Princeton University

Ph205
Mechanics
Problem Set 6

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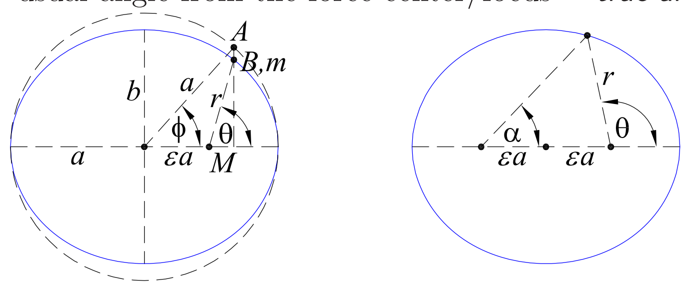
1. (a) **Kepler's Equation of Time** (Secular Equation).

As well as stating his three laws, Kepler also gave an equation of time *vs.* position in elliptical-orbit motion,

$$\Omega t = \phi - \epsilon \sin \phi, \tag{1}$$

where $\Omega = 2\pi / T_{\rm orbit}$ is the average angular velocity of the motion, ϵ = eccentricity, and ϕ = eccentric anomaly = angle from the center of the ellipse, with respect to the major axis of length $2a$, to the point A on a circle of radius a such that its projection, B, onto the ellipse is the position described by time t of eq. (1).

Angle θ = usual angle from the force center/focus = true anomaly.



For the ellipse,

$$\frac{1}{r} = \frac{1 + \epsilon \cos \theta}{a(1 - \epsilon^2)}, \tag{2}$$

where $\theta = 0$ at perihelion, first show that,

$$\cos \phi = \frac{\cos \theta - \epsilon}{1 - \epsilon \cos \theta}. \tag{3}$$

Then, use $L = mr^2 \dot{\theta}$ to derive an expression for $\dot{\phi}$, which can be integrated to give an equation of time.

Show also that,

$$\frac{L}{mab} = \sqrt{\frac{GM}{a^3}} = \Omega, \tag{4}$$

where $b = a\sqrt{1 - \epsilon^2}$ is the semiminor axis, which permits the equation of time to be written in the Keplerian form (1). *Kepler was aware of the second equality in eq. (4), but not the first (due to Newton).*

(b) **The Greek Eccentricity.**

Let α be the angle to the orbiting mass from the empty focus of the elliptical orbit, as shown in the right figure above. For small eccentricity $\epsilon \ll 1$, show that,

$$\dot{\alpha} = \frac{L}{ma} = \text{constant}. \tag{5}$$

This result supports the Greek view that the Sun is in uniform motion about a point at distance $2a\epsilon$ from the Earth. Hence, the Greek eccentricity is twice the Keplerian.

2. **Spin-Orbit Coupling.**

A prominent astronomical fact is that the Moon always shows the same face to the Earth. This means that the Moon rotates once about its axis each Earth month. It turns out that the "days" of Mercury and Venus are nearly equal to their respective "years,"[1] and that the periods of axial and orbital revolution are equal for most of the moons of Jupiter, Saturn, Uranus and Neptune. In 1879, George Darwin (son of Charles) proposed that this has come about due a coupling between the "day" and month/year via tidal friction[2] – resistance of the moon or planet to changes in shape induced by the $1/r^2$ variation of gravity of the body at the focus of its orbit, and that eventually the Earth day will equal one month.[3,4]

In this problem you should deduce a kind of existence proof that a spin-orbit coupling mechanism leads to changes of the "day" and the "month/year" such that these can eventually become equal.

For simplicity, consider a point satellite of mass m that revolves with orbital angular velocity ω around a planet of mass M in a nearly circular orbit of radius R. The planet rotates about its axis with "spin" angular velocity Ω, its moment of inertia about this axis is I, and this axis is perpendicular to the plane of the satellite's orbit.

Find expressions for the total angular momentum L of the system about its center of mass, and for the total (kinetic + potential) energy E. Eliminate R from these expressions to show that,

$$L = I\Omega + \frac{C}{\omega^{1/3}}, \qquad E = \frac{I\Omega^2}{2} - \frac{C\omega^{2/3}}{2}, \tag{6}$$

and deduce the value of C.

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[1] Mercury's "day" is 2/3 of its "year."

[2] G.H. Darwin, *The Determination of the Secular Effects of Tidal Friction by a Graphical Method*, Proc. Roy. Soc. London **29**, 168 (1879), http://kirkmcd.princeton.edu/examples/astro/darwin_prsl_29_168_79.pdf
On the Precession of a Viscous Spheroid, and on the remote History of the Earth, Phil. Trans. Roy. Soc. London **170**, 447 (1879), http://kirkmcd.princeton.edu/examples/astro/darwin_ptrsl_170_447_79.pdf
On the Secular Changes in the Elements of the Orbit of a Satellite revolving about a Tidally distorted Planet, Phil. Trans. Roy. Soc. London **171**, 713 (1880),
http://kirkmcd.princeton.edu/examples/astro/darwin_ptrsl_171_713_80.pdf
On the Analytical Expressions which give the History of a Fluid Planet of Small Viscosity, attended by a Single Satellite, Proc. Roy. Soc. London **30**, 255 (1880),
http://kirkmcd.princeton.edu/examples/astro/darwin_prsl_30_255_79.pdf

[3] This hypothesis was first postulated by Kant (1754), pp. 6-9 of *Whether the Earth Has Undergone an Alteration of Its Axial Rotation*, Wöchentliche Frag- und Anzeigungs-Nachricten (Königsberg), Nos. 23-24 (1754); English translation in W. Hastie, *Kant's Cosmogony*, (James Maclehose, Glasgow, 1900), http://kirkmcd.princeton.edu/examples/astro/kant_cosmogony.pdf. Kant's (verbal) argument was that if the Earth's day does not equal a month, then the tidal bulge caused by the Moon rotates with respect to the Earth and experiences tidal friction, which slows down the Earth's rotation until the day equals a month. The present problem is a slight mathematical elaboration of Kant's argument.

[4] That the length of a month is increasing seems to have been first noted by Halley (1695), p. 174 of *Some Account of the Ancient State of the City of Palmyra*, Phil. Trans. Roy. Soc. London **19**, 160 (1695), http://kirkmcd.princeton.edu/examples/astro/halley_ptrsl_19_160_1695.pdf

In general, the angular velocities ω and Ω are different. If $\omega \neq \Omega$ then tidal friction reduces the (kinetic + potential) energy E while conserving angular momentum. Show that there is a range of initial conditions such that eventually $\omega_0 = \Omega_0$.[5]

For the Earth-Moon system, Ω_E is decreasing with time. Give an expression for R as a function of Ω (and not ω) to show that R increases as Ω decreases. Then, by Kepler's law for the system, ω must be decreasing also.

Darwin noted that extrapolation of the above scenario into the past suggests there may have been a time when $R = R_E$ and the Earth and Moon were part of a single protoplanet.[6]

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[5]Hint: Consider the variable $x = C/\omega^{1/3}$ = orbital angular momentum.

[6]For a popular review, see P. Goldreich, *Tides and the Earth-Moon system*, Sci. Am. April, 42 (1972), http://kirkmcd.princeton.edu/examples/astro/goldreich_sa_4_42_72.pdf,
and also C.L. Coughenour, A.W. Archer and K.J. Lacovara, *Tides, tidalities, and secular changes in the Earth-Moon system*, Earth-Sci. Rev. **97**, 59 (2009),
http://kirkmcd.princeton.edu/examples/astro/coughenour_esr_97_59_09.pdf.
Nowadays, the so-called impact-origin hypothesis enjoys greater favor, although the issue remains unsettled. See, for example, M. Ćuk and S.T. Stewart, *Making the Moon from a Fast-Spinning Earth: A Giant Impact Followed by Resonant Despinning*, Science **338**, 1047 (2012),
http://kirkmcd.princeton.edu/examples/astro/cuk_science_338_1047_12.pdf,
I. Crawford, *The Moon and the early Earth*, A&G **54**, 1.31 (2013),
http://kirkmcd.princeton.edu/examples/astro/crawford_ag_54_1.31_13.pdf.

3. (a) **Skyhook.**

As part of the Equatorial-African Space Program, Idi Amin proposed to launch a "skyhook" satellite. This consists of a long rope which is in orbit such that the rope is along a radius through the center of the Earth, with the lower end just above the Earth's surface. The skyhook orbits the Earth once a day, so the lower end appears suspended in space above a fixed point on the equator. *"I just want a place to hang my hat": I. Amin*

Derive a differential equation for the tension in the rope. Note that the tension in the rope vanishes at its ends to show,

$$r_{\rm max} = \frac{r_E}{2}\left(\sqrt{1 + \frac{8Gm_E}{\omega_E^2 r_E^3}} - 1\right), \tag{7}$$

if $r_{\rm min} = r_E$.

Is the skyhook stable against hanging a hat on it?

To get a sense of this issue, consider its potential energy and the kinetic energy of its center-of-mass motion.

(b) **Ringworld.**

A 1970 science-fiction novel by Larry Niven with the above title considered a band/ring about a Sunlike star at the radius of the Earth, which rotated with (high) angular velocity such that the centrifugal force equaled Earth's gravity.

Is this system stable against a radial perturbation of its center of mass away from the center of the star?

4. Some dimensional analysis:

   (a) **Modeling.**

   We wish to investigate some phenomenon by means of a scale model, in which all lengths are scaled by $l' = \alpha l$ (usually $\alpha < 1$). However, we use the same material to construct the model as in the original, so the density ρ is unchanged. In general, we need to stretch or shrink the time scale, $t' = \beta t$, which we accomplish by scaling the velocities, $v' = \alpha v/\beta$.

   How is the force F' in the scale model related to force F in the original (such that their motions are similar).

   If the force involves gravity, how is the scaling further constrained?

   Example: A model of a ship with $\alpha = 1/n$ is observed to experience a drag force F' when traveling through a tank of water with velocity v'. For what velocity v of the original is the model result relevant?

   (b) If we can write a drag force as $F = C_d v^p A^q$ where A is the area of the object projected onto a plane perpendicular to $\mathbf{v}$, and C_d, p and q are constants, then what are p and q such that scaling holds?

   (c) A particle starts from rest, subject to the force $\mathbf{F} = -A\mathbf{r} - B\mathbf{v}$, where A and B are constants. Show by dimensional analysis that the time to reach the origin is independent of the initial position.

