A, is,

$$T = \int_0^{2\pi} \frac{\rho A}{2} \left[(\delta\dot{r})^2 + r_0^2\, (\delta\dot{\theta})^2\right] r_0\, d\theta = \frac{\rho A r_0^3}{2} \int_0^{2\pi} \left[(\delta\dot{\theta}')^2 + (\delta\dot{\theta})^2\right] d\theta. \qquad (71)$$

   We recall from p. 239 of the Notes that the potential energy for transverse vibrations of a bar that is straight when at rest can be written as,

$$V = \int \frac{YI}{2\rho R^2}\, dx, \qquad (72)$$

   where Y is the Young's modulus, I is the moment of inertia per unit length of a transverse section about the centerline at rest, R is the radius of curvature of the displaced centerline, and x is measured along the centerline at rest. For a circular ring, eq. (72) must be approximately true, with $x = r_0\, d\theta$,

$$V \approx \int_0^{2\pi} \frac{YI}{2\rho R^2}\, r_0\, d\theta. \qquad (73)$$

   The radius of curvature R of the centerline is given in polar coordinates as,[3]

$$\frac{1}{R} = \frac{1 + 2(r')^2/r^2 - (r'')/r}{r(1 + (r')^2/r^2)^{3/2}}\,. \qquad (74)$$

   In the approximate eq. (73) for the potential energy of the deformed ring, we should not use the "exact" expression (74) for the radius of curvature R, but an approximation. It turns out that it is appropriate to write,

$$\frac{1}{R} \approx \frac{1 + 2(r')^2/r^2 - (r'')/r}{r} = \frac{1}{r} + \frac{d^2}{d\theta^2}\frac{1}{r}\,. \qquad (75)$$

[3] See, for example, eq. (6) of https://mathworld.wolfram.com/RadiusofCurvature.html.

For $r = r_0 + \delta r$, recalling (70),

$$\frac{1}{r} = \frac{1}{r_0 + \delta r} \approx \frac{1}{r_0}\left(1 - \frac{\delta r}{r_0}\right) = \frac{1}{r_0}[1 + (\delta\theta)'], \tag{76}$$

$$\frac{1}{R} \approx \frac{1}{r} + \frac{d^2}{d\theta^2}\frac{1}{r} \approx \frac{1}{r_0}\left[1 + (\delta\theta)' + (\delta\theta)'''\right]. \tag{77}$$

An alternative argument is given in H. Lamb, Proc. London Math. Soc. **19**, 365 (1888), http://kirkmcd.princeton.edu/examples/mechanics/lamb_plms_19_365_87.pdf

Hence, the Lagrangian for the vibrating ring is,

$$L = T - V = \frac{\rho A r_0^3}{2}\int_0^{2\pi}\left[(\delta\dot\theta')^2 + (\delta\dot\theta)^2\right] d\theta - \frac{YI}{2\rho r_0}\int_0^{2\pi}\left[1 + (\delta\theta)' + (\delta\theta)'''\right]^2 d\theta$$
$$\equiv \int_0^l \mathcal{L}\, dx. \tag{78}$$

To obtain the equation of motion of the vibrating ring, we apply Hamilton's principle that $\delta\int \mathcal{L}\, dt = 0$ for small variations of the centerline $s + \eta$ about the equilibrium circular line s. In the resulting variational equation, we integrate the various terms by parts (multiple times if necessary, as on pp. 240-241, Lecture 22 of the Notes) and apply appropriate boundary conditions to find that,

$$-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\delta\dot\theta} - \frac{d^2}{d\theta\, dt}\frac{\partial\mathcal{L}}{\partial(\delta\dot\theta)'} - \frac{d}{d\theta}\frac{\partial\mathcal{L}}{\partial(\delta\theta)'} - \frac{d^3}{d\theta^3}\frac{\partial\mathcal{L}}{\partial(\delta\theta)'''} = 0, \tag{79}$$

$$-\rho A r_0^3\,\delta\ddot\theta + \rho A r_0^3\,(\delta\ddot\theta)'' + \frac{YI}{\rho r_0}\left[(\delta\theta)'' + (\delta\theta)''''\right] + \frac{YI}{\rho r_0}\left[(\delta\theta)'''' + (\delta\theta)''''''\right] = 0, \tag{80}$$

$$\delta\ddot\theta - (\delta\ddot\theta)'' - \frac{YI}{\rho^2 A r_0^4}\left[(\delta\theta)'' + 2(\delta\theta)'''' + (\delta\theta)''''''\right] = 0. \tag{81}$$

