Princeton University

Ph205
Mechanics
Problem Set 3

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(1988)

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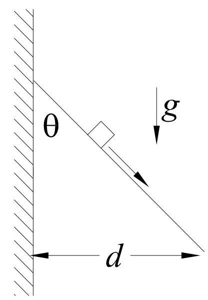
http://kirkmcd.princeton.edu/examples/

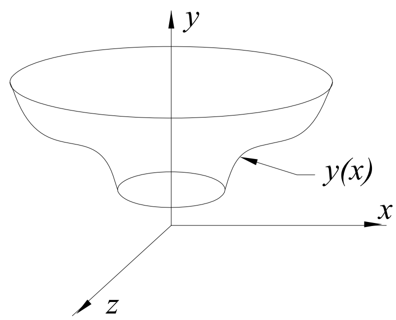
1. We wish to slide objects, initially at rest, down a straight, frictionless chute that begins at a vertical walls and ends at distance d from the wall. At what angle θ to the vertical should the chute be placed to minimize the time of descent?



If the chute were bent into a cycloid, the descent would be the fastest possible. It is agreeable to our intuition that a cusp of the cycloid must be at the wall, and the end point at the bottom of the cycloid. For a derivation, see sec. 3-11 of R. Weinstock, *Calculus of Variations (McGraw-Hill, 1952),*
http://kirkmcd.princeton.edu/examples/mechanics/weinstock_52.pdf
taking the reference curve to be the wall, $x = 0$.

Compare the times of descent for a straight chute at the angle θ found above, and for the cycloidal chute described above.

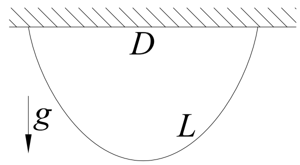
2. A curve $y(x)$ is rotated about the y-axis to make a surface of revolution bounded by two disks. What form of $y(x)$ produces the minimum surface area?



Based on our experience with the case of rotation of the curve about the x-axis, p. 55 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l5.pdf, we might expect the curve to be a catenary, of the form $x = A\cosh(y - B)$.

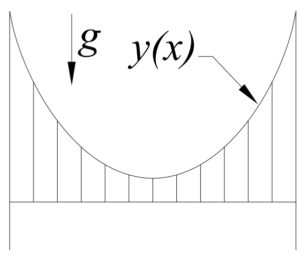
3. (a) **Hanging Rope**

A rope of length L and uniform mass is hung from two fixed points at the same height, separated by distance D. Use elementary methods to examine the condition of static equilibrium for a segment of the rope dx long (with the x-axis horizontal, y-axis vertical). First look at F_x, then F_y, to show that $y''/\sqrt{1+y'^2}$ = constant, $\Rightarrow y(x)$ is a catenary.



(b) **Suspension Bridge**

A massless cable is strung between two fixed points at the same height, separated by distance D, and (massless) vertical cables are attached to it to cary a uniform horizontal load – the bridge. What is the shape of the curving cable such that the tensions are equal in all vertical cables?



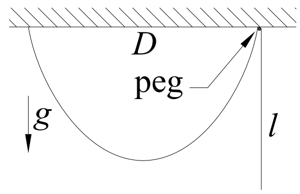
Assume an infinite number of evenly spaced vertical cables.

Use elementary methods to show that y'' = constant, $\Rightarrow$ parabola.

4. (a) **Hanging Rope**

   Use the calculus of variations to show that the form of a hanging rope, Prob. 3(a), is a catenary.

   (b) Suppose the rope of part (a) has one end attached to a fixed point while the other end drapes over a fixed, frictionless peg at the same height. What is the length l of the vertical segment of the rope, assuming no friction?



5. **Geodesics on a Sphere**

Full credit for working either part (a) or (b).

Find the curve on the surface of a sphere that is the shortest distance between two points.

(a) One approach is to reduce the problem to 2 dimensions. Parameterize the surface by two independent coordinates (u, v). Then, $(x(u, v), y(u, v), z(u, v))$ is the surface. Use the equation of the surface of the sphere to find functions $P(u, v)$, $Q(u, v)$, $R(u, v)$ such that an element of arc length on the sphere can be written as $ds^2 = dx^2 + dy^2 + dx^2 = P\, du^2 + 2Q\, du\, dv + R\, dv^2$.

A 1-dimensional curve on the surface of the sphere can be written as $v = v(u)$ so $ds = \sqrt{P + 2Qv' + Rv'^2}\, du$.

Try parameters $a =$ (constant) radius of sphere, $u = \phi =$ azimuthal angle, $v = \theta$ = polar angle. You should eventually find a solution of the form,

$$x \sin C_1 + y \cos C_1 - \frac{z}{\sqrt{a^2/C_2^2 - 1}} = 0, \qquad (1)$$

which is the equation of a plane passing through the center of the sphere. The intersection of this plane with the surface of the sphere is a great circle, which is the geodesic on a sphere.

(b) Stay in 3 dimensions and regard the surface of the sphere as a constraint. Let $(x(t), y(t), z(t))$ be the desired curved, and $g(x, y, z) = 0$ be the equation of the surface of the sphere. Then, $ds = f\, dt$ is an element of arc length, with $f = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$.

Use the calculus of variations to show that,

$$\frac{\frac{d}{dt}\left(\frac{\dot{x}}{f}\right)}{\partial g/\partial x} = \frac{\frac{d}{dt}\left(\frac{\dot{y}}{f}\right)}{\partial g/\partial y} = \frac{\frac{d}{dt}\left(\frac{\dot{z}}{f}\right)}{\partial g/\partial z} \qquad \Rightarrow \qquad \frac{\frac{d^2x}{ds^2}}{\partial g/\partial x} = \frac{\frac{d^2y}{ds^2}}{\partial g/\partial y} = \frac{\frac{d^2z}{ds^2}}{\partial g/\partial z}. \qquad (2)$$

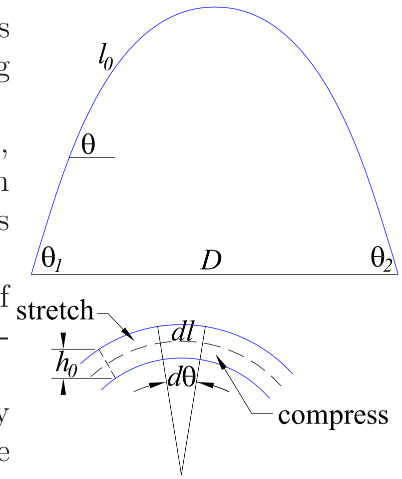
Referring to p. 12 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l1.pdf, the numerators are components of a vector normal to the curve (*i.e.*, $\hat{\mathbf{n}} \propto d\hat{\mathbf{s}}/ds$). The denominators are components of $\boldsymbol{\nabla} g$, which is normal to the surface. The equality of the ratios is consistent with the geometric picture of a Lagrange multiplier given on pp. 58-89 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l5.pdf.

For a sphere use the left form of eq. (2) to find the geodesic curve.

A trick is to write $\dot{f}/f$ two ways, giving expressions that can be integrated to yield logarithms. Rearrange and integrate again to show that $x + Ay + Bz = 0$ as in part (a).

6. **Meanders**

A flexible tape is bent into a planar curve by fixing the positions and slopes at its endpoints. The tape has length l_0 and thickness h_0 when undeformed, and spring constant k for linear stretching along its length.



In the bend, suppose the length along the midline remains l_0, and the thickness h_0, while above the midline (for a bend as in the figure) the tape is stretched, while below the midline it is compressed.

The tape assumes whatever shape requires the least work of deformation. That is, the stored potential energy will be a minimum.

Divide the wedge shown in the lower figure into slices, each a tiny spring. Deduce the spring constant of each slice, and integrate to show that the total potential energy is,

$$\text{PE} = \frac{k l_0 h_0^2}{24} \int_0^{l_0} \left(\frac{d\theta}{dl}\right)^2 dl. \tag{3}$$

We wish to minimize this, subject to a suitable constraint. But, noting that the length of the tape is l_0 won't help here. Instead, consider that the distance D between the endpoints is fixed,

$$D = \int_0^{l_0} \cos\theta \, dl = \text{constant}. \tag{4}$$

The other constraints, on the angles θ_1 and θ_2, can be applied once the general shape is known.