   (d) A particle starts from rest subject to the force $\mathbf{F} = -A\,\hat{\mathbf{r}}/r^n$, where $A > 0$ and n are constants. Show that the time for the particle to fall to the origin from initial distance r_0 scales as $T \propto r_0^{\frac{n+1}{2}}$.

5. **Biomechanics.**

(a) How does the amount of time an animal can survive in a hot, dry desert depend on is height/length L?

(b) Animals are heat engines – they must sweat to live. How does "horsepower" scale with the size L of an animal?

(c) How does the maximum running speed of an animal scale with its size L if air resistance is the limit, and if gravity is the limit (as in running uphill)?

(d) How does the height h that an animal can jump depend on its size L.

6. (a) Calculate the differential cross section in the center-of-mass frame for elastic scattering of spheres of radius b off a sphere of radius a (both with no initial rotation).

   (b) In the lab frame, in which the sphere of radius a is at rest, show that the differential cross section can be written as,

$$\frac{d\sigma}{dT_{\text{lost}}} = \frac{\pi a^2}{T_{\text{lost,max}}}, \tag{8}$$

   where T_{lost} is the kinetic energy lost by the other sphere (and gained by the sphere of radius a) during the scattering.

   What are the extremes of the "lost" energy, $T_{\text{lost,min}}$ and $T_{\text{lost,max}}$?

7. (a) A parallel beam of projectiles is fired from space toward the Moon with initial velocity v_0. What is the collision cross section σ for the projectiles to hit the Moon, neglecting its motion?

   Express σ in terms of the radius R of the Moon, the escape velocity v_e from the Moon, and v_0.

   There is no need to integrate over Rutherford's formula – go back to basics!

   What would the cross section be if gravity were a repulsive force, $F = +GMm/r^2$?

   (b) The interaction between an atom and an ion is described by the potential energy $V(r) = -C/r^4$ for a constant C (and r such that the atom and ion are not in contact).[7]

   Make a sketch of the effective potential energy $V_{\rm eff}(r)$. Note that if the total energy of the ion exceeds the maximum value of the effective potential, then the ion spirals inward to the atom.

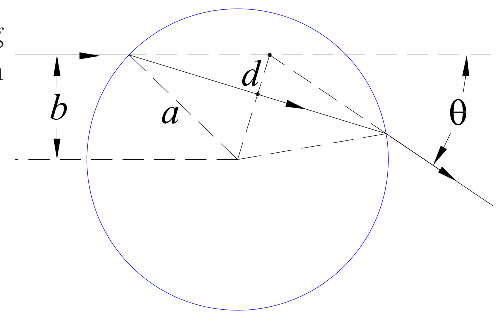
   Find the capture cross section for an ion to strike the atom if it has velocity v_∞ when far from the atom, supposing $m_{\rm ion} \ll m_{\rm atom}$.

[7] $C = P_a^2 e^2/2$, where P_a is the electric polarizability of the atom and e is the electric charge of the ion.

8. Calculate the differential cross section for scattering off the potential,

$$V(r) = \begin{cases} 0 & r > a, \\ -V_0 & r < a. \end{cases} \qquad (9)$$

A trick is to use the problem on p. 16 of L.D. Landau and E.M. Lifshitz, *Mechanics*, 3rd ed. (Pergamon, 1976), http://kirkmcd.princeton.edu/examples/mechanics/landau_mechanics.pdf, to note that this problem is equivalent to light scattering off a sphere of index of refraction $n = \sqrt{T/(T + V}$, such that $n = 1$ outside the sphere and $n > 1$ inside it.



To relate the impact parameter b to the scattering angle θ, you might solve for distance d, shown in the figure, in two ways to find,

$$b^2 = \frac{a^2 n^2 \sin^2 \theta/2}{n^1 + 1 - 2n \cos \theta/2}. \qquad (10)$$

What is $\theta_{\rm max}$?

Then, for $n > 1$ inside the sphere you should find,

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi a^2 n^2}{2\cos\theta/2} \frac{(n\cos\theta/2 - 1)(n - \cos\theta/2)}{(n^1 + 1 - 2n\cos\theta/2)^2}. \qquad (11)$$

The case of $n < 1$ is not possible for light scattering in air, but for scattering of light off an air bubble under water, $n_{\rm outside}/n_{\rm inside} > 1$. Here, there would be no limit to the scattering angle "at first glance". However, as $b \to a$, light won't enter the bubble, but rather just bounces off the surface by "total external reflection". [Try looking at a bubble in a swimming pool sometime.] The reflection scattering is isotropic over the allowed range of angles (as in Prob. 6 above).

9. Calculate the differential cross section for the scattering of point particles off the repulsive central force whose potential is $V = \alpha/r^2$ where α is a positive constant. The beam particles have energy E.

   This problem is well suited to the formal method of sec. 18 of L.D. Landau and E.M. Lifshitz, *Mechanics*, 3rd ed. (Pergamon, 1976),
   http://kirkmcd.princeton.edu/examples/mechanics/landau_mechanics.pdf.

   You should find that the impact parameter b is related to the scattering angle θ by,

$$b^2 = \frac{\alpha}{E} \frac{(\pi - \theta)^2}{\theta(2\pi - \theta)}\,, \tag{12}$$

   and hence, the differential cross section per unit solid angle is,

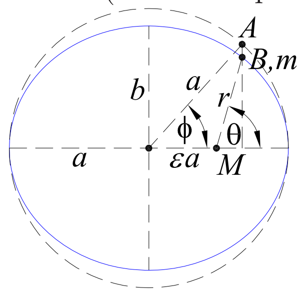
$$\frac{d\sigma}{d\Omega} = \frac{\pi^2 \alpha}{E \sin\theta} \frac{\pi - \theta}{\theta^2 (2\pi - \theta)^2}\,. \tag{13}$$

   Sketch this.

   As for the Rutherford cross section, since the force extends to infinity, the total cross section diverges.

Solutions

1. (a) **Kepler's Equation of Time** (Secular Equation).



For the ellipse $r = a(1-\epsilon^2)/(1+\epsilon\cos\theta)$, we have that,

$$\cos\phi = \frac{r\cos\theta + \epsilon a}{a} = \frac{(1-\epsilon^2)\cos\theta}{1+\epsilon\cos\theta} + \epsilon = \frac{(1-\epsilon^2)\cos\theta + \epsilon + \epsilon^2\cos\theta}{1+\epsilon\cos\theta} = \frac{\cos\theta+\epsilon}{1+\epsilon\cos\theta}, \quad (14)$$

$$\cos\theta = \frac{\cos\phi - \epsilon}{1-\epsilon\cos\phi}, \qquad 1+\epsilon\cos\theta = \frac{1-\epsilon^2}{1-\epsilon\cos\phi}, \quad (15)$$

$$\sin\theta = \sqrt{1 - \left(\frac{\cos\phi-\epsilon}{1-\epsilon\cos\phi}\right)^2} = \frac{\sqrt{1 - 2\epsilon\cos\phi + \epsilon^2\cos\phi^2 - \cos^2\phi + 2\epsilon\cos\phi - \epsilon^2}}{1-\epsilon\cos\phi}$$

$$\frac{\sqrt{(1-\epsilon^2)(1-\cos^2\phi)}}{1-\epsilon\cos\phi} = \frac{\sin\phi\sqrt{1-\epsilon^2}}{1-\epsilon\cos\phi}. \quad (16)$$

Taking the time derivative of eq. (14), and recalling that $L = mr^2\dot\theta$ and $b = a\sqrt{1-\epsilon^2}$,

$$-\sin\phi\,\dot\phi = \frac{-\sin\theta\,\dot\theta}{1+\epsilon\cos\theta} - \frac{(\cos\theta-\epsilon)\epsilon\sin\theta\,\dot\theta}{(1+\epsilon\cos\theta)^2} = -\frac{\sin\theta\,\dot\theta(1-\epsilon^2)}{(1+\epsilon\cos\theta)^2},$$

$$= -\frac{\sin\phi}{1-\epsilon\cos\phi}\frac{L}{mr^2}\frac{1-\epsilon^2}{(1+\epsilon\cos\theta)^2} = -\frac{\sin\phi\sqrt{1-\epsilon^2}}{1-\epsilon\cos\phi}\frac{L}{ma^2\sqrt{1-\epsilon^2}} \quad (17)$$

$$(1-\epsilon\cos\phi)\,\dot\phi = \frac{L}{mab}, \quad (18)$$

$$\phi - \epsilon\sin\phi = \frac{Lt}{mab}. \quad (19)$$

We also note Kepler's 3rd law, $\Omega^2 a^3 = GM$, where $\Omega = 2\pi/T$ is the average angular velocity of mass m in orbit around mass $M \gg m$, and that from p. 105 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l10.pdf,

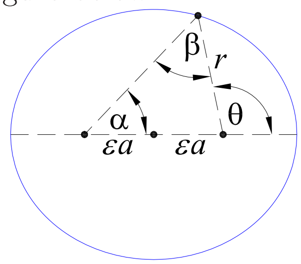
$$L^2 = GMm^2a(1-\epsilon) = \Omega^2m^2a^4(1-\epsilon^2) = (\Omega mab)^2. \quad (20)$$

so Kepler's equation of time (19) can also be written as,

$$\phi - \epsilon\sin\phi = \Omega t. \quad (21)$$

(b) **The Greek Eccentricity.**

Let α be the angle to the orbiting mass from the empty focus of the elliptical orbit, as shown in the figure below.