For an in-plane vibrational mode of the form $\delta\theta = a\cos n\theta\cos\omega t$, we have,

$$-\omega^2 - n^2\omega^2 - \frac{YI}{\rho^2 A r_0^4}(-n^2 + 2n^4 - n^6) = 0, \tag{82}$$

$$\omega^2 = \frac{YI}{\rho^2 A r_0^4}\frac{n^2(n^2-1)^2}{n^2+1}. \tag{83}$$

The modes with $n = 0$ and 1 are suppressed:
$n = 0 \Rightarrow$ rotation of the ring with no deformation;
$n = 1 \Rightarrow$ translation of the ring with no deformation.

This problem was first solved, by other methods, in R. Hoppe, J. Reine Angew. Math. **73**, 158 (1871), http://kirkmcd.princeton.edu/examples/mechanics/hoppe_jram_73_158_71.pdf
The minor effects of shear, and of rotational kinetic energy, and the more significant effects when $\delta r/r_0$ is substantial, are discussed, for example, in R.S. Seidel and E.A. Erdelyi, J. Eng. Ind. **86**, 240 (1964),
http://kirkmcd.princeton.edu/examples/mechanics/seidel_jei_86_240_64.pdf

8. We consider a square drum head of edge length l, mass density ρ_0 per unit area and surface tension T. A small mass M is attached at $(x, y) = (a, b)$ from one corner.

   As on p. 246, Lecture 22 of the Notes, the normal modes of out-of-plane oscillations of a square drum/membrane (clamped along its edges) in the absence of the mass m have the form,

$$s_{m,n}(x,y,t) = f_{m,n}(x,y)\cos\omega_{m,n}t, \qquad f_{m,n} = a_{m,n}\sin\frac{m\pi x}{l}\sin\frac{n\pi y}{l}, \tag{84}$$

$$\omega_{m,n}^2 = \frac{T}{\rho_0}\left[\left(\frac{m\pi}{l}\right)^2 + \left(\frac{b\pi}{l}\right)^2\right]. \tag{85}$$

   Note that modes m, n and n, m have the same frequency, so any linear combination,

$$\frac{Af_{m,n} + Bf_{n,m}}{\sqrt{A^2+B^2}}\cos\omega_{m,n}t, \tag{86}$$

   is also a normal mode (in the absence of mass m).

   When mass M is attached, the mode 1,1 is still a normal mode. However, the degenerate (*i.e.*, having the same frequency) modes 1,2 and 2,1 are no longer normal modes. Rather, a mode of frequency $\omega_{1,2}$, which we denote as $2a$, is such that mass M remains at rest, while other normal mode of frequency $\omega_{1,2}$, denoted as $2b$, is orthogonal to mode $2a$.

   The mode $2a$ obeys,

$$0 = Af_{1,2}(a,b) + Bf_{2,1}(a,b) = Aa_2\sin\frac{\pi a}{l}\sin\frac{2\pi b}{l} + Ba_2\sin\frac{2\pi a}{l}\sin\frac{\pi b}{l}, \tag{87}$$

$$A = a_2C\sin\frac{2\pi a}{l}\sin\frac{\pi b}{l}, \qquad B = -a_2C\sin\frac{\pi a}{l}\sin\frac{2\pi b}{l}, \tag{88}$$

$$C = \frac{1}{\sqrt{\sin^2\frac{2\pi a}{l}\sin^2\frac{\pi b}{l} + \sin^2\frac{\pi a}{l}\sin^2\frac{2\pi b}{l}}}, \tag{89}$$

   and the orthogonal mode, $2b$, has $A' = -B$, $B' = A$. That is,

$$f_{2a} = a_2C\sin\frac{2\pi a}{l}\sin\frac{\pi b}{l}\sin\frac{\pi x}{l}\sin\frac{2\pi y}{l} - a_2C\sin\frac{\pi a}{l}\sin\frac{2\pi b}{l}\sin\frac{2\pi x}{l}\sin\frac{\pi y}{l}, \tag{90}$$

$$f_{2b} = a_2C\sin\frac{\pi a}{l}\sin\frac{2\pi b}{l}\sin\frac{\pi x}{l}\sin\frac{2\pi y}{l} + a_2C\sin\frac{2\pi a}{l}\sin\frac{\pi b}{l}\sin\frac{2\pi x}{l}\sin\frac{\pi y}{l}. \tag{91}$$