Use the calculus of variations to find $\theta(l)$. For small θ_1, show that $\theta \approx \theta_1 \cos(kl)$ for some constant k.

To get a sense for the more exact solution at large θ_1, Sketch 1 or 2 periods of the curve for $\theta_1 = 120°$. This curve appears on maps – as the shape of meandering rivers. See, for example,

http://kirkmcd.princeton.edu/examples/mechanics/leopold_sa_214-6_60_66.pdf

http://kirkmcd.princeton.edu/examples/mechanics/einstein_natur_14_223_26_english.pdf

7. **Components of Acceleration in a Non-Cartesian Coordinate System**

Suppose we use an orthogonal coordinate system (q_1, q_2, q_3) to describe the position of a point particle. Then, we can write the line element as,

$$ds^2 = ds_1^2 + ds_2^2 + ds_3^2, \qquad \text{where} \qquad ds_i = f_i(q_1, q_2, q_3)\, dq_i. \tag{5}$$

The velocity v is just $v = ds/dt$, so the kinetic energy is,

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 = \frac{1}{2}m\left[f_1^2\left(\frac{dq_1}{dt}\right)^2 + f_2^2\left(\frac{dq_2}{dt}\right)^2 + f_3^2\left(\frac{dq_3}{dt}\right)^2\right]. \tag{6}$$

Lagrange's equations of motion provide a quick way to deduce the components of acceleration in this coordinate system.

One form is,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j, \tag{7}$$

where for small displacements the generalized force Q_j obeys,

$$\sum_j Q_j\, \delta q_j = \mathbf{F} \cdot \delta \mathbf{r}. \tag{8}$$

If we decompose vectors $\mathbf{F}$ and $\delta\mathbf{r}$ in our new coordinate system, we can write,

$$\mathbf{F} = F_1\, \hat{\mathbf{q}}_1 + F_2\, \hat{\mathbf{q}}_2 + F_3\, \hat{\mathbf{q}}_3, \qquad \text{and} \qquad \delta\mathbf{r} = \delta r_1\, \hat{\mathbf{q}}_1 + \delta r_2\, \hat{\mathbf{q}}_2 + \delta r_3\, \hat{\mathbf{q}}_3. \tag{9}$$

But, by definition of the line element,

$$\delta r_j = \delta s_j = f_j\, \delta q_j, \qquad \text{such that} \qquad \sum_j Q_j\, \delta q_j = \sum_j F_j f_j\, \delta q_j, \tag{10}$$

so,

$$Q_j = F_j f_j, \tag{11}$$

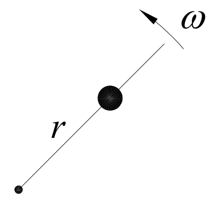
relates the generalized force $\mathbf{Q}$ to the ordinary force $\mathbf{F}$.

Meanwhile, Newton tells us that $F_j = m a_j$, so the component of the acceleration in the $\hat{\mathbf{q}}_j$ direction is,

$$a_j = \frac{Q_j}{m f_j} = \frac{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}}{m f_j}. \tag{12}$$

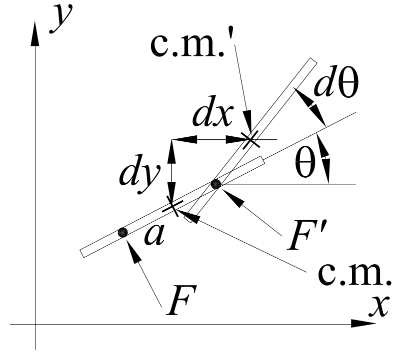
Use these tricks to compute the forms of the a_j in cylindrical and spherical coordinates systems. Compare with the results given on pp. 9-10 of the Ph205 Lecture Notes.

8. Find the constraint force on a mass m slides without friction along a rod that is constrained to rotate in a plane with constant angular velocity ω (Ph205 Set 5, Prob.). Use both elementary methods, and the method of Lagrange multipliers.



9. **The Return of the One-Legged Ice Skater**

Consider the model of a one-legged ice skater introduced on p. 70 of the Notes, but now suppose that the constraint force $\mathbf{F}$ is not applied at the center of mass, but at distance a from it. Recall that the motion of the skate consists of a slide along the direction of the skate combined with a rotation about the point of application of the constraint force. We still have that $\mathbf{F} \cdot \mathbf{v}_{\rm c.m.} = 0$, so the constraint force does no work, and the kinetic energy of the ice skater is constant. If $v_{\rm c.m.}$ decreases, $\dot\theta$ increases, and the possibility exists that the skater comes to rest, with a higher angular velocity.



Find the equations of motion of the ice skate by elementary methods, or by Lagrange's method (using a Lagrange multiplier to include the non-holonomic constraint force).[1]

To integrate the equations of motion, you may wish to replace $\dot x_{\rm c.m.}$ and $\dot y_{\rm c.m.}$ by functions of $\mathbf{v}_F$ and θ where $\mathbf{v}_F$ is the velocity of the point of application of the constraint force (rather than $\mathbf{v}_{\rm c.m.}$). Write the moment of inertia about the center of mass as $I = mb^2$ and let $k^2 = 1 + b^2/a^2$. In general, k is not constant, but it suffices to suppose that it is.

You should find that,

$$ak^2\ddot\theta + v_F\dot\theta = 0, \qquad \text{and} \qquad \dot v_F = a\dot\theta^2. \tag{13}$$

Differentiate and combine, then multiply by $\ddot\theta/\dot\theta$ to find,

$$k^2\frac{d}{dt}\left(\frac{\ddot\theta}{\dot\theta}\right)^2 = -\frac{d\dot\theta^2}{dt}. \tag{14}$$

Integrate twice (and then a third time) to show that,

$$\dot\theta = \frac{c}{\cosh(ct/k)} = c\cos\frac{\theta}{k}, \tag{15}$$

where c is a constant. Hence,

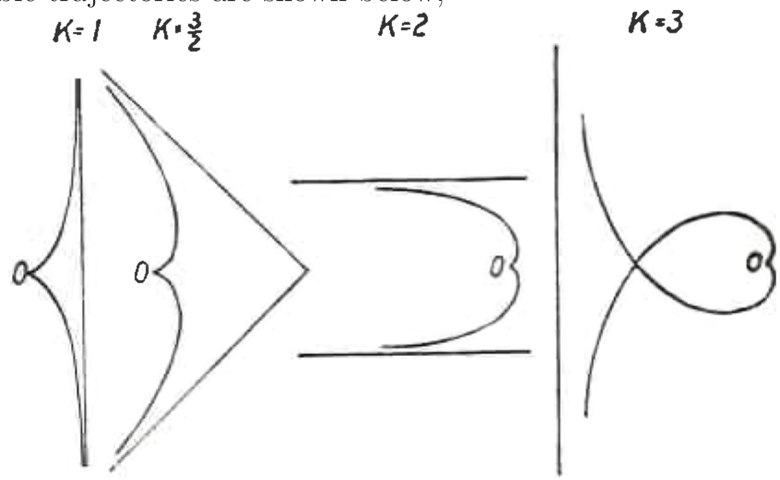
$$v_F = ack\tanh\frac{ct}{k} = ack\sin\frac{\theta}{k}, \tag{16}$$

[1]The term holonomic was introduced by Hertz on p. 91 of *Die Prinzipien der Mechanik* (Barth, Leipzig, 1894), http://kirkmcd.princeton.edu/examples/mechanics/hertz_mechanik_94.pdf

which implies that v_F and θ go to constant values as $t \to \pm\infty$.

Show that at $t = 0$, $dx_F/d\theta = dy_F/d\theta = d^2y_F/d\theta^2 = 0$, while $d^2x_F/d\theta^2 \neq 0$ and $d^3y_F/d\theta^3 \neq 0$. This implies that the trajectory has a cusp at the origin at $t = 0$, pointing in the negative-x direction.

Some possible trajectories are shown below,[2]



Can real ice skaters produce these shapes?