By the law of sines,

$$\frac{\sin\alpha}{r} = \frac{\sin\beta}{2\epsilon a} = \frac{\sin(\pi - \alpha - (\pi - \theta))}{2\epsilon a} = \frac{\sin(\theta - \alpha)}{2\epsilon a} = \frac{\sin\theta\cos\alpha - \cos\theta\sin\alpha}{2\epsilon a}, \quad (22)$$

$$\tan\alpha = \frac{\sin\theta}{\cos\theta + 2\epsilon a/r}. \quad (23)$$

In the rest of the analysis, we consider small ϵ, and neglect terms of order ϵ^2. The time derivative of eq. (23) is,

$$(1 + \tan^2\alpha)\dot{\alpha} = \frac{\cos\theta\,\dot{\theta}}{\cos\theta + 2\epsilon a/r} - \frac{\sin\theta}{(\cos\theta + 2\epsilon a/r)^2}\left(-\sin\theta\,\dot{\theta} - \frac{2\epsilon a\dot{r}}{r^2}\right). \quad (24)$$

The equation of the ellipse is,

$$\frac{1}{r} = \frac{1 + \epsilon\cos\theta}{a(1 - \epsilon^2)} \approx \frac{1 + \epsilon\cos\theta}{a}, \qquad r \approx \frac{a}{1 + \epsilon\cos\theta}, \quad (25)$$

$$\dot{r} \approx \frac{a\epsilon\sin\theta\,\dot{\theta}}{(1 + \epsilon\cos\theta)^2} \approx a\epsilon\sin\theta\,\dot{\theta}, \quad (26)$$

$$\cos\theta + \frac{2\epsilon a}{r} \approx \cos\theta + 2\epsilon a\frac{1 + \epsilon\cos\theta}{a} \approx \cos\theta + 2\epsilon, \quad (27)$$

$$1 + \tan^2\alpha = \frac{(\cos\theta + 2\epsilon a/r)^2 + \sin^2\theta}{(\cos\theta + 2\epsilon a/r)^2} \approx \frac{1 + 4\epsilon\cos\theta}{(\cos\theta + 2\epsilon a/r)^2}. \quad (28)$$

Using these approximations in eq. (24), we have,

$$(1 + 4\epsilon\cos\theta)\dot{\alpha} \approx (\cos\theta + 2\epsilon)\cos\theta\,\dot{\theta} + \sin\theta\left(\sin\theta\,\dot{\theta} + 2\epsilon a(a\epsilon\sin\theta\,\dot{\theta})\frac{(1 + \epsilon\cos\theta)^2}{a^2}\right)$$

$$\approx \dot{\theta}(1 + 2\epsilon\cos\theta), \quad (29)$$

$$\dot{\alpha} \approx \dot{\theta}\frac{1 + 2\epsilon\cos\theta}{1 + 4\epsilon\cos\theta} \approx \dot{\theta}(1 - 2\epsilon\cos\theta) = \frac{L}{mr^2}(1 - 2\epsilon\cos\theta),$$

$$\approx \frac{L}{ma^2}(1 + \epsilon\cos\theta)^2(1 - 2\epsilon\cos\theta) \approx \frac{L}{ma^2} = \text{constant}, \quad (30)$$

where $L = mr^2\dot{\theta}$ is the conserved angular momentum of mass m about the force center.

2. **Spin-Orbit Coupling.**

The center of mass of the planet-satellite system are at distances,

$$r_M = \frac{m}{M+m} R, \qquad \text{and} \qquad r_m = \frac{M}{M+m} R \tag{31}$$

from the centers of these bodies, respectively, where $R = r_M + r_m$. The total angular momentum of the system (ignoring possible angular momentum associated with rotation of the satellite about its axis) in the rest frame of the center of mass of the system is the constant,

$$L = I\Omega + (Mr_M^2 + mr_m^2)\,\omega = I\Omega + \mu R^2 \omega, \qquad \omega = \frac{L - I\Omega}{\mu R^2}\,, \tag{32}$$

which provides a relation between the orbital angular velocity ω and the "spin" angular velocity Ω, where,

$$\mu = \frac{mM}{M+m} \tag{33}$$

is the reduced mass of the system. The total kinetic + potential energy of the system is,

$$E = KE + PE = \frac{I\Omega^2}{2} + \frac{(Mr_M^2 + mr_m^2)\omega^2}{2} - \frac{GMm}{R} = \frac{I\Omega^2}{2} + \frac{\mu R^2 \omega^2}{2} - \frac{GMm}{R}\,, \tag{34}$$

where G is Newton's gravitational constant.

The equations of motion,

$$M\ddot{\mathbf{r}}_M = -m\ddot{\mathbf{r}}_m = -\frac{GMm\mathbf{R}}{R^2}\,, \qquad \mathbf{R} = \mathbf{r}_M - \mathbf{r}_m, \tag{35}$$

lead readily for circular orbits to,

$$\frac{\mu R^2 \omega^2}{2} = \frac{GMm}{2R} = -\frac{PE}{2}\,, \qquad R^3 = \frac{GMm}{\mu\omega^2}\,. \tag{36}$$

The first form of eq. (36) is true in general for a $1/r^2$ attractive force according to the so-called virial theorem,[8] while the second form is Kepler's (3rd) law for the system when the orbits are nearly circular, as assumed here.

If we accept Kant's comment that the eventual effect of tidal friction is to make the (final) "spin" angular velocity Ω_0 "locked" to the final orbital angular velocity ω_0, at which time the bodies are distance R_0 apart, then conservation of angular momentum (32) and Kepler's 3rd law (36) that $R_i^3 \omega_i^2 = R_0^3 \omega_0^2$ suffice to determine ω_0 and R_0, according to,

$$L = I\Omega_i + \mu R_i^2 \omega_i = (I + \mu R_0^2)\omega_0 \approx \mu R_0^2 \omega_0 = \mu R_i^2 \omega_i \frac{\omega_0^{1/3}}{\omega_i^{1/3}}\,. \tag{37}$$

[8] See, for example, sec. 10 of L.D. Landau and E.M. Lifshitz, *Mechanics*, 3rd ed. (Pergamon, 1976), http://kirkmcd.princeton.edu/examples/mechanics/landau_mechanics.pdf.

Thus,

$$\omega_0 = \Omega_0 \approx \omega_i \left(1 + \frac{I\Omega_i}{\mu R_i^2 \omega_i}\right)^2 \qquad \text{for} \qquad (I \ll \mu R_0^2). \tag{38}$$

For the Earth-Moon system, this analysis predicts that the eventual day/month will be 48 present days,[9] *The only known example of a two-body system that has evolved to a final state in which both "days" equal their common "month" is Pluto and Charon.*

To establish analytically that a final state can exist with $\omega_0 = \Omega_0$, we use eq. (36) in eq. (32) to write,[10]

$$L = I\Omega + \frac{(G^2 \mu M^2 m^2)^{1/3}}{\omega^{1/3}} \equiv I\Omega + \frac{C}{\omega^{1/3}} \equiv I\Omega + x, \qquad \omega = \frac{C^3}{(L - I\Omega)^3}\,, \tag{39}$$

where x is the orbital angular momentum (which can be taken as positive by suitable choice of direction of the polar axis),

$$x = \mu R^2 \omega = \frac{C}{\omega^{1/3}} > 0, \qquad C = (G^2 \mu M^2 m^2)^{1/3}, \qquad \Omega = \frac{L - x}{I}\,. \tag{40}$$

Since the angular momentum is constant we can write,

$$0 = \frac{dL}{dx} = I\frac{d\Omega}{dx} + 1, \qquad \frac{d\Omega}{dx} = -\frac{3\omega^{4/3}}{C}\frac{d\Omega}{d\omega} = -\frac{1}{I}\,, \tag{41}$$

which implies that if Ω decreases then so does ω.

The energy (34) can now be written as,

$$E = \frac{I\Omega^2}{2} - \frac{GMm}{2R} = \frac{I\Omega^2}{2} - \frac{C\omega^{2/3}}{2} = \frac{I\Omega^2}{2} - \frac{C^3}{2x^2} = \frac{(L-x)^2}{2I} - \frac{C^3}{2x^2}\,. \tag{42}$$

Note that $E(x = 0) = -\infty$ and that $E(x = \infty) = \infty$, but that $E(x)$ is not necessarily a monotonic function. Taking the derivative of eq. (42), we have that,

$$\frac{dE}{dx} = -\frac{L - x}{I} + \frac{C^3}{x^3} = -\Omega + \omega = \frac{x}{I} + \frac{C^3}{x^3} - \frac{L}{I}\,. \tag{43}$$

Hence, if an equilibrium exists, where $dE(x_0)/dx = 0$, we have that $\Omega_0(x_0) = \omega_0(x_0)$, and the equilibrium "spin" and orbital angular velocities are "locked".

If we suppose that M represents the Moon and m represents the Earth, the above argument suggests that the period of rotation of the Moon about its axis should be equal to its orbital period once a certain kind of equilibrium was established in the

[9]First computed in sec. 276 of W. Thomson and P.G. Tait, *A Treatise on Natural Philosophy*, (Cambridge U. Press, 1879, 1896), http://kirkmcd.princeton.edu/examples/astro/thomson_tait_sec276.pdf.

[10]The remainder of this solution follows G.H. Darwin, *The Determination of the Secular Effects of Tidal Friction by a Graphical Method*, Proc. Roy. Soc. London **29**, 168 (1879), http://kirkmcd.princeton.edu/examples/astro/darwin_prsl_29_168_79.pdf.

past.[11] *However, we can also suppose that M represents the Earth and m represents the Moon, in which case we anticipate that the Earth-Moon system can evolve until the Earth day equals one month, and both the Earth and the Moon present the same face to one another at all times.*

The equilibrium, $dE/dx = 0$, exists in eq. (43) only if,

$$\text{Min}\left(\frac{x}{I} + \frac{C^3}{x^3}\right) = 4\left(\frac{C}{3I}\right)^{3/4} < \frac{L}{I}, \tag{44}$$

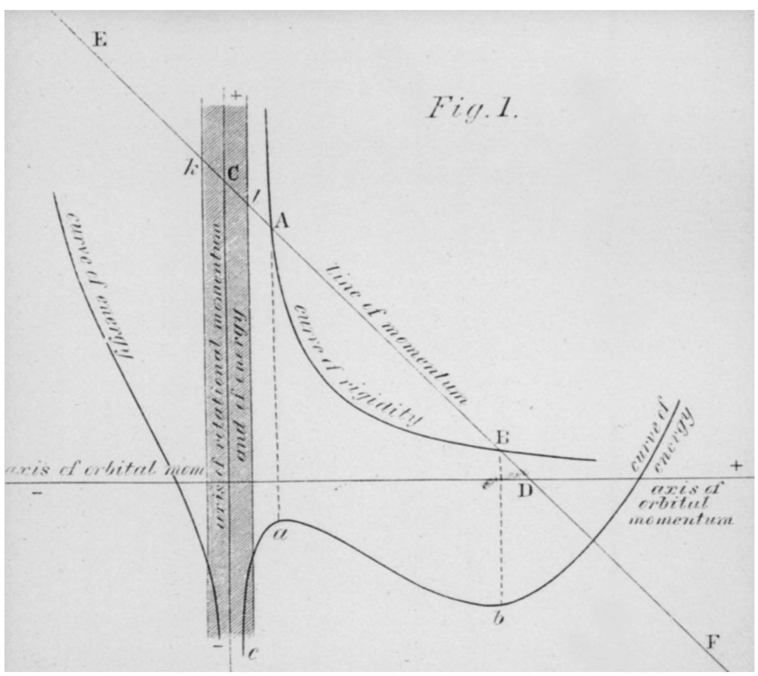
$$L = I\Omega_i + x_i > L_{\text{min}} = 4I\left(\frac{C}{3I}\right)^{3/4} = \frac{4(3C^3 I)^{1/4}}{3} = \frac{4x_{0,\text{min}}}{3} > 0, \tag{45}$$

noting that the minimum occurs for $x_{0,\text{min}} = (3C^3 I)^{1/4}$. The requirement that L be positive (in the sense of the orbital angular momentum) means that if the "spin" angular momentum $I\Omega$ is opposite to the orbital angular momentum ($\mu R^2\omega = x$) and large, no equilibrium will exist.[12] Furthermore, if the evolution is to involve increasing orbital angular momentum x, as in the Earth-Moon system, the initial "spin" angular momentum $I\Omega_i$ must be a substantial fraction of the total for eventual equilibrium with $\omega_0 = \Omega_0$ to exist.