   We estimate the angular frequencies of the normal modes in the presence of mass m via Rayleigh's perturbation method, sketched on p. 236, Lecture 22 of the Notes.

To extend this method to waves on a 2-d membrane, we recall its essence to be that the time-average kinetic and potential energies of oscillatory modes are equal. Then, for the 2-d case we have, from p. 245, Lecture 22 of the Notes, with $s(x,y,t) = f(x,y)\cos\Omega t$,

$$\langle \text{KE} \rangle = \frac{1}{2}\int\int \frac{\Omega^2\rho}{2} f^2(x,y)\,dx\,dy = \langle \text{PE} \rangle = \frac{1}{2}\int\int \frac{T}{2}(f_x^2 + f_y^2)\,dx\,dy, \tag{92}$$

where we ignore the small changes in the surface tension T during the oscillation, and $f_x = \partial f/\partial x$, *etc.* We also note that different modes are orthogonal, in the sense that their spatial parts, $f(x,y)$ and $g(x,y)$ obey,

$$\int_0^l \int_0^l f g\,dx\,dy = 0. \tag{93}$$

Likewise, the spatial derivatives of the modes are orthogonal,

$$\int_0^l \int_0^l f_x g_x\,dx\,dy = 0 = \int_0^l \int_0^l f_y g_y\,dx\,dy. \tag{94}$$

The potential energy, eq. (92), of a mode is unaffected by the perturbation,

$$\langle \text{PE}_{m,n} \rangle = \frac{T}{4}\frac{(m^2+n^2)\pi^2 a_{m,n}^2}{l^2}\frac{l}{2}\frac{l}{2} = \frac{(m^2+n^2)\pi^2 a_{m,n}^2 T}{16} = \frac{a_{m,n}^2 \rho_0 l^2\,\omega_{m,n}^2}{16}. \tag{95}$$

$$\tag{96}$$

This also holds for linear combinations (86) when $a_{m,n} = a_{n,m}$, using the orthogonality (94) of the spatial derivatives of the modes, and recalling eqs. (89) and (91).

The perturbed mass density can be written as,

$$\rho(x,y) = \rho_0 + \delta\rho(x,y) = \rho_0 + M\,\delta(x-a)\,\delta(y-b), \tag{97}$$

such that the time-average kinetic energy of a perturbed mode $f\cos\Omega t$ is,

$$\langle \text{KE} \rangle = \frac{\Omega^2\rho_0}{4}\int_0^l \int_0^l f^2(x,y)\,dx\,dy + \frac{\Omega^2 M}{4} f^2(a,b) = \frac{\Omega^2 a^2 \rho_0 l^2}{16} + \frac{\Omega^2 M}{4} f^2(a,b) \tag{98}$$

where a is the amplitude of the mode. This holds for modes (such as f_{2a} and f_{2b}) (with the same amplitude a) that are linear combinations of $f_{m,n}$ and $f_{n,m}$, recalling the orthogonality relation (93).

Equating $\langle \text{KE} \rangle$ with $\langle \text{KE} \rangle$, the perturbed angular frequency Ω is given by,

$$\frac{\Omega^2 a^2 \rho_0 l^2}{16}\left(1 + \frac{4M}{a^2\rho_0 l^2} f^2(a,b)\right) = \langle \text{PE} \rangle, \tag{99}$$

$$\Omega \approx \sqrt{\frac{16\,\langle \text{PE} \rangle}{a^2\rho_0 l^2}}\left(1 - \frac{2M}{a^2\rho_0 l^2} f^2(a,b)\right) = \omega\left(1 - \frac{2M}{a^2\rho_0 l^2} f^2(a,b)\right), \tag{100}$$

where ω is the unperturbed angular frequency of the mode.