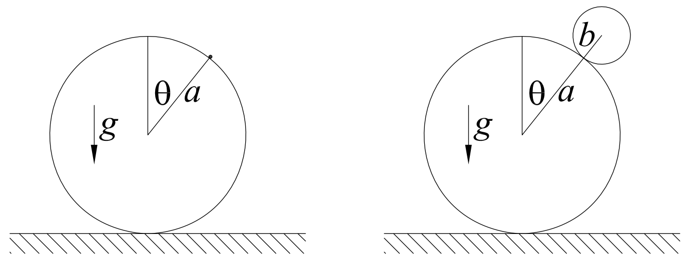
You should be able to verify that the kinetic energy is,

$$T = \frac{ma^2c^2k^2}{2}, \qquad \text{and that} \qquad F = \frac{c^2 I}{2k} \sin\frac{2\theta}{k}. \tag{17}$$

[2]C. Carathéodory, *Der Schlitten*, Z. Angew. Math. Mech. **13**, 71 (1933), http://kirkmcd.princeton.edu/examples/mechanics/caratheodory_zamm_13_71_33.pdf
A. Sommerfeld, *Mechanics* (Academic Press, 1952), p. 251, http://kirkmcd.princeton.edu/examples/mechanics/sommerfeld_mechanics_52.pdf

10. (a) A particle starts from rest at the top of a frictionless sphere of radius a and slides down. When it flies off, the normal force vanishes. Use the method of Lagrange multipliers to show that this occurs when $\cos\theta = 2/3$, where θ is the polar angle of the particle.

This problem was posed on a Ph103 Learning Guide, sans Lagrange.



(b) A uniform sphere of radius b starts at rest from the top of a fixed sphere of radius a, and rolls without slipping down the latter. Show by any method that the upper sphere flies off when $\cos\theta = 10/17 = 0.59$, so at a larger angle than for case (a).

For the more complicated case where the lower sphere/cylinder can roll without slipping, see http://kirkmcd.princeton.edu/examples/2cylinders.pdf
Perhaps surprisingly, there are cases where the inital rotations of the two spheres/cylinders have opposite senses, but change to the same sense before the upper sphere/cylinder flies off the lower.

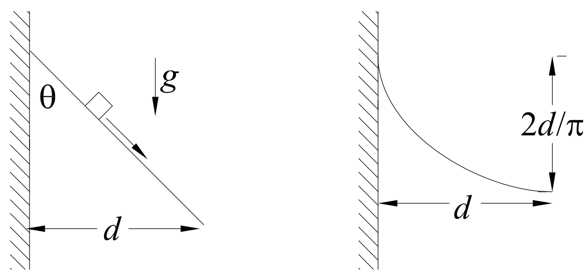
Solutions

1. The acceleration of an object down a frictionless chute at angle θ to the vertical is $a = g\cos\theta$, where g is the acceleration due to gravity. If the chute ends at distance d from the vertical wall, its length is $l = d/\sin\theta$. The time of the descent (from rest at the top of the chute) is related by $l = at^2/2$, *i.e.*,

$$t^2 = \frac{4d}{g\sin 2\theta}\,, \qquad 2t\frac{dt}{d\theta} = -\frac{8d\cos 2\theta}{g\sin^2 2\theta}\,, \tag{18}$$

so the time is minimal when $dt/d\theta = 0$, *i.e.*, for $\cos 2\theta = 0$, $\theta = 45°$. The time of descent is,

$$t = 2\sqrt{\frac{d}{g}}\,. \tag{19}$$



The least time of descent from the wall to a point at distance $x = d$ from is obtained a cycloid, with a cusp at the top of the (curved) chute, say $(x, y) = (0, 0)$, and its bottom at $(d, -2b)$. This cycloid can be parameterized by,

$$x = b(\phi - \sin\phi), \qquad y = -b(1 - \cos\phi). \tag{20}$$

The bottom of the cycloid is at $\phi = \pi$, such that $b = d/\pi$.

When the object is at height $y < 0$, after starting from rest at $y = 0$, its speed is $v = \sqrt{2g\,|y|} = \sqrt{2gb(1 - \cos\phi)}$.

The time to slide down a segment ds of the cycloid at height y is,

$$dt = \frac{ds}{v} = \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gb(1 - \cos\phi)}} = \frac{b\sqrt{(1 - \cos\phi)^2 + \sin^2\phi}}{\sqrt{2gb(1 - \cos\phi)}}\,d\phi = \sqrt{\frac{b}{g}}\,d\phi = \sqrt{\frac{d}{\pi g}}\,d\phi. \tag{21}$$

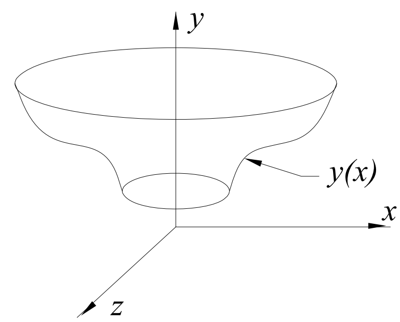
The total travel time from $\phi = 0$ to π is,

$$t = \int_0^\pi \frac{dt}{d\phi}\,d\phi = \sqrt{\frac{\pi d}{g}} = 1.77\sqrt{\frac{d}{g}} < 2\sqrt{\frac{d}{g}}\,. \tag{22}$$

However, the descent using a cycloid is only 11% faster than that using a 45° straight chute.

2. The surface of revolution of curve $y(x)$ about the y-axis has area,

$$A = 2\pi \int_{x_1}^{x_2} x\sqrt{1 + y'^2}\, dx = 2\pi \int f(x, y')\, dx. \tag{23}$$



To minimize the area, we consider the Euler-Lagrange equation for the function f,

$$\frac{\partial f}{\partial y} = 0 = \frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{d}{dx}\frac{xy'}{\sqrt{1 + y'^2}}\,. \tag{24}$$

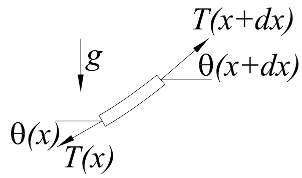
Hence,

$$\frac{xy'}{\sqrt{1 + y'^2}} = C, \qquad y' = \frac{C}{\sqrt{x^2 - C^2}}\,, \qquad y = \cosh^{-1}\frac{x}{C} + B\,, \qquad x = C\cosh(y - B), \tag{25}$$

using Dwight 260.01.

3. (a) **Hanging Rope**

We consider an element of the hanging rope, as sketched below.



For static equilibrium, the horizontal force equation is,

$$T(x)\cos\theta(x) = T(x+dx)\cos\theta(x+dx) \approx T(x)\cos\theta(x) + \frac{d}{dx}(T\cos\theta)\,dx, \quad (26)$$

which tells us that,

$$d(T\cos\theta) = 0, \qquad T\cos\theta = T_0. \quad (27)$$

Similarly, the vertical force equation is, with $\tan\theta = dy/dx = y'$,

$$\frac{Mg}{L}\sqrt{1+y'^2}\,dx = T(x+dx)\sin\theta(x+dx) - T(x)\sin\theta(x)$$

$$= T_0\tan\theta(x+dx) - T_0\tan\theta(x) \approx T_0\frac{d}{dx}(\tan\theta)\,dx = T_0\frac{dy'}{dx}\,dx = T_0 y''\,dx, \quad (28)$$

$$\frac{y''}{\sqrt{1+y'^2}} = \frac{Mg}{LT_0} = \text{constant}. \quad (29)$$

This the differential equation for a catenary = shape of the hanging rope.

(b) **Suspension Bridge**

The analysis for part(a) holds for a suspension bridge as well, except that the left side of eq. (28) is just $Mg\,dx/L$ for a bridge of length M and mass M which is uniform in x. Hence $y'' =$ constant, and the shape of the long supporting cable is a parabola.

4. (a) **Hanging Rope**

To analyze the hanging rope using the calculus of variations, we minimize the potential energy,

$$V = \int \rho g y \sqrt{1 + y'^2}\, dx = \rho g \int F\, dx, \tag{30}$$

where ρ is the mass density of the rope, subject to the constraint on the length of the rope,

$$L = \int \sqrt{1 + y'^2}\, dx = \int G\, dx. \tag{31}$$

We consider the "Lagrangian" function $F^*(x, y, y') = \rho g F + \rho g \lambda G$, where λ is a to-be-determined Lagrange multiplier (written so that we can ignore the constant factor ρg in the following).