In greater detail, the equilibrium value x_0 of the orbital angular momentum is a root of the quartic equation obtained by setting eq. (43) to zero,

$$x^4 - Lx^3 + C^3 I = 0. \tag{46}$$

When the condition (45) is satisfied, the so-called discriminant Δ of the quartic equation (46) is negative, which implies that there are two real roots and two complex roots.



[11]If the Moon consists of matter somehow ejected from the Earth, it is probable that the Moon was created with the lunar day equal to a month (at that early time).

[12]In planetary systems where all objects have a common origin in an initial gas cloud, the sense of the angular momenta of all objects is typically the same.

The figure on the previous page, from Darwin (1879), shows (among others) the lines labeled "curve of energy" which correspond to eq. (42); the curve on the left is for a case where no equilibrium exists, while for the curve on the right the stable equilibrium is at b, corresponding to the root x_0 of eq. (46), and the unstable equilibrium is at a, corresponding to the root x_1.

When condition (45) is satisfied, the two real roots are,[13]

$$x_{0,1} = \frac{L}{4} - S \pm \sqrt{\frac{3L^2}{4} + \frac{L^3}{8S} - 4S^2}\,, \qquad (47)$$

where,

$$S = \frac{1}{2}\sqrt{\frac{L^2}{4} + \frac{1}{3}\left(Q + \frac{81L_{\rm min}^4}{256Q}\right)}, \qquad Q = \left(\frac{729L_{\rm min}^4}{256}\right)^{1/3}\left(\frac{L^2 + \sqrt{L^4 - L_{\rm min}^4}}{2}\right)^{1/3}. \qquad (48)$$

As tidal friction decreases the energy of the system, the equilibrium at x_0 (where $\omega_0 = \Omega_0$) can only be reached if the initial value of x is greater than x_1; otherwise the system evolves towards $x = 0$, which implies increasing ω, increasing Ω, and decreasing R until the two masses merge.[14] Hence, we deduce a condition on the initial orbital angular velocity ω_i for the existence of an equilibrium final state where $\omega_0 = \Omega_0$,

$$\omega_i < \frac{C^3}{x_1^3}\,. \qquad (49)$$

If $\omega_i < C^3/x_0^3$, then as the energy decreases with time x decreases, ω increases, and R decreases; whereas if $C^3/x_0^3 < \omega_i < C^3/x_1^3$, then as the energy decreases x increases, ω decreases, and R increases with time. *The Earth-Moon system is of the latter type.*

Lastly, we equate the expressions for ω in eqs. (32) and (39) to obtain,

$$R = \frac{(L - I\Omega)^2}{\mu^{1/2}C^{3/2}}\,, \qquad \frac{dR}{d\Omega} = -\frac{2I(L - I\Omega)}{\mu^{1/2}C^{3/2}} = -\frac{2\mu^{1/2}I\omega R^2}{C^{3/2}}\,, \qquad (50)$$

which implies that if Ω decreases then R increases. Taking the derivative of the first form of eq. (42), we find,

$$\frac{dE}{d\Omega} = I\Omega + \frac{GMm}{2R^2}\frac{dR}{d\Omega} = I(\Omega - \omega), \qquad d\Omega = \frac{dE}{I(\Omega - \omega)}\,. \qquad (51)$$

[13]We use the notation of https://en.wikipedia.org/wiki/Quartic_function

[14]This behavior has come to be called the satellite paradox, that the effect of an energy-dissipation mechanism in a (gravitational) two-body system can be to increase the kinetic energies of the bodies. For example, the effect of atmospheric drag on a satellite in a low orbit about the Earth is to increase the speed of the satellite as it slowly spirals inwards towards the Earth's surface, Prob. 5 of http://kirkmcd.princeton.edu/examples/ph205set5.pdf.
Thus, the intuitive argument of Kant, as seconded by Thomson and Tait, that tidal friction lengthens the "day" and the "month/year" is not true in general.

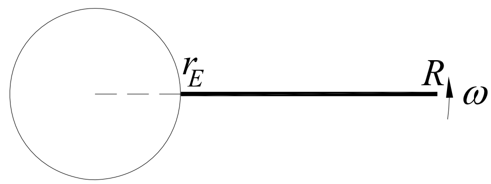
Hence, as tidal friction reduces the kinetic + potential energy E, the "spin" angular velocity Ω decreases if $\Omega > \omega$, as holds for the Earth-Moon system. At the same time, the Earth-Moon distance R increases according to eq. (50), and the orbital angular frequency ω decreases according to eq. (41).

Additional discussion of spin-orbit coupling and "locking" is given in, for example, P. Goldreich and S.J. Pearle, *Resonant Spin States in the Solar System*, Nature **209**, 1078 (1966), http://kirkmcd.princeton.edu/examples/astro/goldreich_nature_209_1078_66.pdf
The Dynamics of Planetary Rotations, Ann. Rev. Astro. Astrophys. **6**, 287 (1968),
http://kirkmcd.princeton.edu/examples/astro/goldreich_araa_6_287_68.pdf
P. Goldreich, *History of the Lunar Orbit*, Rev. Geophys. **4**, 411 (1966),
http://kirkmcd.princeton.edu/examples/astro/goldreich_rg_4_411_66.pdf,
(*which also considered effects of tilts of the axes of the spinning bodies with respect to the orbital plane*).
C. Clouse, A. Ferroglia and M.C.N. Fiolhais, *Spin-orbit gravitational locking—an effective potential approach*, Eur. J. Phys. **43**, 035602 (2022),
http://kirkmcd.princeton.edu/examples/mechanics/clouse_ejp_43_035602_22.pdf

This simplified model leaves open the question of the very early history of the Earth-Moon system, when the Earth day was much shorter than at present, and the Earth-Moon distance was comparable to the Earth's radius. The contemporary view that the Moon was ejected from the Earth during a collision with a large asteroid first gained prominence in W.A. Hartmann and D.R. Davis, *Satellite-Sized Planetesimals and Lunar Origin*, Icarus 24, 504 (1975),
http://kirkmcd.princeton.edu/examples/astro/hartmann_icarus_24_504_75.pdf

3. (a) **Skyhook.**

The concept of the skyhook became well known following the publication of J.D. Isaacs *et al.*, *Satellite Elongation into a True "Sky-Hook"*, Science **151**, 682 (1966), http://kirkmcd.princeton.edu/examples/mechanics/isaacs_science_151_682_66.pdf. This problem follows J. Gersten, H. Soodak and M. Tiersten, *Jack and the skyhook: The beanstalk revisited*, Am. J. Phys. **49**, 118 (1981),
http://kirkmcd.princeton.edu/examples/mechanics/gersten_ajp_49_118_81.pdf
For a review, see Y. Chen *et al.*, *History of the Tether Concept and Tether Missions: A Review*, IRSN A.A. **2013**, 502973,
http://kirkmcd.princeton.edu/examples/mechanics/chen_irsnaa_2013_502973.pdf



For a length dr of the rope of density ρ to be in a geosynchronous orbit with angular velocity ω at distance r we must have,

$$F_{\rm inwards} = T(r+dr) - T(r) + \frac{GM_E\rho\, dr}{r^2} = ma = \rho\, dr\, \omega_E^2 r, \tag{52}$$

$$\frac{dT}{dr} = \rho\,\omega_E^2 r - \frac{GM_E\rho}{r^2}, \qquad T = T_0 + \frac{\rho\,\omega_E^2 r^2}{2} + \frac{GM_E\rho}{r}. \tag{53}$$

where $T(r)$ is the tension in the rope which vanishes at its ends, $r_{\rm min} = r_E$ and $r_{\rm max} = R$,

$$0 = \frac{\rho\,\omega_E^2}{2}(R^2 - r_E^2) + GM_E\rho\left(\frac{1}{R} - \frac{1}{r_E}\right), \tag{54}$$

$$\frac{\omega_E^2}{2}R^3 - \left(\frac{\omega_E^3 r_E^2}{2} + \frac{GM_E}{r_E}\right)R + GM_E = 0. \tag{55}$$

Since $R - r_e$ must be a solution, we can factorize the cubic equation,

$$(R - r_E)\left(\frac{\omega_E^2}{2}R^2 + \frac{r_E\,\omega_E^2}{2}R - \frac{GM_E}{r_E}\right) = 0. \tag{56}$$

the positive root of the quadratic equation gives the nontrivial solution for R,

$$R = \frac{-r_E\,\omega_E^2/2 + \sqrt{(r_E\,\omega_E^2/2)^2 + 4(\omega_E^2/2)(GM_e/r_E)}}{\omega_E^2} = \frac{r_E}{2}\left(\sqrt{1 + \frac{8Gm_E}{\omega_E^2 r_E^3}} - 1\right) \tag{57}$$

Numerically,

$$r_E \approx 6\times 10^6 \text{ m}, \quad \omega_E = \frac{2\pi}{24\text{ hr}} = \frac{2\pi}{8.6\times 10^4} = 7.3\times 10^{-5}, \quad \frac{GM_E}{r_E^2} = g \approx 10, \tag{58}$$

$$R \approx \frac{r_E}{\omega_E}\sqrt{\frac{2g}{r_E}} \approx \frac{6\times 10^6}{7.3\times 10^{-5}}\sqrt{\frac{20}{6\times 10^6}} \approx 1.5\times 10^8 \text{ m} \approx 25 r_E. \tag{59}$$

The skyhook extends almost half the way to the Moon!