Finally, we obtain,

$$\Omega_{1,1} = \omega_{1,1}\left(1 - \frac{2M}{a_{1,1}^2\rho_0 l^2} f_{1,1}^2(a,b)\right) = \omega_{1,1}\left(1 - \frac{2M}{\rho_0 l^2}\sin^2\frac{\pi a}{l}\sin^2\frac{\pi b}{l}\right), \tag{101}$$

$$\Omega_{2a} = \omega_{1,2}\left(1 - \frac{2M}{a_2^2\rho_0 l^2} f_{2a}^2(a,b)\right) = \omega_{1,2} = \omega_{2,1}, \tag{102}$$

$$\Omega_{2b} = \omega_{1,2}\left(1 - \frac{2M}{a_2^2\rho_0 l^2} f_{2b}^2(a,b)\right) = \omega_{1,2}\left(1 - \frac{2M}{\rho_0 l^2}\frac{(\sin^2\frac{\pi a}{l}\sin^2\frac{2\pi b}{l} + \sin^2\frac{2\pi a}{l}\sin^2\frac{\pi b}{l})^2}{\sin^2\frac{2\pi a}{l}\sin^2\frac{\pi b}{l} + \sin^2\frac{\pi a}{l}\sin^2\frac{2\pi b}{l}}\right)$$
$$= \omega_{1,2}\left[1 - \frac{8M}{\rho_0 l^2}\sin^2\frac{\pi a}{l}\sin^2\frac{\pi b}{l}\left(\cos^2\frac{\pi a}{l} + \cos^2\frac{\pi b}{l}\right)\right], \tag{103}$$

recalling that $f_{2a}(a,b) = 0$, and eqs. (89) and (91).

That is, the degeneracy of the unperturbed modes 1,2 and 2,1 has been broken by the perturbation.

9. We consider transverse vibrations of a circular membrane of radius a, mass density ρ per unit area, and surface tension T.

   For an area element $r\,dr d\theta$, $F = ma$ for transverse displacement s is, recalling that surface tension T is a force per unit length,

$$\rho\, r\, dr\, d\theta\, \ddot{s} = T\left(r\, d\theta \frac{\partial s}{\partial r}\bigg|_{r+dr} - r\, d\theta \frac{\partial s}{\partial r}\bigg|_{r} \right) + T\left(\frac{dr}{r}\frac{\partial s}{\partial \theta}\bigg|_{\theta+d\theta} - \frac{dr}{r}\frac{\partial s}{\partial \theta}\bigg|_{\theta} \right)$$

$$= T\left[d\theta\, dr \frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right) + \frac{dr}{dr}\, d\theta \frac{\partial}{\partial \theta}\left(\frac{\partial s}{\partial \theta}\right)\right] = dr\, d\theta\, T\left(r\frac{\partial^2 s}{\partial r^2} + \frac{\partial s}{\partial r} + \frac{1}{r}\frac{\partial^2 s}{\partial \theta}\right) \qquad (104)$$

$$\frac{1}{c^2}\frac{\partial^2 s}{\partial t^2} = \frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{1}{r^2}\frac{\partial^2 s}{\partial \theta^2}, \qquad \text{with} \qquad c^2 = \frac{T}{\rho}. \qquad (105)$$

We try a separation-of-variables solution, $s = f(r)g(\theta)h(t)$, to the wave equation (105),

$$\frac{fg\ddot{h}}{c^2} = \left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr}\right) gh + f\frac{d^2 g}{d\theta^2}h, \qquad (106)$$

$$\frac{\ddot{h}}{c^2 h} = \frac{1}{f}\left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr}\right) + \frac{1}{gr^2}\frac{d^2 g}{d\theta^2}. \qquad (107)$$

The left side of eq. (107) depends only on t, while the right side depends only on r and $theta$, so each side must be constant. We choose,

$$\frac{\ddot{h}}{c^2 h} = -\frac{\omega^2}{c^2} \equiv -k^2, \qquad h \propto \cos\omega t \text{ or } \sin\omega t. \qquad (108)$$