Since $\partial F^*/\partial x = 0$, we know there exists a conserved (constant) quantity, the "Hamiltonian",

$$H = y' \frac{\partial F^*}{\partial y'} - F^* = \frac{y'^2(y+\lambda)}{\sqrt{1+y'^2}} - (y+\lambda)\sqrt{1+y'^2}, \tag{32}$$

$$H\sqrt{1+y'^2} = y'^2(y+\lambda) - (y+\lambda)(1+y'^2) = -(y+\lambda), \tag{33}$$

$$y' = \frac{dy}{dx} = \frac{\sqrt{(y+\lambda)^2 - H^2}}{H}, \tag{34}$$

$$x = H \int \frac{dy}{\sqrt{(y+\lambda)^2 - H^2}} = H \cosh^{-1} \frac{y+\lambda}{H} + C, \tag{35}$$

$$y = H \cosh \frac{x - C}{H} - \lambda, \tag{36}$$

using Dwight 260.01, which is the form of a catenary.

If we had not noticed the "conservation law" (32), use of the Euler-Lagrange equation would have led to a second-order differential equation, and the same result (36) after somewhat greater effort. See the handwritten solution at http://kirkmcd.princeton.edu/examples/Ph205/ph205sol3.pdf

The shape of the rope is symmetric about the line $x = D/2$, so $C = D/2$.

$y(0) = 0 \;\Rightarrow\; \lambda = H\cosh(D/2H)$.

H can be determined from the transcendental equation,

$$L = \int_0^D \sqrt{1+y'^2}\, dx = \int_{-D/2}^{D/2} \cosh \frac{x}{H}\, dx = 2H \sinh \frac{D}{2H}\,. \tag{37}$$

(b) For the rope of total length L with length $l < L$ hanging over a peg with one end fixed, we minimize the potential energy,

$$\text{PE} = \int_0^D \rho g y\, dl - \rho l g \frac{l}{2} = \int_0^D \rho g y \sqrt{1+y'^2}\, dx - \frac{\rho l^2 g}{2}, \tag{38}$$

subject to the constraint on the length,

$$L = \int_0^D dl + l = \int_0^D \sqrt{1 + y'^2}\, dx + l. \tag{39}$$

For the calculus of variations, we consider,

$$F^*(x, y, y', l) = \int_0^D \rho g y \sqrt{1 + y'^2}\, dx - \frac{\rho l^2 g}{2} + \lambda \rho g \int_0^D \sqrt{1 + y'^2}\, dx + \lambda l, \tag{40}$$

where λ is a Lagrange multiplier to be determined.
The variational analysis in the independent variable l is simply $dF^*/dl = 0$, which implies that $\lambda = l$.
The variational analysis in y is now the same as in part (a), so,

$$y = H \cosh \frac{x - C}{H} - \lambda = H \cosh \frac{x - D/2}{H} - l. \tag{41}$$

We can use eq. (35) with $(x, y) = (0, 0)$ to find,

$$\lambda = l = H \cosh \frac{D}{2H}\,. \tag{42}$$

The length calculation now gives the transcendental equation,

$$L - l = L - H \cosh \frac{D}{2H} = \int_0^D \sqrt{1 + y'^2}\, dx = 2H \sinh \frac{D}{2H}\,. \tag{43}$$

H is less than in part (a).

5. (a) We write the surface of the sphere of radius a as $x = (x(\theta,\phi),\ y = y(\theta,\phi,\ z = z(\theta),\phi)$ where θ and ϕ are the polar and azimuthal angles of a spherical coordinate system with z as the polar axis. An element of arc length on the surface is,

$$ds^2 = dx^2 + dy^2 + dz^2. \tag{44}$$

With,

$$dx = \frac{\partial x}{\partial\theta}\,d\theta + \frac{\partial x}{\partial\phi}\,d\phi, \qquad dx^2 = \left(\frac{\partial x}{\partial\theta}\right)^2 d\theta^2 + 2\frac{\partial x}{\partial\theta}\frac{\partial x}{\partial\phi}\,d\theta\,d\phi + \left(\frac{\partial x}{\partial\phi}\right)^2 d\phi^2, \tag{45}$$

etc., we can write,

$$ds^2 = P\,d\theta^2 + 2Q\,d\theta\,d\phi + R\,d\phi^2, \tag{46}$$

where,

$$P = \left(\frac{\partial x}{\partial\theta}\right)^2 + \left(\frac{\partial y}{\partial\theta}\right)^2 + \left(\frac{\partial z}{\partial\theta}\right)^2, \tag{47}$$

$$Q = \frac{\partial x}{\partial\theta}\frac{\partial x}{\partial\phi} + \frac{\partial y}{\partial\theta}\frac{\partial y}{\partial\phi} + \frac{\partial z}{\partial\theta}\frac{\partial z}{\partial\phi} \tag{48}$$

$$R = \left(\frac{\partial x}{\partial\phi}\right)^2 + \left(\frac{\partial y}{\partial\phi}\right)^2 + \left(\frac{\partial z}{\partial\phi}\right)^2. \tag{49}$$

On the surface of the sphere,

$$x = a\sin\theta\cos\phi, \qquad y = a\sin\theta\sin\phi, \qquad z = a\cos\theta, \tag{50}$$

so,

$$P = a^2\sin^2\theta, \qquad Q = 0, \qquad R = a^2. \tag{51}$$

and for a curve $\theta = \theta(\phi)$ on the surface of the sphere, its arc length is,

$$L = \int ds = a\int_{\phi_1}^{\phi_2} \sqrt{\sin^2\theta + \theta'^2}\,d\phi, \tag{52}$$

where $\theta' = d\theta/d\phi$. To minimize L, we consider the function $F(\phi,\theta') = \sqrt{\sin^2\theta + \theta'^2}$ in the Euler-Lagrange analysis, with ϕ as the independent variable.
Since $\partial F/\partial\phi = 0$, the "Hamiltonian" H is a conserved quantity,

$$H = \phi'\frac{\partial F}{\partial\theta'} - F = \frac{\theta'^2}{\sqrt{\sin^2\theta + \theta'^2}} - \sqrt{\sin^2\theta + \theta'^2}. \tag{53}$$

$$\sin^2\theta = H\sqrt{\sin^2\theta + \theta'^2}, \qquad \sin^4\theta = H^2(\sin^2\theta + \theta'^2), \tag{54}$$

$$\theta' = \frac{d\theta}{d\phi} = \sqrt{\frac{\sin^4\theta}{H^2} - \sin^2\theta}, \qquad d\phi = \frac{d\theta}{\sin\theta\sqrt{\frac{\sin^2\theta}{H^2} - 1}} \tag{55}$$

$$\int d\phi = \phi + C = -\tan^{-1}\frac{\cos\theta}{\sqrt{\frac{\sin^2\theta}{H^2} - 1}}, \tag{56}$$

using 2.599.6 of Gradshteyn and Ryzhik,
https://physics.princeton.edu/~mcdonald/examples/mechanics/gradshteyn_80.pdf.
Then, recalling that identity that,

$$\tan^{-1}\frac{a}{b} = \sin^{-1}\frac{a}{\sqrt{a^2+b^2}}\,, \tag{57}$$

we have that

$$\phi + C = -\sin^{-1}\frac{\cos\theta}{\sqrt{\cos^2\theta + \frac{\sin^2\theta}{H^2} - 1}} = -\sin^{-1}\frac{\cot\theta}{\sqrt{\frac{1}{H^2}-1}}\,, \tag{58}$$

$$\sin(\phi + C) = \sin\phi\cos C + \cos\phi\sin C = \frac{\cot\theta}{\sqrt{\frac{1}{H^2}-1}}. \tag{59}$$

Multiplying this by $a\sin\theta$, we obtain an equation for the curve as a function of (x, y, z),

$$0 = a\sin\theta(\sin\phi\cos C + \cos\phi\sin C) - \frac{a\ cos\theta}{\sqrt{\frac{1}{H^2}-1}}x\cos C + y - \frac{z}{\sqrt{\frac{1}{H^2}-1}}\,, \tag{60}$$

which is a plane passing through the origin. The geodesic curve is the intersection of this plane with the surface of the sphere, *i.e.*, a great circle.