To assess the stability of the skyhook, we note that for a point mass to be in a bound orbit about the Earth, its energy other than that of rotation about its center of mass (*i.e.*, kinetic energy of its c.m. motion plus potential energy) must be negative.

$$\mathrm{KE_{cm}} = \frac{1}{2}\rho(R - r_E)\,\omega_E^2\,\left(\frac{R + r_e}{2}\right)^2 \approx \frac{\rho\,\omega_E^2 R^3}{8} \approx \frac{\rho(7.3\times10^{-5})^2(1.5\times10^8)^3}{8} \approx 2.2\times10^{15}\rho, \qquad (60)$$

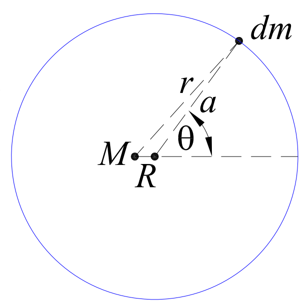
$$\mathrm{PE} = -\int_{r_E}^{R}\frac{GM_E\rho\,dr}{r} = -\frac{GM_e\rho}{r_E^2}r_e^2\ln\frac{R}{r_E} \approx 10\rho(6\times10^6)^2\ln 25 \approx -1.2\times10^{15}\rho. \qquad (61)$$

The energy $\mathrm{KE_{cm}} + \mathrm{PE}$ is positive, which suggests that the orbit is not bound (although in principle it could be stabilized by a very large mass at its upper end, in geosynchronous orbit).

For a more detailed analysis, we can note that although angular momentum of the skyhook about the Earth is not conserved, there is a conserved generalized momentum, which permits an effective potential to be constructed. This is more straightforward for a dumbell satellite than for the skyhook, and shows that a long dumbell is unstable in orbit, http://kirkmcd.princeton.edu/examples/dumbell.pdf

(b) **Ringworld.**

We consider the force on the center of mass of the ring, of mass m and radius a when it is displaced by small distance $R \ll a$ from the center of the star of mass M.



To deduce the force, we compute the gravitational potential energy of the ring,

$$V(R) = -GM\int_0^{2\pi}\frac{m}{2\pi}\frac{d\theta}{r} = -\frac{GMm}{2\pi}\int_0^{2\pi}\frac{d\theta}{\sqrt{a^2 - 2aR\cos\theta + R^2}}$$

$$\approx -\frac{GMm}{2\pi a}\int_0^{2\pi}d\theta\left(1 + \frac{2R\cos\theta}{R} - \frac{R^2}{a^2} + \frac{3R^2\cos^2\theta}{2a^2}\right) = -\frac{GMm}{a}\left(1 + \frac{R^2}{2a^2}\right), \qquad (62)$$

$$\text{then,} \qquad F = -\frac{dV}{dR} = \frac{GMmR}{a^3} > 0. \qquad (63)$$

So the perturbation grows until a point on the ring collides with the star.

This result was first established by Laplace, who noted thereby that the rings of Saturn cannot be solid; Chap. 6, Book 2 of

http://kirkmcd.princeton.edu/examples/mechanics/laplace_mc_v2.pdf

http://kirkmcd.princeton.edu/examples/mechanics/laplace_mc_v2_english.pdf.

That an annular ring of particles can be stable was established by Maxwell (1858) in his second major work,

http://kirkmcd.princeton.edu/examples/mechanics/maxwell_saturn_59.pdf.

4. (a) **Modeling.**

This problem is from Art. 369, p. 296 of E.J. Routh, *The Elementary Part of a Treatise on the Dynamics of a System of Rigid Bodies*, 7th ed. (Macmillan, 1905), http://kirkmcd.princeton.edu/examples/mechanics/routh_elementary_rigid_dynamics.pdf

In our models, we scale lengths according to,

$$l' = \alpha l, \tag{64}$$

while keeping the density ρ unchanged. Then, masses scale as,

$$m' \propto \rho l'^3 = \alpha^3 \rho l^3, \qquad m' = \alpha^3 m. \tag{65}$$

We scale time according to,

$$t' = \beta t, \tag{66}$$

such that velocities and accelerations scale as,

$$v' \propto \frac{l'}{t'} \propto \frac{\alpha l}{\beta t}, \qquad v' = \frac{\alpha}{\beta} v, \qquad a' \propto \frac{l'}{t'^2} \propto \frac{\alpha l}{\beta^2 t^2}, \qquad a' = \frac{\alpha}{\beta^2} a. \tag{67}$$

Then, forces scale as,

$$F' = m'a' = \alpha^3 m \cdot \alpha\beta^2 a = \frac{\alpha^4}{\beta^2} F. \tag{68}$$

However, we can't scale the acceleration g due to gravity. Hence,

$$F = mg \to F' = m'g = \alpha^3 mg = \alpha^3 F. \tag{69}$$

So, if gravity is among the relevant forces, we must have that,

$$\alpha^3 = \frac{\alpha^4}{\beta^2}, \qquad \beta = \alpha^{1/2}, \tag{70}$$

and

$$v' = \alpha^{1/2} v. \tag{71}$$

Example: A model of a ship with $\alpha = 1/n$ is observed to experience a drag force F' when traveling through a tank of water with velocity v'. The drag force depends on gravity, which determines how much of the ship is below to water line and subject to drag. Then, according to eq. (71),

$$v = \alpha^{-1/2} v' = \sqrt{n}\, v'. \tag{72}$$

(b) If we can write a drag force as $F = C_d v^p A^q$ where $A \propto l^2$ is the area of the object projected onto a plane perpendicular to $\mathbf{v}$, and C_d, p and q are constants, then the force scales as,

$$\frac{\alpha^4}{\beta^2} F = F' = C_d v'^p A'^q = C_d \frac{\alpha^p}{\beta^p} v^p \alpha^{2q} A^q = \frac{\alpha^{p+2q}}{\beta^p} F, \tag{73}$$

using eqs. (66)-(68) since F does not necessarily depend on g, and noting that $A' = \alpha^2 A$. Hence, we must have $p = 2$ and $q = 1$, *i.e.*, $F = C_d v^2 A$,

(c) A particle of mass m starts from rest, subject to the force $\mathbf{F} = -A\mathbf{r} - B\mathbf{v}$, where A and B are constants. Then, the dimensions of A and B are,

$$[A] = \frac{[F]}{[r]} = \frac{[m]}{[t]^2}, \qquad [B] = \frac{[F]}{[v]} = \frac{[m]}{[t]}. \tag{74}$$

The time T for the particle to reach the origin is a function of m, r_0, A and B, say $T = km^\alpha r_0^\beta A^\gamma B^\delta$ for some dimensionless constant k. Then, by dimensional analysis,

$$[T] = [t] = [m]^\alpha \cdot [l]^\beta \cdot \frac{[m]^\gamma}{[t]^{2\gamma}} \cdot \frac{[m]^\delta}{[t]^\delta} = [l]^\beta [m]^{\alpha+\gamma+\delta}[t]^{-2\gamma-\delta}, \tag{75}$$

and we must have,

$$\beta = 0, \qquad \alpha + \gamma + \delta = 0, \qquad -2\gamma - \delta = 1, \qquad \gamma = \alpha - 1, \qquad \delta = 1 - 2\alpha, \tag{76}$$

$$T = km^\alpha A^{1-\alpha} B^{1-2\alpha}, \tag{77}$$

so the time to reach the origin is independent of the initial distance r_0.

(d) A particle of mass m starts from rest subject to the force $\mathbf{F} = -A\,\hat{\mathbf{r}}/r^n$, where $A > 0$ and n are constants. Here, the dimensions of A are,

$$[A] = [F] \cdot [r]^n = \frac{[m][l]^{n+1}}{[t]^2}. \tag{78}$$

The time T for the particle to reach the origin is a function of m, r_0, and A, say $T = km^\alpha r_0^\beta A^\gamma$ for some dimensionless constant k. Then, by dimensional analysis,

$$[T] = [t] = [m]^\alpha \cdot [l]^\beta \cdot \frac{[m]^\gamma [l]^{\gamma(n+1)}}{[t]^{2\gamma}} = [l]^{\alpha+\beta+n\beta}[m]^\beta [t]^{-2\beta}, \tag{79}$$

and we must have,

$$\beta + \gamma(n+1) = 0, \qquad \alpha + \gamma = 0, \qquad \gamma = -\frac{1}{2}, \qquad \alpha = \frac{1}{2}, \qquad \beta = \frac{n+1}{2}, \tag{80}$$

$$T = k\sqrt{\frac{m}{A}} r_0^{\frac{n+1}{2}}. \tag{81}$$

The time for the particle to fall to the origin from initial distance r_0 scales as $T \propto r_0^{\frac{n+1}{2}}$.

5. **Biomechanics.**

(a) An animal can store an water proportional to its volume, *i.e.*, $V_{\rm water} \propto L^3$ where L is the characteristic size of the animal. The rate of loss of water is proportional to its surface area, $dV_{\rm water}/dt \propto L^2$. Hence, the survival time of the animal against dehydration scales as $V_{\rm water}/\dot{V}_{\rm water} \propto L$.

(b) The mechanical power output P of an animal is limited by the rate it can exhaust heat energy – by sweating, which process scales as the surface area of the animal. Hence, $P \propto L^2$.

Horses have size $\approx$ twice that of humans, whose sustained power output is about 1/4 horsepower.

(c) If air resistance limits the running speed of an animal, then the limiting force is $F \propto v_{\rm max} A \propto v_{\rm max} L^2$, recalling Prob. 3(b) of this Set. The power required is $P_{\rm max} = F v_{\rm max} \propto v_{\rm max}^2 L^2$, and according to part (b) above, the maximum power that an animal can exert steadily scales as L^2.

Hence, the maximum running speed of an animal is largely independent of L (on level ground).

However, when running uphill, the power required is $F v_{\rm max} \propto m g v_{\rm max} \propto L^3 g v_{\rm max}$, whose limit scales as L^2 according to part (b). In this case, $v_{\rm max} \propto 1/L$.

Smaller animals can run faster uphill.

(d) Animals can jump to height $h = v_0^2/2g$, where v_0 is the peak vertical velocity an animal can give itself before losing contact with the ground.

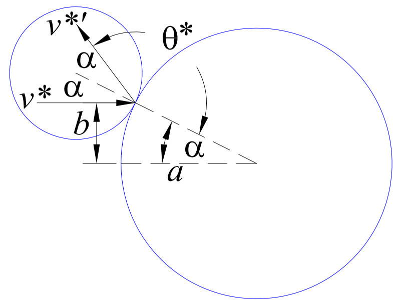
This velocity is related by $mv_0^2/2 = Fd$ where F is the peak force the animal can exert for a short time, and d is a relevant vertical scale of the animal, $\propto L$. The peak force F is exerted by some relevant muscle and is limited by the cross sectional area of the muscle, which scales as L^2. That is, $h \propto v_0^2 \propto Fd/m \propto L^2 \cdot L/L^3 = L^0$.