Then, eq. (107) can be rewritten as,

$$\frac{r^2}{f}\left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr}\right) + \frac{\omega^2 r^2}{c^2} = -\frac{1}{g}\frac{d^2 g}{d\theta^2}. \qquad (109)$$

The left side of eq. (109) depends only on r while the right side depends only on θ, so each side must be constant. We choose,

$$\frac{1}{g}\frac{d^2 g}{d\theta^2} = -n^2, \qquad g \propto \cos n\theta \text{ or } \sin n\theta, \qquad (110)$$

where the condition that $g(\theta) = g(\theta + 2\pi)$ implies that n is an integer. Now, the left side of eq. (109) can be rewritten as, with $k = \omega/c$

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \left(k^2 - \frac{n^2}{r^2}\right) f = 0, \quad \text{or} \quad \frac{d^2 f}{d(kr)^2} + \frac{1}{kr}\frac{df}{d(kr)} + \left(1 - \frac{n^2}{(kr)^2}\right) f = 0, (111)$$

which is Bessel's equation for (integer) order n, whose solutions that are finite at the origin are the Bessel functions of the first kind, $J_n(kr)$.

The boundary condition that $s(r = a) = 0$ requires that $J_n(ka) = 0$.

The lowest frequency corresponds to the smallest ka that is a zero of a Bessel function,[4] which is the first zero of J_0, *i.e.*, $ka = 2.4048$, so the lowest frequency is,

$$\omega = 2.4048\frac{c}{a}\,. \tag{112}$$

Without recourse to Bessel functions, we can estimate the lowest frequency using Rayleigh's method, pp. 235-236, Lecture 22 of the Notes.

The lowest-frequency mode will have the smoothest wavefunction, *i.e.*, $n = 0$, and hence have the form $s = f(r)\cos\omega t$. In addition to the boundary condition that $f(a) = 0$, the derivative of f must vanish at the origin so the waveform is smooth there. A simple polynomial form that satisfies these conditions is,

$$f(r) = a^p - r^p, \tag{113}$$

for some constant p (not necessarily an integer).

In Rayleigh's method, we equate the time-average kinetic and potential energies to find an expression for the angular frequency $\omega(p)$, which we then minimize with respect to p to find the lowest angular frequency ω.

The kinetic energy is,

$$\mathrm{KE} = \frac{\rho}{2}\int_0^a r\,dr\int_0^{2\pi} d\theta\,\dot{s}^2, \tag{114}$$

whose time-average for the form (113) is,

$$\begin{aligned}\langle \mathrm{KE}\rangle &= \frac{2\pi\rho\,\omega^2}{4}\int_0^a r\,dr\,(a^{2p} - 2a^p r^p + r^{2p}) = \frac{\pi\rho\,\omega^2}{2}\left(\frac{a^{2p+2}}{2} - \frac{2a^{2p+2}}{p+2} + \frac{a^{2p+2}}{2p+2}\right)\\ &= \frac{\pi\rho\,a^{2p+2}\,\omega^2}{4(p+1)(p+2)}((p+1)(p+2) - 4(p+1) + p + 2) = \frac{\pi\rho\,a^{2p+2}p^2\,\omega^2}{4(p+1)(p+2)}\,.\end{aligned} \tag{115}$$

The stored potential energy is the work done by the surface tension T (assumed to be independent of the small displacement s of the membrane) in stretched it from its rest configuration, $s = 0$, to a nonzero $s(r, \theta, t)$. This equals the surface tension times (the area of the membrane at time t minus the rest area πa^2),

$$\mathrm{PE} = T(\mathrm{Area}(t) - \pi a^2) = T\left(\int\int dl_r\,dl_\theta - \pi a^2\right), \tag{116}$$

where the arc lengths dl_r and dl_θ bounding an area element of the membrane that is displaced from the rest element $r\,dr\,d\theta$ are related by,

$$(dl_r)^2 = (dr)^2 + \left(\frac{\partial s}{\partial r}\right)^2 (dr)^2, \qquad dl_r \approx \left[1 + \frac{1}{2}\left(\frac{\partial s}{\partial r}\right)^2\right] dr, \tag{117}$$

$$(dl_\theta)^2 = (r\,d\theta)^2 + \left(\frac{\partial s}{\partial \theta}\right)^2 (d\theta)^2, \qquad dl_\theta \approx \left[1 + \frac{1}{2r^2}\left(\frac{\partial s}{\partial \theta}\right)^2\right] r\,d\theta. \tag{118}$$

[4]A table of zeroes of Bessel functions is at https://mathworld.wolfram.com/BesselFunctionZeros.html.