(b) For an analysis that uses the three spatial coordinates (x, y, x), we consider a curve $(x(t), y(t), z(t))$ whose length is,

$$L = \int_{t_1}^{t_2} F\,dt, \qquad \text{where} \qquad F = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}, \tag{61}$$

with the constraint,

$$G(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0, \tag{62}$$

that the curve lie on a sphere of radius a centered on the origin. For the Euler-Lagrange method we consider the function,

$$F^*(x, y, y, \dot{x}, \dot{y}, \dot{z}; t) = F + \lambda G, \tag{63}$$

where λ is a Lagrange multiplier. Then,

$$\frac{d}{dt}\frac{\partial F^*}{\partial \dot{x}} = \frac{d}{dt}\frac{\dot{x}}{F} = \frac{\ddot{x}}{F} - \frac{\dot{x}\dot{F}}{F^2} = \frac{\partial F^*}{\partial x} = 2\lambda x, \qquad \ddot{x} = \dot{x}\frac{\dot{F}}{F} + 2\lambda x F, \tag{64}$$

together with similar equations for y and z.
We now perform a "hat trick", and introduce the 3 quantities,

$$J_x = y\dot{z} - z\dot{y}, \qquad J_y = z\dot{x} - x\dot{z}, \qquad J_z = x\dot{y} - y\dot{x}, \tag{65}$$

Their derivatives are, using all three versions of eq. (64),

$$\dot{J}_x = \dot{y}\dot{z} + y\ddot{z} - \dot{z}\dot{y} - z\ddot{y} = y\dot{z}\frac{\dot{F}}{F} + 2\lambda yzF - z\dot{y}\frac{\dot{F}}{F} - 2\lambda zyF = J_x\frac{\dot{F}}{F}\,, \tag{66}$$

$$\dot{J}_y = J_y\frac{\dot{F}}{F}\,, \qquad \dot{J}_z = J_z\frac{\dot{F}}{F}\,. \tag{67}$$

We can integrate these to find,

$$\ln J_i = \ln F + C_i, \qquad J_i = D_i F, \tag{68}$$

where the D_i are constants. Another trick is to note that,

$$\begin{aligned} xJ_x + yJ_y + zJ_z &= x(y\dot{z} - z\dot{y}) + y(z\dot{z} - x\dot{z}) + z(x\dot{y} - y\dot{x}) = 0 \\ &= xD_xF + yD_yF + zD_zF. \end{aligned} \tag{69}$$

Finally, we obtain,

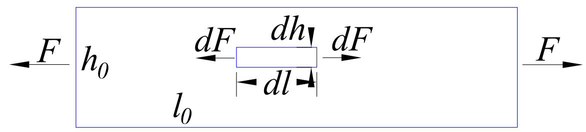
$$D_x x + D_y y + D_z z = 0,. \tag{70}$$

which is the equation of a plane that passes through the origin. Again, the geodesic curve is the intersection of such a plane with the sphere = a great circle.

6. The elastic tape an unstretched length l_0, thickness h_0, and spring constant k for stretching along its length.

   If the tape is stretched by force F, an element of the tape of size $dh \times dl$ is stretched by force,

$$dF = F\frac{dh}{h_0}, \tag{71}$$



The entire tape is stretched by length $\Delta l = F/k$, while the small element is stretched by,

$$\Delta(dl) = \Delta l \frac{dl}{l_0} = \frac{dF}{k'}, \tag{72}$$

where k' is the spring constant of the element,

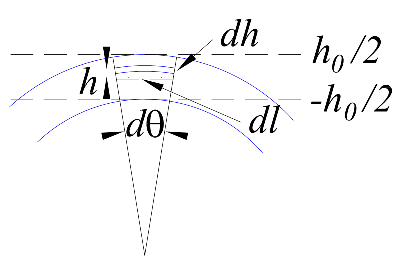
$$k' = \frac{dF}{\Delta(dl)} = F\frac{dh}{h_0}\frac{l_0}{\Delta l\, dl} = k\Delta l \frac{dh}{h_0}\frac{l_0}{\Delta l\, dl} = k\frac{dh}{h_0}\frac{l_0}{dl}. \tag{73}$$

A segment of length dl of the bent tape is illustrated in the figure below, assuming that the bend lies in a plane. We subdivide this segment into slices of thickness dh, at height h relative to the centerline of the tape. The stretch/compression of such a slice is,

$$\Delta(dl) = d\theta\left(\frac{dl}{d\theta} + h\right) - dl = h\, d\theta, \tag{74}$$

so the potential energy stored in the slice is,

$$d^2\mathrm{PE} = \frac{1}{2}k'\Delta^2(dl) = k\frac{dh}{h_0}\frac{l_0}{dl}\frac{dh}{h_0}\frac{l_0}{dl}h^2\, d\theta^2 = \frac{kl_0}{2h_0}\left(\frac{d\theta}{dl}\right)^2 dh\, dl. \tag{75}$$



Integrating over the slices in the segment of length dl,

$$d\mathrm{PE} = \frac{kl_0}{2h_0}\left(\frac{d\theta}{dl}\right)^2 dl \int_{-h_0/2}^{h_0/2} h^2\, dh = \frac{kh_0^2 l_0}{24}\left(\frac{d\theta}{dl}\right)^2 dl, \tag{76}$$

and the total potential energy in the bent tape is,

$$\mathrm{PE} = \frac{kh_0^2 l_0}{24}\int_0^{l_0}\left(\frac{d\theta}{dl}\right)^2 dl. \tag{77}$$

We can now consider the shape of the bent tape as described by the function $\theta(l)$, and minimize the potential energy subject to the constraint that the ends of the tape are separated by distance,

$$D = \int \cos\theta \, dl, \tag{78}$$

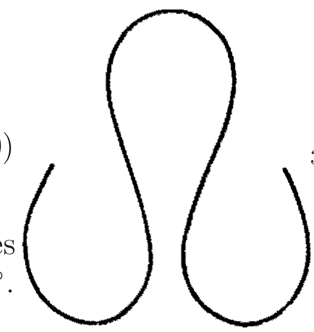
for angle θ measured with respect to the line joining the endpoints of the tape. The function F is $F(\theta'; l) = \theta'^2$ and the constraint function is $G(\theta; l) = \cos\theta$, so we use the calculus of variations for the combined function $F^* \ \theta'^2 + \lambda\cos\theta$, where λ is a Lagrange multiplier. The Euler-Lagrange equation for this is,

$$\frac{d}{dl}\frac{\partial F^*}{\partial\theta'} = 2\theta'' = \frac{\partial F^*}{\partial\theta} = -\lambda\sin\theta, \qquad \theta'' + \frac{\lambda}{2}\sin\theta = 0, \tag{79}$$

which equation is familiar from the simple pendulum at large amplitudes.

For small amplitudes, an oscillatory solution has the form,

$$\theta = \theta_{\max}\cos\left(\sqrt{\frac{\lambda}{2}}l\right). \tag{80}$$


[3]

While the form (80) does not hold exactly for large $\theta_{\max}$, it gives a sense of the solution. The figure to the right is for $\theta_{\max} = 110°$.