The height an animal can jump is largely independent of its size!

A more fanciful application of dimensional analysis to animals is at
http://kirkmcd.princeton.edu/examples/pigs_can_fly.pdf

6. (a) The elastic scattering of two spheres takes place in the plane containing their line of centers and the relative velocity. The two final-state velocities have four components to be determined, but conservation of energy and momentum provide only three relations among the four components.

In the center-of-mass frame, the magnitudes of the velocities are unchanged by the elastic collision. The usual premise is that in the center-of-mass frame, the velocity components along the line of centers at the moment of contact are reversed.



Then, a sphere with center-of-mass velocity $\mathbf{v}^*$ that collides with the sphere of radius a with impact parameter $b = a\sin\alpha$, scatters by angle $\theta^* = \pi - 2\alpha$,

$$b = a\sin\alpha = a\sin\left(\pi - \frac{\theta^*}{2}\right) = a\cos\frac{\theta}{2}\,, \qquad db = -\frac{a}{2}\sin\frac{\theta^*}{2}\,d\theta^*. \tag{82}$$

If there are N such spheres per area A perpendicular to the direction of $\mathbf{v}^*$, then the number in an annulus of thickness db about radius b are,

$$dN = \frac{N}{A}2\pi b\,db = \frac{N}{A}\pi a^2\cos\frac{\theta^*}{2}\sin\frac{\theta^*}{2}\,d\theta^* = \frac{N}{A}\frac{\pi a^2}{2}\sin\theta^*\,d\theta^* = \frac{N}{A}\frac{\pi a^2}{2}\,d\cos\theta^*. \tag{83}$$

The (center-of-mass-frame) scattering differential cross section is,

$$d\sigma = \frac{dN}{N/A} = \frac{\pi a^2}{2}\sin\theta^*\,d\theta^* = \frac{\pi a^2}{2}\,|d\cos\theta^*|\,. \tag{84}$$

The total scattering cross section is $\sigma = \pi a^2$, the geometric cross section of the sphere of radius a, which result holds in any frame. Also, the scattering is isotropic in the center-of-mass frame, $d\sigma/d\Omega = (1/2\pi)d\sigma/d\cos\theta = \pi a^2/4\pi$.

(b) In the lab frame, the sphere of radius a and mass M has velocity $V = 0$, while the other sphere has velocity v and mass m.

The velocity of the center of mass (in the lab frame) is,

$$v_{\rm cm} = \frac{mv + MV}{m+M} = \frac{mv}{m+M}\,. \tag{85}$$

The velocities of the spheres in their center-of-mass frame are,

$$v^* = v - v_{\rm cm} = \frac{Mv}{m+M}\,, \qquad V^\star = V - v_{\rm cm} = -v_{\rm cm} = -\frac{mv}{m+M}\,. \tag{86}$$

In a scatter of mass m by angle θ^* in the center-of-mass frame, the final velocity components are,

$$v'^*_{\parallel} = v^* \cos\theta^*, \qquad v'^*_{\perp} = v^* \sin\theta^*, \qquad V'^*_{\parallel} = V^* \cos\theta^*, \qquad V'^*_{\perp} = V^* \sin\theta^*. \quad (87)$$

In the lab frame, the final velocity components are,

$$v'_{\parallel} = v'^*_{\parallel} + v_{\rm cm} \qquad v'_{\perp} = v'^*_{\perp}, \qquad V'_{\parallel} = V'^*_{\parallel} v_{\rm cm}, \qquad V'_{\perp} = V'^*_{\perp}. \quad (88)$$

The final velocities in the lab frame are given by,

$$v'^2 = v^{*2} + 2v^* v_{\rm cm}\cos\theta^* + v_{\rm cm}^2 \quad (89)$$

$$V'^2 = V^{*2} + 2V^* v_{\rm cm}\cos\theta^* + v_{\rm cm}^2 = 2v_{\rm cm}^2(1 - \cos\theta^\star) \quad (90)$$

The kinetic energy gained in the lab frame by the sphere of radius a is the same as that lost by the other sphere, such that,

$$T_{\rm lost} = \frac{m(v^2 - v'^2)}{2} = \frac{MV'^2}{2} = Mv_{\rm cm}^2(1 - \cos\theta^\star), \quad (91)$$

$$T_{\rm lost,min} = 0, \qquad T_{\rm lost,max} = 2Mv_{\rm cm}^2. \quad (92)$$

Further, since the scattering is isotropic in the center-of-mass frame, the distribution of $T_{\rm lost}$ is uniform between 0 and $T_{\rm lost,max}$, so the differential cross section can be written as,

$$\frac{d\sigma}{dT_{\rm lost}} = \frac{d\sigma}{d\cos\theta^*}\frac{d\cos\theta^*}{dT_{\rm lost}} = \frac{\pi a^2}{T_{\rm lost,max}} = \frac{\sigma}{T_{\rm lost,max}}. \quad (93)$$

7. (a) We consider a projectile with impact parameter b and velocity v_0 when far from the Moon (and ignore the gravity of the Earth, Sun, *etc.*), and compute the distance $r_{\rm min}$ of closest approach to the moon. The collision cross section is then $\sigma = \pi b_0^2$, where b_0 is the impact parameter corresponding to $r_{\rm min} = R$, the radius of the Moon.

Conservation of energy tells us that,

$$\frac{mv_0^2}{2} = \frac{mv_{\rm min}^2}{2} - \frac{GMm}{r_{\rm min}}, \tag{94}$$

where $v_{\rm min}$ is the velocity of the projectile when at $r_{\rm min}$, and M is the mass of the Moon. Conservation of angular momentum about the center of the Moon tells us that,

$$mv_0 b = mv_{\rm min} r_{\rm min}, \qquad v_{\rm min} = v_0 \frac{b}{r_{\rm min}}. \tag{95}$$

Combining eqs. (94) and (95),

$$v_0^2 = v_{\rm min}^2 - \frac{2GM}{r_{\rm min}} = v_0^2 \frac{b^2}{r_{\rm min}^2} - \frac{2GM}{r_{\rm min}}, \tag{96}$$

$$r_{\rm min}^2 + \frac{2GM}{v_0^2} r_{\rm min} - b^2 = 0, \tag{97}$$

$$r_{\rm min} = -\frac{GM}{v_0^2} + \sqrt{\left(\frac{GM}{v_0^2}\right)^2 + b^2}. \tag{98}$$

$$b^2 = r_{\rm min}^2 + \frac{2GM}{v_0^2} r_{\rm min}. \tag{99}$$

For $r_{\rm min} = R$, the radius of the Moon, we have that,

$$b_0^2 = R^2 + \frac{2GM}{v_0^2} R = R^2 \left(1 + \frac{v_e^2}{v_0^2}\right). \tag{100}$$

noting that the escape velocity from the Moon is related by $v_e^2 = 2GM/R$. Hence, the collision cross section is,

$$\sigma = \pi b_0^2 = \pi R^2 \left(1 + \frac{v_e^2}{v_0^2}\right). \tag{101}$$

If the force of gravity were repulsive, $F = GMm/r^2$, the sign of the gravitational potential energy would be reversed, and the above formalism applies with the change $G \to -G$. In this case the collision cross section would be,

$$\sigma = \pi R^2 \left(1 - \frac{v_e^2}{v_0^2}\right) \qquad \text{(repulsive gravity, } v_0 > v_e\text{)}. \tag{102}$$

For $v_0 < v_e = \sqrt{2GM/R}$, even a projectile with impact parameter $b \approx 0$ would not strike the Moon.

(b) The effective potential associated with the potential $V(r) = -C/r^4$ for a constant C, for the interaction of a point mass/ion m with angular momentum L about the fixed force center/atom (with mass $M \gg m$), is,

$$V_{\rm eff}(r) = \frac{L^2}{2mr^2} + V(r) = \frac{L^2}{2mr^2} - \frac{C}{r^4}\,, \qquad (103)$$

$$\frac{dV_{\rm eff}}{dr} = -\frac{L^2}{mr^3} + \frac{4C}{r^5}\,, \qquad (104)$$

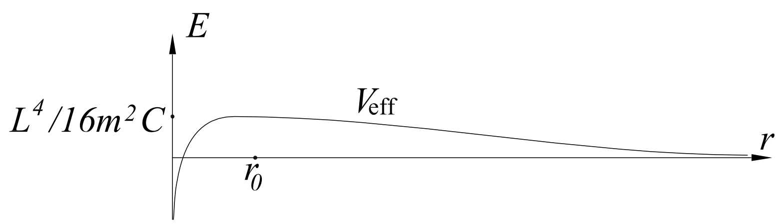
$$\frac{d^2V_{\rm eff}}{dr^2} = \frac{3L^2}{mr^4} - \frac{20C}{r^6}\,. \qquad (105)$$

An equilibrium circular orbit exists at radius r_0, with equilibrium angular velocity $\Omega = L/mr_0^2$, related by ,

$$r_0^2 = \frac{4mC}{L^2}\,, \quad \Omega = \frac{L}{4m^2C}\,, \quad E_0 = \frac{mr_0^2\Omega^2}{2} - \frac{C}{r_0^4} = \frac{L^4}{8m^2C} - \frac{L^4}{16m^2C} = \frac{L^4}{16m^2C}\,. \qquad (106)$$

but this equilibrium is unstable as,

$$k_{\rm eff} = \frac{d^2V_{\rm eff}(r_0)}{dr^2} = \frac{3L^6}{16m^2C^2} - \frac{20L^6}{64m^2C^2} = -\frac{L^6}{8m^2C^2} < 0. \qquad (107)$$



Ions with energy $E = mv_\infty^2/2$ when at large distances from the force center/atom can reach the origin, and be captured by the atom, if $E > E_0 = L^4/16m^2C$.
The angular momentum of an ion with impact parameter b with respect to the atom is $L = mv_\infty b$, so the ion is captured if,

$$E = \frac{mv_\infty^2}{2} > E_0 = \frac{L^4}{16m^2C} = \frac{m^2v_\infty^4 b_{\rm capture}^4}{16C}\,, \qquad b_{\rm capture}^4 < \frac{8C}{mv_\infty^2}\,. \qquad (108)$$

The capture cross section is,

$$\sigma_{\rm capture} = \pi b_{\rm capture,max}^2 = \frac{1}{v_\infty}\sqrt{\frac{8C}{m}}\,. \qquad (109)$$

8. This is Prob. 2, p. 54 of L.D. Landau and E.M. Lifshitz, *Mechanics*, 3rd ed. (Pergamon, 1976), http://kirkmcd.princeton.edu/examples/mechanics/landau_mechanics.pdf

The differential cross section for scattering can be written as,

$$d\sigma = 2\pi b\, db = \pi db^2, \tag{110}$$

where b is the impact parameter of the particle that is scattered. Then,

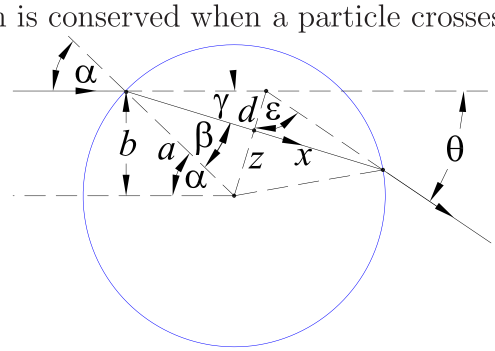
$$\frac{d\sigma}{d\cos\theta} = \pi \left| \frac{db^2}{d\cos\theta} \right|, \tag{111}$$

and the computation of the cross section becomes that of relating the impact parameter b to the scattering angle θ.