Hence,

$$\text{PE} \approx \frac{T}{2}\int_0^a r\,dr \int_0^{2\pi} d\theta \left[\left(\frac{\partial s}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial s}{\partial \theta}\right)^2\right]. \tag{119}$$

The time-average potential energy for the form (113) is,

$$\langle \text{PE} \rangle \approx \frac{2\pi T}{4}\int_0^a r\,dr \left(-p\,r^{p-1}\right)^2 = \frac{\pi p^2 T}{2}\frac{a^{2p}}{2p} = \frac{\pi a^{2p}\,p\,T}{4}. \tag{120}$$

For oscillatory modes, we equate the time-average energies ⟨KE⟩ and ⟨PE⟩ of eqs. (115) and (120) to find that,

$$\omega^2 = \frac{T}{\rho a^2}\frac{(p+1)(p+2)}{p} = \frac{T}{\rho a^2}\frac{p^2+3p+2}{p} = \frac{c^2}{a^2}\left(p+3+\frac{2}{p}\right). \tag{121}$$

To find the lowest frequency ω, we minimize eq. (121) with respect to p, which implies that $p^2 = 2$. Then, from eq. (121) we have,

$$\omega = \frac{c}{a}\sqrt{\sqrt{2}+3+\frac{2}{\sqrt{2}}} = 2.414\frac{c}{a}, \tag{122}$$

in good agreement with the "exact" result (112).

Once we have the expressions (114) and (119) for the kinetic and potential energies of the vibrating membrane, we can consider the Lagrangian,

$$L = \text{KE} - \text{PE} = \frac{\rho}{2}\int_0^a r\,dr\int_0^{2\pi} d\theta\,\dot{s}^2 - \frac{T}{2}\int_0^a r\,dr\int_0^{2\pi} d\theta\left[\left(\frac{\partial s}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial s}{\partial \theta}\right)^2\right]$$
$$\equiv \int_0^l \mathcal{L}\,dx. \tag{123}$$

The equation of motion follows from Hamilton's principle that $\delta\int \mathcal{L}\,dt$ for variations $s \to s+\eta$ around the physical displacement s, as on pp.. 240-241 of the Notes,

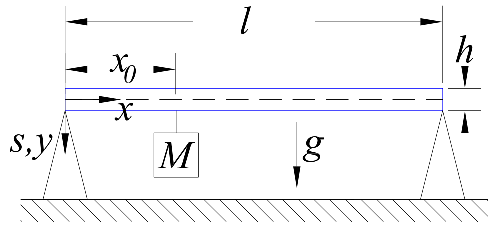
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{s}} - \frac{d}{dr}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{d}{d\theta}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial s}, \tag{124}$$

$$r\rho\,\ddot{s} - T\frac{d}{dr}\left(r\frac{\partial s}{\partial r}\right) - \frac{T}{r}\frac{\partial^2 s}{\partial \theta^2} = 0, \tag{125}$$

$$\ddot{s} = \frac{T}{\rho}\left(\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{\partial^2 s}{\partial \theta^2}\right), \tag{126}$$

as previously found in eq. (105).

10. We consider a rectangular beam of length l, width w and height h is supported at the same height at both ends. The supports do not constrain the slope of the beam at its end (such that $s''(\text{end}) = 0$). Mass M is hung at distance x_0 from one end.



We recall from p. 240, Lecture 22 of the Notes that the elastic potential energy of the displaced beam is,

$$V = \frac{YI}{2\rho} \int (s'')^2 \, dx, \tag{127}$$

where Y is the Young's modulus of the beam, I is the moment of inertial per unit length of a cross section of the beam about its horizontal midline, and ρ is the mass density per unit length.