CATASTROPHIC EXAMPLE of a sine-generated curve on a much larger scale was provided by the wreck of a Southern Railway freight train near Greenville, S.C., on May 31, 1965. Thirty adjacent flatcars carried as their load 700-foot sections of track rails chained in a bundle to the car beds. The train, pulled by five locomotives, collided with a bulldozer and was derailed. The violent compressive strain folded the trainload of rails into the drastically foreshortened configuration shown in this aerial photograph. The elastic properties of the steel rails tended to minimize total bending exactly as in the case of the spring-steel strip shown at top of these two pages, and the wrecked train assumed the shape of a sine-generated curve that distributed the bending as uniformly as possible. [4]

[3]http://kirkmcd.princeton.edu/examples/mechanics/leopold_gspp_422-h_66.pdf

[4]http://kirkmcd.princeton.edu/examples/mechanics/leopold_sa_214-6_60_66.pdf

7. In an orthogonal coordinate system (q_1, q_2, q_3), the line element has the form,

$$ds^2 = f_1^2\,dq_1^2 + f_2^2\,dq_2^2 + f_3^2\,dq_3^2. \tag{81}$$

In cylindrical coordinates (r, ϕ, z), the line element is,

$$ds^2 = dr^2 + r^2\,d\phi^2 + dz^2, \tag{82}$$

so,

$$f_r = 1, \qquad f_\phi = r, \qquad f_z = 1. \tag{83}$$

The kinetic energy of a point particle is,

$$T = \frac{m}{2}\left[f_1^2\left(\frac{dq_1}{dt}\right)^2 + f_2^2\left(\frac{dq_2}{dt}\right)^2 + f_3^2\left(\frac{dq_3}{dt}\right)^2\right] = \frac{m}{2}(\dot r^2 + r^2\dot\phi^2 + \dot z^2), \tag{84}$$

The components of the acceleration are given by,

$$a_j = \frac{\frac{d}{dt}\frac{\partial T}{\partial \dot q_j} - \frac{\partial T}{\partial q_j}}{m f_j}, \tag{85}$$

$$a_r = \ddot r - r\dot\phi^2, \qquad a_\phi = \frac{r^2\ddot\phi + 2r\dot r\dot\phi}{r} = r\ddot\phi + 2\dot r\dot\phi, \qquad a_z = \ddot z. \tag{86}$$

In spherical coordinates (r, θ, ϕ), the line element is,

$$ds^2 = dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2, \tag{87}$$

so,

$$f_r = 1, \qquad f_\theta = r, \qquad f_\phi = r\sin\theta. \tag{88}$$

The kinetic energy of a point particle is,

$$T = \frac{m}{2}(\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\phi^2), \tag{89}$$

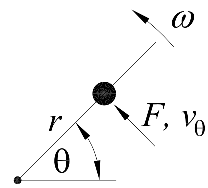
The components of the acceleration are given by eq. (85) as,

$$a_r = \ddot r - 2r\dot\theta^2 - r\sin^2\theta\,\dot\phi^2, \tag{90}$$

$$a_\theta = \frac{r^2\ddot\theta + 2r\dot r\dot\theta - r^2\sin\theta\cos\theta\,\dot\phi^2}{r} = r\ddot\theta + 2\dot r\dot\theta - r\sin\theta\cos\theta\,\dot\phi^2, \tag{91}$$

$$a_\phi = \frac{r^2\sin^2\theta\,\ddot\phi + 2r\dot r\sin^2\theta\,\dot\phi + 2r^2\sin\theta\cos\theta\,\dot\theta\dot\phi}{r\sin\theta} = r\sin\theta\,\ddot\phi + 2\dot r\sin\theta\,\dot\phi + 2r\cos\theta\,\dot\theta\dot\phi. \tag{92}$$

8. The constraint force $\mathbf{F}$ is parallel to the velocity component $\mathbf{v}_\theta = \boldsymbol{\omega} \times \mathbf{r}$, and $\mathbf{F} \cdot \mathbf{v}_\theta$ equals the rate of change of kinetic energy of the sliding mass m,



$$T = \frac{mv^2}{2} = \frac{1}{2}m(v_r^2 + v_\theta^2) = \frac{1}{2}m(\dot{r}^2 + \omega^2 r^2), \tag{93}$$

$$\mathbf{F} \cdot \mathbf{v}_\theta = F\omega r = \frac{dT}{dt} = m(\dot{r}\ddot{r} + \omega^2 r\dot{r}) = 2m\omega^2 r\dot{r}, \tag{94}$$

$$F = 2m\omega\dot{r}, \tag{95}$$

recalling that the centripetal acceleration is $\ddot{r} = \omega^2 r$.

Taking the generalized coordinates of mass m to be r and θ, the Lagrangian is,

$$L = T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right), \tag{96}$$

subject to the constraint $g = \dot{\theta} - \omega = 0$. We define $a_r = \partial g/\partial r = 0$ and $a_\theta = \partial g/\partial\theta = 1$. Then, in Ferrers' variant of the method of Lagrange,[5] we associate a Lagrange multiplier λ with the constraint force, and write the equations of motion as,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = m\ddot{r} - mr\dot{\theta}^2 = Q_r = \lambda a_r = 0, \tag{97}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = Q_\theta = \lambda a_\theta = \lambda. \tag{98}$$

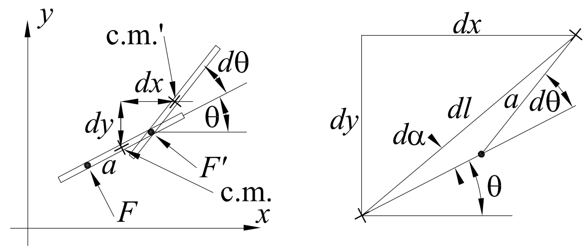
where $Q_\theta = \lambda$ is the generalized constraint force associated with coordinate θ. With $\dot{\theta} = \omega =$ constant, $\ddot{\theta} = 0$, we obtain $\ddot{r} = \omega^2 r$ and,

$$Q_\theta = 2m\omega r\dot{r}. \tag{99}$$

This generalized force is a torque, r times the constraint force $F = 2m\omega\dot{r}$.

[5] N.M. Ferrers, *Extension of Lagrange's Equations*, Quart. J. Pure Appl. Math. **12**, 1 (1872), http://kirkmcd.princeton.edu/examples/mechanics/ferrers_qjpam_12_1_72.pdf

9. We describe the skater by three coordinates, (x, y) of its center of mass, and the angle θ of the skate to the x-axis, as shown in the figure below.



To deduce a constraint on the motion when the center of mass of the skater moves by $dl = \sqrt{dx^2 + dy^2}$, we have that,

$$dx = dl\cos(\theta + d\alpha) \approx dl\cos\theta - dl\, d\alpha \sin\theta \approx dl\cos\theta - a\sin\theta\, d\theta, \tag{100}$$

$$dy = dl\sin(\theta + d\alpha) \approx dl\sin\theta + dl\, d\alpha \cos\theta \approx dl\sin\theta + a\cos\theta\, d\theta, \tag{101}$$

using the law of sines for the small triangle,

$$\frac{d\alpha}{a} \approx \frac{\sin(d\alpha)}{a} = \frac{\sin(\pi - d\theta)}{dl} \approx \frac{d\theta}{dl}\,. \tag{102}$$

Then, the non-holonomic constraint is,

$$dl \approx \frac{dx + a\sin\theta\, d\theta}{\cos\theta} \approx \frac{dy - a\cos\theta\, d\theta}{\sin\theta}\,, \tag{103}$$

$$dg(x, y, \theta) = \sin\theta\, dx - \cos\theta\, dy + a\, d\theta = 0. \tag{104}$$

An alternative derivation of the constraint notes that the constraint force $\mathbf{F}$ *does no work, so*

$$0 = \mathbf{F}\cdot\mathbf{v}_F = \mathbf{F}\cdot(\mathbf{v}_{\text{c.m.}} - a\dot\theta\,\hat{\mathbf{F}}) = F(-\sin\theta\,\dot x + \cos\theta\,\dot y - a\,\dot\theta), \tag{105}$$

from which eq. (104) follows (more quickly).

We use Ferrers extension of Lagrange's method, with Lagrange multiplier λ,

$$L = T = \frac{1}{2}(m\dot x^2 + m\dot y^2 + I\dot\theta^2),\ a_x = \frac{dg}{dx} = \sin\theta,\ a_y = \frac{dg}{dy} = -\cos\theta,\ a_\theta = \frac{dg}{d\theta} = a, \tag{106}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} = \lambda a_j, \tag{107}$$

$$m\ddot x = \lambda\sin\theta, \qquad m\ddot y = -\lambda\cos\theta, \qquad I\ddot\theta = \lambda a. \tag{108}$$

Note that these equations also follow from elementary methods, with the constraint force $F = -\lambda$.