For scattering by a force center associated with the potential,

$$V(r) = \begin{cases} 0 & r > a, \\ -V_0 & r < a, \end{cases} \tag{112}$$

the force is radial, and nonzero only at $r = a$. Hence, the motion of a (point) particle in this potential consists of straight line segments, that are deflected if the particle crosses the surface $r = a$. The tangential force is zero everywhere, so in particular, the tangential momentum is conserved when a particle crosses the surface.



When a particle of mass m and momentum $p_{\rm out}$ crosses into the sphere at angle of incidence α as in the figure, its angle changes to β related by,

$$p_{\rm out} \sin\alpha = p_{\rm in} \sin\beta, \tag{113}$$

which has the form of Snell's law in optics, with the indices of refraction inside and outside of the sphere proportional to the particle's momenta there. If we define the index of refraction to be 1 outside the sphere, then the index inside is $n = p_{\rm in}/p_{\rm out}$.

Also, mechanical energy $E = T + V = p^2/2m + V$ is conserved so that,

$$E_{\rm out} = T_{\rm out} = \frac{p_{\rm out}^2}{2m} = E_{\rm in} = T_{\rm in} - V_0 = \frac{p_{\rm in}^2}{2m} - V_0, \tag{114}$$

$$n = \sqrt{\frac{p_{\rm in}^2/2m}{p_{\rm out}^2/2m}} = \sqrt{\frac{T_{\rm in}}{T_{\rm out}}} = \sqrt{\frac{T_{\rm in}}{T_{\rm in} - V_0}} = \sqrt{\frac{T}{T+V}}, \tag{115}$$

where the last form describes the index n both inside and outside the sphere.

We now relate the impact parameter b to the index n and the scattering angle θ. First, we note that,

$$\theta = \pi - 2\epsilon = \pi - 2\left(\frac{\pi}{2} - \gamma\right) = 2\gamma = 2(\alpha - \beta), \qquad \epsilon = \frac{\pi}{2} - \frac{\theta}{2}. \tag{116}$$

Next, we find two expressions for the distance d, as well as expressions for distances $b = a\sin\alpha$, x and z,

$$x = a\cos\beta = a\sqrt{1 - \frac{\sin^2\alpha}{n^2}} = \sqrt{a^2 - \frac{b^2}{n^2}}, \qquad z = a\sin\beta = \frac{a\sin\alpha}{n} = \frac{b}{n}, \tag{117}$$

$$d = x\tan\gamma = x\tan\frac{\theta}{2} = \sqrt{a^2 - \frac{b^2}{n^2}}\tan\frac{\theta}{2}, \tag{118}$$

and also via the sin rule,

$$\frac{d+z}{\sin\alpha} = \frac{a}{\sin\epsilon}, \qquad d = \frac{a\sin\alpha}{\cos\theta/2} - \frac{b}{n} = b\left(\frac{1}{\cos\theta/2} - \frac{1}{n}\right) = b\frac{n - \cos\theta/2}{n\cos\theta/2}. \tag{119}$$

From eqs. (118) and (119) we have,

$$b\left(n - \cos\frac{\theta}{2}\right) = n\sqrt{a^2 - \frac{b^2}{n^2}}\sin\frac{\theta}{2}, \tag{120}$$

$$b^2\left(n^2 - 2n\cos^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right) = n^2\left(a^2 - \frac{b^2}{n^2}\right)\sin^2\frac{\theta}{2}, \tag{121}$$

$$b^2 = \frac{a^2n^2\sin^2\theta/2}{n^2 + 1 - 2n\cos\theta/2} = \frac{a^2n^2(1-\cos\theta)}{2\left(n^2 + 1 - 2n\sqrt{(1+\cos\theta)/2}\right)}. \tag{122}$$

We digress slightly to note that $\theta_{\rm max}$ occurs when $b = a$,

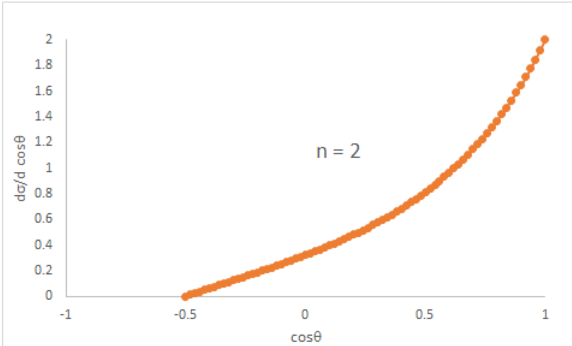
$$n^2 + 1 - 2n\cos\theta_{\rm max}/2 = n^2(1 - \cos^2\theta_{\rm max}/2), \tag{123}$$

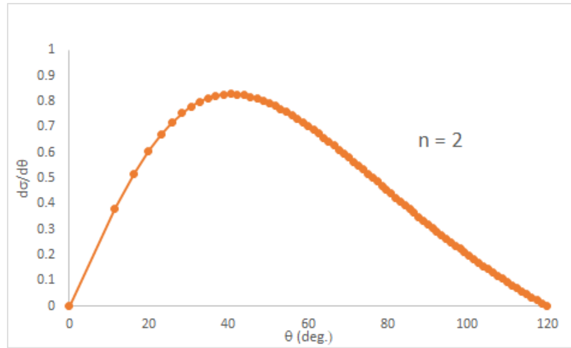
$$n^2\cos^2\frac{\theta_{\rm max}}{2} - 2n\cos\frac{\theta_{\rm max}}{2} + 1 = 0, \qquad \cos\frac{\theta_{\rm max}}{2} = \frac{1}{n}. \tag{124}$$

Finally, the differential cross section is,

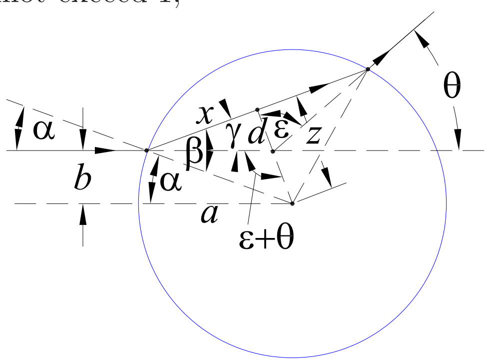
$$\begin{aligned}\frac{d\sigma}{d\cos\theta} = \pi\left|\frac{db^2}{d\cos\theta}\right| &= \left|-\frac{\pi a^2n^2}{2\,(n^2 + 1 - 2n\cos\theta/2)} + \frac{\pi a^2n^2\sin^2\theta/2}{(n^2+1-2n\cos\theta/2)^2}\frac{n}{2\cos\theta/2}\right| \\ &= \left|\pi\frac{a^2n^3\sin^2\theta/2 - a^2n^2\cos\theta/2(n^2+1-2n\cos\theta/2)}{2\cos\theta/2(n^2+1-2n\cos\theta/2)^2}\right| \\ &= \left|\pi\frac{a^2n^3 - a^2n^2\cos\theta/2(n^2+1-n\cos\theta/2)}{2\cos\theta/2(n^2+1-2n\cos\theta/2)^2}\right| \\ &= \left|\frac{\pi a^2n^2}{2\cos\theta/2}\frac{n - n^2\cos\theta/2 - \cos\theta/2 + n\cos^2\theta/2}{(n^2+1-2n\cos\theta/2)^2}\right| \\ &= \frac{\pi a^2n^2}{2\cos\theta/2}\frac{(n\cos\theta/2 - 1)(n - \cos\theta/2)}{(n^2+1-2n\cos\theta/2)^2}.\end{aligned} \tag{125}$$

For $n = 2$, $\cos(\theta_{\rm max}/2) = 0.5$, $\theta_{\rm max} = 120^\circ$. The cross sections $(1/\sigma_{\rm tot})d\sigma/d\cos\theta$ and $(1/\sigma_{\rm tot})d\sigma/d\theta$ are shown in the figures below.