Ignoring the deflection of the beam due to its own weight, and ignoring the variation in the deflection across the width of the beam, the Lagrangian of the system can be written as,

$$L = T - V_{\text{total}} = \int_0^l \left[\rho A(\ddot{s})^2 - \frac{YI}{2\rho} \int (s'')^2 - Mgs\delta(x - x_0) \right] dx \equiv \int_0^l \mathcal{L} \, dx, \tag{128}$$

taking displacement s to be positive downwards, and A to be the area of the cross section of the beam. Using Hamilton's principle, the equation of motion is,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{s}} - \frac{d^2}{dt\,dx}\frac{\partial \mathcal{L}}{\partial \dot{s}'} - \frac{d^2}{dx^2}\frac{\partial \mathcal{L}}{\partial s''} = \frac{\partial \mathcal{L}}{\partial s}, \tag{129}$$

$$\rho A\,\ddot{s} - I\ddot{s}'' + \frac{YI}{\rho} s'''' = Mg\,\delta(x - x_0). \tag{130}$$

For static equilibrium, this reduces to,

$$s'''' = \frac{Mg\rho}{YI}\,\delta(x - x_0). \tag{131}$$

The displacement $s(x)$, which vanishes at $x = 0$ and at $x = l$, can be written as a Fourier series of the form,

$$s(x) = \sum_n A_n \sin\frac{n\pi x}{l}. \tag{132}$$

The equation of motion (131) then tells us,

$$\sum_n \left(\frac{n\pi}{l}\right)^4 A_n \sin\frac{n\pi x}{l} = \frac{Mg\rho}{YI}\,\delta(x - x_0). \tag{133}$$

To evaluate the Fourier coefficients A_n, we multiply eq. (133) by $\sin m\pi x/l$ and integrate from 0 to l,

$$\sum_n \left(\frac{n\pi}{l}\right)^4 A_n \int_0^l \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = \frac{l}{2}\sum_n \delta_{mn}\left(\frac{n\pi}{l}\right)^4 A_n = \frac{l}{2}\left(\frac{m\pi}{l}\right)^4 A_m$$

$$= \frac{Mg\rho}{YI}\int_0^l \sin\frac{m\pi x}{l}\,\delta(x-x_0)\,dx = \frac{Mg\rho}{YI}\sin\frac{m\pi x_0}{l}\,, \qquad (134)$$

$$A_n = \frac{2}{l}\left(\frac{l}{n\pi}\right)^4 \frac{Mg\rho}{YI}\sin\frac{n\pi x_0}{l}\,. \qquad (135)$$

The moment of inertia I per unit length is given by,

$$I = \int_{-h/2}^{h/2} \rho w y^2\,dy = \frac{\rho w h^3}{12}\,, \qquad (136)$$

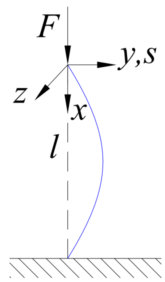
so the Fourier series for the displacement is,

$$s(x) = \frac{24Mgl^3}{\pi^4 Y w h^3}\sum_n \frac{1}{n^4}\sin\frac{n\pi x_0}{l}\sin\frac{n\pi x}{l}\,. \qquad (137)$$

The terms go as $1/n^4$, so the series converges quickly.

11. **Charlie Chaplin's Cane**

This problem was first solved by Euler (1744), sec. 25 of,
http://kirkmcd.princeton.edu/examples/mechanics/euler_E065g_44.pdf
http://kirkmcd.princeton.edu/examples/mechanics/euler_E065g_44_english.pdf
This solution is abstracted from § 17-21 of Landau and Lifshitz, *Theory of Elasticity*,
http://kirkmcd.princeton.edu/examples/mechanics/landau_e_70.pdf



For compressive force F, less than the critical force $F_{\rm crit}$ on the cane/beam, extending over $0 < x < l$, it remains straight. At the critical force, any small transverse displacement, of the form,

$$s(x) = s_0 \sin \frac{\pi x}{l}\,, \tag{138}$$

in the x-y plane is also at static equilibrium. Therefore, the solution investigates the conditions for static equilibrium of such a displacement.