Whatever method used to obtain eq. (108), to go further it is useful to write the constraint (104) as

$$\dot x\sin\theta - \dot y\cos\theta + a\,\dot\theta = 0. \tag{109}$$

We also introduce the velocity v_F of the point of application of the constraint force,

$$x_F = x - a\cos\theta, \qquad y_F = y - a\sin\theta, \tag{110}$$

$$\dot x_F = \dot x + a\sin\theta\,\dot\theta = v_F\cos\theta, \qquad \dot y_F = \dot y - a\cos\theta\,\dot\theta = v_F\sin\theta, \tag{111}$$

$$v = \dot x\cos\theta + \dot y\sin\theta, \tag{112}$$

$$\dot v_F = \ddot x\cos\theta + \ddot y\sin\theta + (-\dot x\sin\theta + \dot y\cos\theta)\,\dot\theta = a\dot\theta^2, \tag{113}$$

where the last equality is obtained using eqs. (108)-(109). For later use we note that,

$$v_F^2 = \dot x_F^2 + \dot y_F^2 = \dot x^2 + \dot y^2 + a^2\dot\theta^2 + 2(a\sin\theta\,\dot x - a\cos\theta\,\dot y)\,\dot\theta = \dot x^2 + \dot y^2 - a^2\dot\theta^2. \tag{114}$$

Furthermore, we can differentiate the constraint (108) with respect to time,

$$\ddot x\sin\theta - \ddot y\cos\theta\ + a\,\ddot\theta + (\dot x\cos\theta + \dot y\sin\theta)\,\dot\theta = 0, \tag{115}$$

$$\ddot x\sin\theta - \ddot y\cos\theta\ + a\,\ddot\theta + (\dot x\cos\theta + \dot y\sin\theta)\,\dot\theta = 0, \tag{116}$$

$$\frac{I\ddot\theta}{ma} + a\ddot\theta + v_F\dot\theta = 0, \tag{117}$$

using eqs. (108)-(109) again. We also write,

$$I = mb^2, \qquad k^2 = 1 + \frac{b^2}{a^2}\,, \tag{118}$$

such that the equation of motion (117) becomes,

$$ak^2\ddot\theta + v_F\dot\theta = 0, \qquad ak^2\frac{\ddot\theta}{\dot\theta} = -v_F, \tag{119}$$

We take the time derivative of this,

$$ak^2\frac{d}{dt}\frac{\ddot\theta}{\dot\theta} = -\dot v_F = -a\dot\theta^2, \tag{120}$$

recalling eq. (113). Multiply this by $\ddot\theta/\dot\theta$,

$$k^2\frac{\ddot\theta}{\dot\theta}\frac{d}{dt}\frac{\ddot\theta}{\dot\theta} = \frac{k^2}{2}\frac{d}{dt}\left(\frac{\ddot\theta}{\dot\theta}\right)^2 = -\ddot\theta\dot\theta = -\frac{1}{2}\frac{d\dot\theta^2}{dt}\,. \tag{121}$$

Integrating this,

$$k^2\left(\frac{\ddot\theta}{\dot\theta}\right)^2 = c - \dot\theta^2, \qquad k\ddot\theta = \dot\theta\sqrt{c - \dot\theta^2}, \qquad \frac{kd\dot\theta}{\dot\theta\sqrt{c-\dot\theta^2}} = dt, \tag{122}$$

for some constant c. We integrate this using Dwight 341.01,[6] setting $t_0 = 0$,

$$-\frac{k}{c}\cosh^{-1}\frac{c}{\dot\theta} = t + t_0 = t, \qquad \dot\theta = \frac{c}{\cosh(ct/k)} = c\cos\frac{\theta}{k}\,, \tag{123}$$

$$\theta = k\tan^{-1}\sinh\frac{ct}{k}\,, \qquad \sinh\frac{ct}{k} = \tan\frac{\theta}{k}\,, \qquad \cosh\frac{ct}{k} = \frac{1}{\cos(\theta/k)}\,, \tag{124}$$

[6] http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf

defining $\theta(t=0)=0$. Using eq. (123) in eq. (113) we find (also using eq. (124)),

$$\dot{v}_F = a\dot{\theta}^2 = \frac{ac^2}{\cosh^2(ct/k)}\,, \qquad v_F = ack\tanh\frac{ct}{k} = ack\sin\frac{\theta}{k}\,, \tag{125}$$

defining $v_F(t=0)=0$.

The kinetic energy is constant, using eqs. (114), (123) and (125),

$$T = \frac{m\dot{x}^2 + m\dot{y}^2 + I\dot{\theta}^2}{2} = \frac{mv_F^2 + ma^2\dot{\theta} + I\dot{\theta}^2}{2} = \frac{m}{2}(v_F^2 + a^2k^2\dot{\theta}^2) = \frac{ma^2c^2k^2}{2}\,. \tag{126}$$

The constraint force is, using eqs. (108) and (123),

$$F = -\lambda = -I\ddot{\theta} = \frac{cI}{k}\sin\frac{\theta}{k}\dot{\theta} = \frac{c^2I}{2k}\sin\frac{2\theta}{k}\,. \tag{127}$$

To characterize the motion for very large times, we note from eq. (125) that $v_F(t\to\pm\infty)\to\pm ack$, and hence that $\theta(t\to\pm\infty)\to\pm k\pi/2$. That is, the trajectories are asymptotically straight lines at a constant angle.

For very small times where $\theta\approx 0$, we consider various derivatives, more simply for (x_F, y_f) than for (x,y) of the center of mass:

$$\frac{dx_F}{d\theta} = \frac{\dot{x}_F}{\dot{\theta}} = \frac{v_F\cos\theta}{\dot{\theta}} = \frac{ak\sin(\theta/k)}{c\cos(\theta/k)}\cos\theta = \frac{ak}{c}\tan\frac{\theta}{k}\cos\theta \overset{t\to 0}{\to} 0, \tag{128}$$

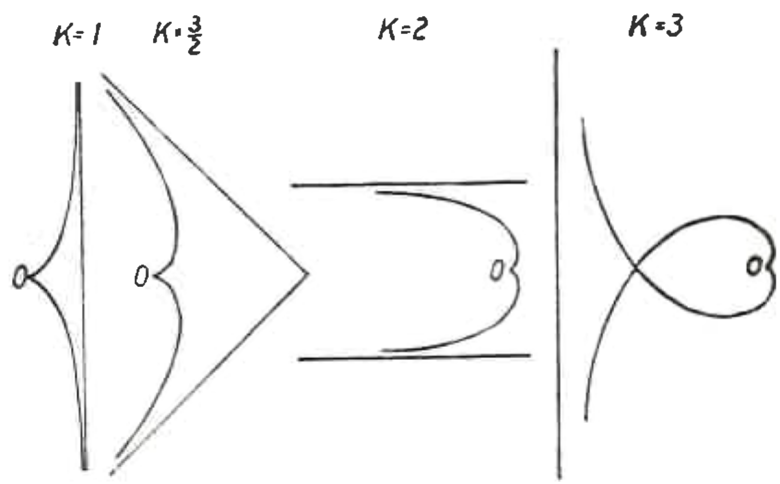
$$\frac{d^2x_F}{d\theta^2} = \frac{ak}{c}\left(\frac{\cos\theta}{k\cos^2(\theta/k)} - \tan\frac{\theta}{k}\sin\theta\right) \overset{t\to 0}{\to} \frac{a}{c} \neq 0, \tag{129}$$

$$\frac{dy_F}{d\theta} = \frac{\dot{y}_F}{\dot{\theta}} = \frac{v_F\cos\theta}{\dot{\theta}} = \frac{ak\sin(\theta/k)}{c\cos(\theta/k)}\sin\theta = \frac{ak}{c}\tan\frac{\theta}{k}\sin\theta \overset{t\to 0}{\to} 0, \tag{130}$$

$$\frac{d^2y_F}{d\theta^2} = \frac{ak}{c}\left(\frac{\sin\theta}{k\cos^2(\theta/k)} + \tan\frac{\theta}{k}\cos\theta\right) \overset{t\to 0}{\to} 0, \tag{131}$$

$$\frac{d^3y_F}{d\theta^3} = \frac{ak}{c}\left(\frac{\cos\theta}{k\cos^2(\theta/k)} - \frac{2\sin\theta\sin(\theta/k)}{k^2\cos^3(\theta/k)} + \frac{\cos\theta}{k\cos^2(\theta/k)} - \tan\frac{\theta}{k}\sin\theta\right) \overset{t\to 0}{\to} \frac{2a}{c}. \tag{132}$$

All these imply the curve has a cusp at the origin, with its tip pointing to the $-x$ axis.