If $V_0 < 0$ then angle β is larger than α, the angle of incidence, as shown in the figure below. Since $\sin\beta$ cannot exceed 1,



Since $\sin\beta = (\sin\alpha)/n$ cannot exceed 1, the figure is only relevant for $\sin\alpha < n$, *i.e.*, for,

$$b = a\sin\alpha < an \equiv b_{\rm max}. \tag{126}$$

At the limiting case, $\sin\alpha_{\rm max} = n$, the scattering angle is,

$$\theta_{\rm max} = \pi - 2\alpha_{\rm max}, \qquad \cos\theta_{\rm max} = -\cos(2\alpha_{\rm max}) = 2\sin^2\alpha_{\rm max} - 1 = 2n^2 - 1, \tag{127}$$

and the scattering is the same as mirror reflection off the surface of the sphere. Indeed, for $\sin\alpha > n$, the particle does not enter the sphere and just reflects off its surface. The total cross section for this mirror scattering is,

$$\sigma_{\rm mirror} = \pi(a^2 - b_{\rm max}^2) = \pi a^2(1 - n^2), \tag{128}$$

and this occurs for $\cos\theta$ in the interval between $\cos\theta = -1$ (backscattering) and $\cos\theta_{\rm max}$, which interval has width $2n^2$. Hence, the differential cross section for mirror scattering can be written as,

$$\frac{d\sigma_{\rm mirror}}{d\cos\theta} = \frac{\sigma_{\rm mirror}}{2n^2} = \pi a^2 \frac{1 - n^2}{2n^2}. \tag{129}$$

For $b < b_{\rm max}$ we relate b to θ in a manner similar to the analysis above when $n < 1$.

$$\theta = \pi - 2\epsilon = \pi - 2\left(\frac{\pi}{2} - \gamma\right) = 2\gamma = 2(\beta - \alpha), \qquad \epsilon = \frac{\pi}{2} - \frac{\theta}{2}. \tag{130}$$

Next, we find two expressions for the distance d, as well as expressions for distances $b = a \sin\alpha$, x and z,

$$x = a\cos\beta = a\sqrt{1 - \frac{\sin^2\alpha}{n^2}} = \sqrt{a^2 - \frac{b^2}{n^2}}, \qquad z = a\sin\beta = \frac{a\sin\alpha}{n} = \frac{b}{n}, \tag{131}$$

$$d = x\tan\gamma = x\tan\frac{\theta}{2} = \sqrt{a^2 - \frac{b^2}{n^2}}\tan\frac{\theta}{2}, \tag{132}$$

and also via the sin rule,

$$\frac{z-d}{\sin\alpha} = \frac{a}{\sin(\epsilon+\theta)} = \frac{a}{\sin(\pi/2+\theta/2)} = \frac{a}{\cos\theta/2}, \tag{133}$$

$$d = \frac{b}{n} - \frac{a\sin\alpha}{\cos\theta/2} = b\left(\frac{1}{n} - \frac{1}{\cos\theta/2}\right) = b\frac{\cos\theta/2 - n}{n\cos\theta/2}. \tag{134}$$

From eqs. (132) and (134) we have,

$$b\left(\cos\frac{\theta}{2} - n\right) = n\sqrt{a^2 - \frac{b^2}{n^2}}\sin\frac{\theta}{2}, \tag{135}$$

$$b^2\left(n^2 - 2n\cos^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right) = n^2\left(a^2 - \frac{b^2}{n^2}\right)\sin^2\frac{\theta}{2}, \tag{136}$$

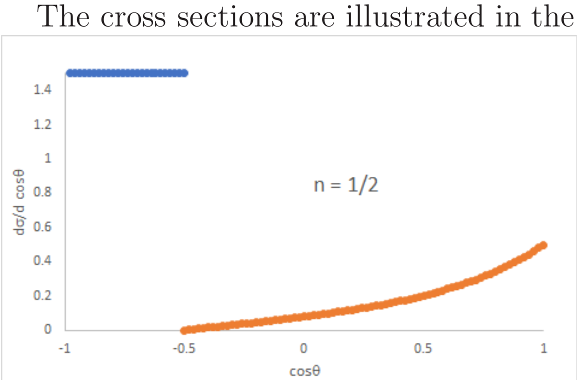
$$b^2 = \frac{a^2n^2\sin^2\theta/2}{n^2+1-2n\cos\theta/2} = \frac{a^2n^2(1-\cos\theta)}{2\left(n^2+1-2n\sqrt{(1+\cos\theta)/2}\right)}, \tag{137}$$

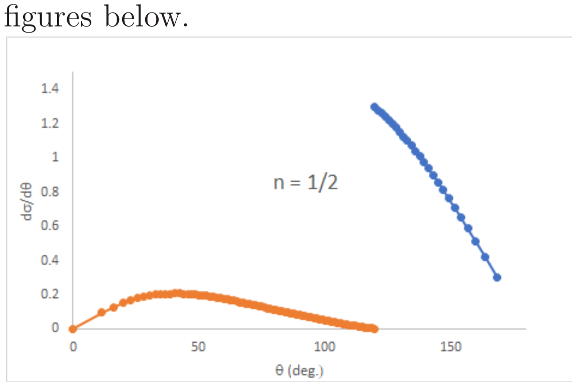
which is the same form as eq. (124). Hence, for $b < na$ and $\cos\theta > \cos\theta_{\rm max} = 2n^2 - 1$, the differential cross section is given by eq. (125).

We illustrate this for $n = 1/2$, $\cos\theta_{\rm max} = -1/2$, $\theta_{\rm max} = 120°$. Then, the mirror scattering cross section is,

$$\frac{1}{\sigma}\frac{d\sigma_{\rm mirror}}{d\cos\theta} = \frac{1-n^2}{2n^2} = 1.5, \qquad \frac{1}{\sigma}\frac{d\sigma_{\rm mirror}}{d\theta} = 1.5\sin\theta, \tag{138}$$

The cross sections are illustrated in the figures below.





9. This is Prob. 1, p. 54 of L.D. Landau and E.M. Lifshitz, *Mechanics*, 3rd ed. (Pergamon, 1976), http://kirkmcd.princeton.edu/examples/mechanics/landau_mechanics.pdf.

As before, the differential cross section for scattering by a fixed center of a central force can be computed from the relation between the impact parameter b and the scattering angle θ,

$$\frac{d\sigma}{d\cos\theta} = \pi\left|\frac{db^2}{d\cos\theta}\right| = 2\pi\frac{d\sigma}{d\Omega}\,, \tag{139}$$

where $d\Omega = 2\pi\, d\cos\theta$ is the differential of solid angle.

The motion is in a plane, which we describe with coordinates r and ϕ where the line $\phi = 0$ is parallel to the velocity $\mathbf{v}_\infty$ of the scattered particle (of mass m) when at large distances from the force center. The conserved angular momentum about the force center is,

$$L = mr^2\,\dot\phi = mbv_\infty \tag{140}$$

The conserved energy E of the scattered particle can be written as,

$$E = \frac{m(\dot r^2 + r^2\dot\phi^2)}{2} + mV(r) = \frac{m\dot r^2}{2} + \frac{m}{2}\frac{b^2v_\infty^2}{r^2} + V = \frac{mv_\infty^2}{2} \tag{141}$$

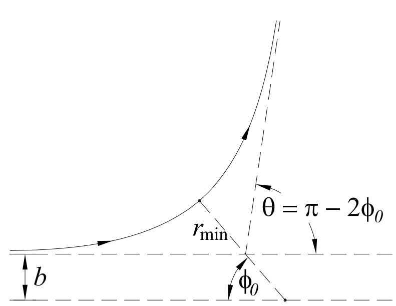
where $V(r)$ is the potential energy of the mass in the central force, assuming that the latter is negligible at large distances. Then,

$$\dot r = \frac{dr}{d\phi}\dot\phi = \frac{L}{mr^2}\frac{dr}{d\phi}\,, \tag{142}$$

$$\frac{d\phi}{dr} = \frac{bv_\infty}{r^2\dot r} = \frac{bv_\infty}{r^2\sqrt{v_\infty^2(1-b^2/r^2) - 2V/m}} = \frac{b}{r^2\sqrt{1-b^2/r^2-2V/mv_\infty^2}} = \frac{b}{r^2\sqrt{1-b^2/r^2-2V/E}}\,. \tag{143}$$

For the point of closest approach to the force center we define $r = r_{\rm min}$ and $\phi = \phi_0$, and note that ϕ_0 is related to the scattering angle θ by,

$$\theta - \pi - 2\phi_0. \tag{144}$$



And, we can relate ϕ_0 to the impact parameter b by integrating eq. (143),

$$\phi_0 = b\int_{r_{\rm min}}^\infty \frac{dr}{r^2\sqrt{1 - b^2/r^2 - V(r)/E}}\,, \tag{145}$$

For the central force with potential energy $V = \alpha/r^2$ for positive constant α, we note that the energy (141) at $r_{\rm min}$, where $\dot r = 0$ is,

$$E = \frac{b^2E}{r_{\rm min}^2} + \frac{\alpha}{r_{\rm min^2}}\,, \qquad r_{\rm min}^2 = b^2 + \frac{\alpha}{E}\,. \tag{146}$$

Then,

$$\phi_0 = b\int_{r_{\rm min}}^\infty \frac{dr}{r^2\sqrt{1 - b^2/r^2 - \alpha/r^2E}} = b\int_{r_{\rm min}}^\infty \frac{dr}{r\sqrt{r^2 - r_{\rm min}^2}}$$
$$= \frac{b}{r_{\rm min}}\cos^{-1}\sqrt{1 - \frac{r_{\rm min}^2}{r^2}}\Bigg|_{r_{\rm min}}^\infty = \frac{\pi b}{2r_{\rm min}}\,, \tag{147}$$

using 281.01 of H.B. Dwight, *Tables of Integrals and Other Mathematical Data*, 4th ed. (Macmillan, 1961), http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf. Hence, the scattering angle is related by,

$$\theta - \pi - 2\phi_0 = \pi\left(1 - \frac{b}{r_{\rm min}}\right), \qquad b = r_{\rm min}\frac{\pi - \theta}{\pi}\,, \qquad b^2 = \left(b^2 + \frac{\alpha}{E}\right)\frac{(\pi-\theta)^2}{\pi^2}\,, \tag{148}$$

$$b^2\left(1 - \frac{(\pi-\theta)^2}{\pi^2}\right) = \frac{\alpha}{E}\frac{(\pi-\theta)^2}{\pi^2}\,, \qquad b^2 = \frac{\alpha}{E}\frac{(\pi-\theta)^2}{\theta(2\pi-\theta)}\,. \tag{149}$$

Finally, the differential cross section is,

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left|\frac{db^2}{d\cos\theta}\right| = \frac{1}{2\sin\theta}\left|\frac{db^2}{d\theta}\right| = \frac{\alpha}{2E\sin\theta}\left|\frac{-2(\pi-\theta)}{\theta(2\pi-\theta)} - \frac{(\pi-\theta)^2}{\theta^2(2\pi-\theta)} + \frac{(\pi-\theta)^2}{\theta(2\pi-\theta)^2}\right|$$
$$= \frac{\alpha}{2E\sin\theta}\left|\frac{(\pi-\theta)[(\pi-\theta)2(\theta-\pi) - 2\theta(2\pi-\theta)]}{\theta(2\pi-\theta)^2}\right| = \frac{\pi^2\alpha}{E\sin\theta}\frac{\pi-\theta}{\theta^2(2\pi-\theta)^2}\,. \tag{150}$$

The figure below shows $(E/\alpha)d\sigma/d\theta$.