For this, both the total force and torque on any segment of the beam must be zero.

On p. 239, Lecture 22 of the Notes we found that the potential energy stored in a short section of length dl of a bent, elastic beam is,

$$dV = \frac{YI}{2\rho}\frac{dl}{R^2}\,, \tag{139}$$

where Y is Young's modulus of elasticity, I is the moment of inertia per unit length of a cross section of the beam about its midline,[5] ρ is (volume) mass density and $1/R \approx s''$ is the radius of curvature of the displacement s from the undisturbed (straight) configuration.

Reviewing the argument leading to eq. (139), we see that,

$$\mathbf{B} = \frac{YI\,\hat{\mathbf{z}}}{\rho R} \approx \frac{YIs''\,\hat{\mathbf{z}}}{\rho}\,, \tag{140}$$

is the torque (often called the *bending moment*) about the centerline of a cross-sectional slice, due to the internal forces acting over positive-x side of its area (for a beam along x when undisturbed).

For a short segment of the beam, bounded by the cross sections at x and $x + dx$ when undeflected, the total torque due to the bending moments acting on the two cross sections is $d\mathbf{B} = \mathbf{B}(x + dx) - \mathbf{B}(x)$.

[5]Many authors, including Landau, define I as our I/ρ.

The positive-x sides of bounding cross sections of the segment are also acted upon by net forces $\mathbf{F}(x)$ and $\mathbf{F}(x+dx)$. The torque due to these forces, about, say, the center of the bounding cross section at $x+dx$ segment is $-d\mathbf{l} \times -\mathbf{F}(x) = d\mathbf{l} \times \mathbf{F}$, where $d\mathbf{l}$ is the vector element of arc length along the displaced beam.

If we can neglect external forces along the beam (as in the present problem), then $d\mathbf{F}/dl = 0$.

The torque equation for static equilibrium of a short segment of the beam is therefore,

$$d\mathbf{B} + d\mathbf{l} \times \mathbf{F} = 0, \tag{141}$$

where this equation is nontrivial only for the desired critical force $F_{\rm crit}$.

We divide eq. (141) by dl and denote $\hat{\mathbf{t}} = d\mathbf{l}/dl$ as the unit vector tangent to the beam,

$$\frac{d\mathbf{B}}{dl} + \hat{\mathbf{t}} \times \mathbf{F}_{\rm crit} = 0, \tag{142}$$

Next, we take the derivative of eq. (142) with respect to $\mathbf{l}$,

$$\frac{d^2\mathbf{B}}{dl^2} + \frac{d\hat{\mathbf{t}}}{dl} \times \mathbf{F}_{\rm crit} + \hat{\mathbf{t}} \times \frac{d\mathbf{F}_{\rm crit}}{dl} = 0, \tag{143}$$

For a small displacement s in the x-y plane, $\hat{\mathbf{t}} \approx (1, s', 0)$, and $d\hat{\mathbf{t}}/dl \approx d\hat{\mathbf{t}}/dx \approx (0, s'', 0)$. For compressive force $F_{\rm crit}$ applied in the x-direction at $x = 0$, the force $\mathbf{F}_{\rm crit}$ along the beam is approximately $-F_{\rm crit}\,\hat{\mathbf{x}}$,[6] so the z-component of eq. (143) is, recalling eq. (140) and that $d\mathbf{F}/dl = 0$ in this problem,

$$\frac{YIs''''}{\rho} + F_{\rm crit}s'' = 0. \tag{144}$$

Finally, for the form (138), we obtain the critical force as,

$$F_{\rm crit} = \frac{\pi^2 YI}{\rho l^2}\,. \tag{145}$$

This result assumed bending in the x-y plane. In case the moment of inertia I is smaller for bending in the x-z plane, that smaller value of I should be used in eq. (145). Of course, for a circular cane, the moment I is the same in the x-z and y-z planes.

[6]Remember that the internal force $\mathbf{F}$ is defined as that acting on the $+x$ side of a cross section of the beam, so a compressive force, which is positive on the $-x$ side of a cross section corresponds to $\mathbf{F}$ in the $-x$ direction.