10. (a) The particle moves in a vertical plane through the center of the sphere, subject to the constraint that,

$$g(r,\theta) = r - a = 0. \tag{133}$$

The Lagrangian is, for coordinates r and θ,

$$L = T - V = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta, \tag{134}$$

taking $V = 0$ at the center of the sphere. Lagrange's equations for the constrained motion are,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta = \lambda\frac{\partial g}{\partial r} = \lambda, \tag{135}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = mr^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - mgr\sin\theta = \lambda\frac{\partial g}{\partial \theta} = 0, \tag{136}$$

Applying the constraint (133) to eq. (136), we have,

$$\ddot{\theta} = \frac{g}{a}\sin\theta, \qquad \dot{\theta}\ddot{\theta} = \frac{g}{a}\dot{\theta}\sin\theta, \qquad \frac{\dot{\theta}^2}{2} = \frac{g}{a}(1-\cos\theta), \tag{137}$$

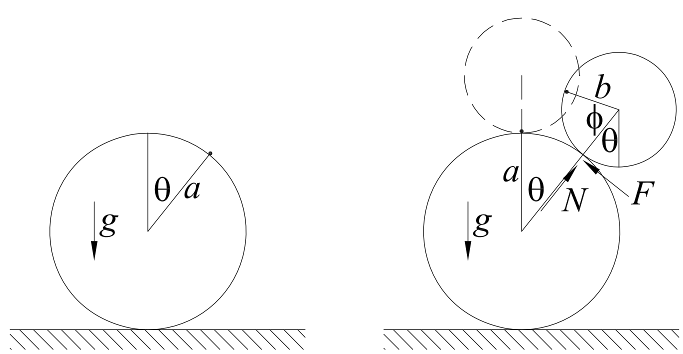
for the initial conditions that $\theta(t=0) = 0 = \dot{\theta}(t=0)$.
Then, from eq. (135),

$$\lambda = -ma\frac{2g}{a}(1-\cos\theta) + mg\cos\theta = mg(3\cos\theta - 2). \tag{138}$$

Since λ has the physical significance of the constraint force, this vanishes when

$$\cos\theta = \frac{2}{3}, \tag{139}$$

at which condition the mass flies off the sphere.



(b) A uniform sphere of radius b, mass m and moment of inertia (about its center of mass) $I = 2mb^2/5$ starts at rest from the top of a fixed sphere of radius a, and rolls without slipping down the latter.

The constraint of rolling without slipping can be written as,

$$b\phi = a\theta, \qquad \phi = \frac{a}{b}\theta, \qquad g_\phi = \phi - \frac{a}{b}\theta \tag{140}$$

where ϕ is the angle between the original point of contact between the spheres when $\theta = 0$ and the line of center of the two spheres.

Associated with this constraint is a tangential force F between the spheres at their point of contact, as well as the normal force N there. The upper sphere flies off the lower when the normal force vanishes.

Following pp. 114-115 of Routh, *Elementary Rigid Dynamics*,
http://kirkmcd.princeton.edu/examples/mechanics/routh_elementary_rigid_dynamics.pdf,
we consider the radial and tangential accelerations of the center of mass of the upper sphere,

$$m\,a_r = m\,\dot\theta^2(a+b) = mg\cos\theta - N, \tag{141}$$

$$m\,a_t = m\,\ddot\theta(a+b) = mg\sin\theta - F. \tag{142}$$

Also, energy $E = T + V$ is conserved in the rolling motion,

$$T = \frac{m\,v^2}{2} + \frac{I}{2}(\dot\theta + \dot\phi)^2 = \frac{m\,\dot\theta^2}{2}\left[(a+b)^2 + \frac{2b^2}{5}\frac{(a+b)^2}{b^2}\right] = \frac{7}{10}m(a+b)^2\dot\theta^2, \tag{143}$$

$$V = mgy = mg(a+b)\cos\theta, \tag{144}$$

$$E = \frac{7}{10}m(a+b)^2\dot\theta^2 + mg(a+b)\cos\theta = mg(a+b), \quad \dot\theta^2 = \frac{10g}{7}\frac{1-\cos\theta}{a+b}, \tag{145}$$

$$\frac{dE}{dt} = 0 = \frac{7}{5}m(a+b)^2\dot\theta\ddot\theta - mg(a+b)\sin\theta\,\dot\theta, \qquad \ddot\theta = \frac{5g\sin\theta}{7(a+b)}. \tag{146}$$

From eqs. (141) and (145), the normal force is,

$$N = mg\left[\cos\theta - \frac{10}{7}(1-\cos\theta)\right] = \frac{mg}{7}(17\cos\theta - 10), \tag{147}$$

which vanishes when,

$$\cos\theta = \frac{10}{17}, \tag{148}$$

and the upper sphere flies off. For what it's worth, the tangential (friction) force is, using eqs. (142) and (146),

$$F = mg\sin\theta\left(1 - \frac{5}{7}\right) = \frac{2mg\sin\theta}{7}. \tag{149}$$

In a Lagrangian analysis, we deduce the normal force N as a constraint force, related to a constraint on the radius of the lower sphere,

$$g_r = r - a = 0, \tag{150}$$

while keeping the radius b of the upper sphere fixed.[7] Then, we can use the constraint (140) as is.[8] That is, we use $\phi = a\theta/b$ in Lagrange's equations for coordinates r and θ with one Lagrange multiplier, λ_r, associated with the constraint (150). Again, $\dot\theta + \dot\phi = (1 + a/b)\dot\theta$, and we write eqs. (143)-(144) as,

$$T = \frac{mv^2}{2} + \frac{I}{2}(\dot\theta + \dot\phi)^2 = \frac{m}{2}\left[\dot r^2 + (r+b)^2\dot\theta^2 + \frac{2}{5}(a+b)^2\dot\theta^2\right], \tag{151}$$

$$V = mgy = mg(a+r)\cos\theta, \tag{152}$$

Lagrange's equations for this system are,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot r} - \frac{\partial L}{\partial r} = m\ddot r - m(r+b)\dot\theta^2 - mg\cos\theta = \lambda_r \frac{\partial g_r}{\partial r} = \lambda_r, \tag{153}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = m(r+b)^2\ddot\theta + m(r+b)\dot r\dot\theta + \frac{2m}{r}(a+b)^2 + mg(a+r)\sin\theta$$
$$= \lambda_r \frac{\partial g_r}{\partial \theta} = 0. \tag{154}$$

Having deduced the equations of motion with the constraint (150) relaxed, we now enforce it, which implies $\dot r = 0 = \ddot r$,

$$m(a+b)\dot\theta^2 - mg\cos\theta = \lambda_r, \tag{155}$$

$$\frac{7m}{5}(a+b)^2\ddot\theta + mg(a+b)\sin\theta = 0. \tag{156}$$

To complete the analysis we note that energy is conserved, so we can use eq. (145) in eq. (155), where λ_r is the constraint force associated with coordinate r, *i.e.*, the normal force $-N$ on the lower sphere,

$$\lambda_r = -N = m(a+b)\frac{10g}{7}\frac{1-\cos\theta}{a+b} - mg\cos\theta = \frac{mg}{7}(10 - 17\cos\theta). \tag{157}$$

The upper sphere flies off when this normal force vanishes,

$$\cos\theta = \frac{10}{17}, \tag{158}$$

as found more easily by elementary methods.

[7]It does not work to use the constraint $g_r = r - b = 0$ on the radius of the upper sphere, which would make the moment of inertia of that sphere variable when this constraint is relaxed.

[8]This is discussed on pp. 374-378 of Symon, *Mechanics*, 3[rd] ed.
http://kirkmcd.princeton.edu/examples/mechanics/symon_71.pdf