Princeton University

Ph304 Final Examination
Electrodynamics

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(1:30 - 4:30 pm, May 22, 2002)

Do all work you wish graded in the exam booklets provided.

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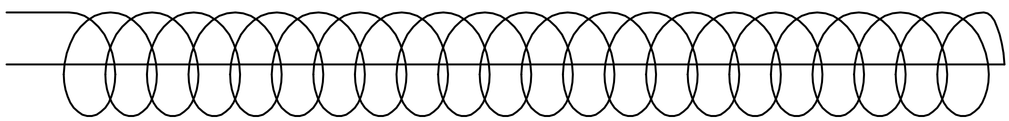
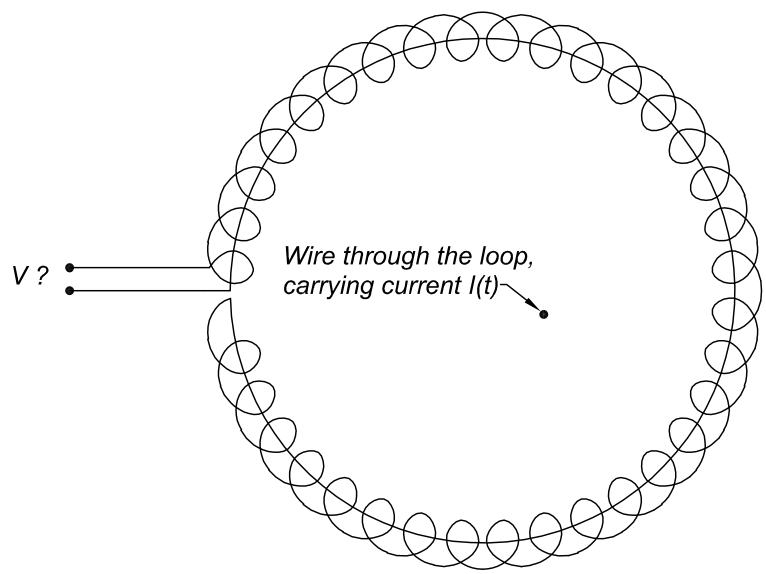
You may use either Gaussian or SI units on this exam.

1. (10 pts.) A device for measuring the magnitude of AC current in a wire is based on a helical coil (solenoid) wound with N turns each of area A. The length L of the coil obeys $L^2 \gg A$. The return lead passes back along the axis:



The coil is then bent so as to surround a wire that carries an alternating current $I(t) = I_0 \cos \omega t$.



What is the voltage $V(t)$ induced at the leads of the bent solenoid coil? Show that this voltage is independent of the exact shape of the coil, and independent of the position of the current-carrying wire. Give a physics reason why the return wire should pass down the center of the coil.

2. (20 pts.) Consider the Earth's ionosphere to be a dilute plasma of uniform density (i.e., the interactions between electrons, and between electrons and ions can be ignored) with a static, uniform magnetic field $\mathbf{B}_E$ (the Earth's field) in the $+z$ direction. Discuss the propagation of circularly polarized plane radio waves parallel to $\mathbf{B}_E$, *i.e.*,

$$\mathbf{E}_\pm = E_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})e^{i(kz-\omega t)}, \tag{1}$$

where the + sign corresponds to left-handed circular polarization.

Deduce the time dependence $\mathbf{r}(t)$ of the position of an ionized electron of charge $-e$ and mass m, assuming the electron to be at rest on average (and the wave weak enough that the motion is nonrelativistic; also $E_0 = cB_0 \ll cB_E$). You may wish to express $\mathbf{r}$

in terms of the so-called Larmor (or cyclotron) frequency $\omega_B = eB_E/m$ ($= eB_E/mc$ in Gaussian units).

Give the frequency-dependent dielectric "constant" $\epsilon_\pm$ and the index of refraction $n_\pm$ for both polarizations, supposing the electron density is N per unit volume, so the plasma frequency is $\omega_p^2 = Ne^2/\epsilon_0 m$ ($= 4\pi Ne^2/m$ in Gaussian units).

Deduce the phase velocity v_p and the group velocity v_g for waves of the two polarizations and frequencies small compared to ω_B and ω_p.

It turns out that $\omega_B \approx \omega_p \approx 10^7$ Hz in the ionosphere. Estimate the difference in arrival times for signals of 10^5 and 2×10^5 Hz originating simultaneously (in lightning flashes) at the opposite side of the Earth. [The curvature of the Earth permits the ionosphere to guide the waves in a circle of radius $R_E \gg \lambda$.] This illustrates the "whistler" or "chirp" effect well known to ham radio operators.

3. (20 pts.) A plane wave of frequency ω propagates in the $+z$ direction in vacuum and is incident on a free electron (charge e, mass m) whose average position is at the origin. Deduce the differential scattering cross section $d\sigma/d\Omega$, and the total cross section σ, assuming that the motion of the electron is nonrelativistic. The wave is unpolarized, which can be taken to mean that it consists of a wave linearly polarized in the x direction plus a wave linearly polarized in the y direction, each with one half the total wave intensity.

   In particular, consider an observer in the x-z plane whose line of sight to the origin makes angle θ to the $+z$ axis. Give the differential scattering cross section separately for the incident waves polarized in the x and y directions; the cross section for the unpolarized wave is then the average of these two results.

   Also, give the differential scattering cross section for left- and right-handed circular polarization of the incident wave (as defined in problem 2).

   Recall that the differential scattering cross section is defined to be ratio of the power scattered into unit solid angle and the incident power per unit area:

$$\frac{d\sigma}{d\Omega} = \frac{dP_{\text{scattered}}/d\Omega}{P_{\text{incident}}}\,. \tag{2}$$

   [Not for credit: A medium with a dielectric constant and an index of refraction, as in problem 2, consists of many scattering centers, one of which was considered in this problem. Hence, the combined behavior of incident wave + scattered waves must provide an alternative explanation of the index of refraction....]

Solutions

1. **The Amp Clamp.**

The changing current in the "test" wire causes a changing magnetic field, which induces an electric field, according to Faraday. The voltage V at the leads to the amp clamp coil is (in Gaussian units)

$$V = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{1}{c}\frac{d\Phi_M}{dt}\,, \tag{3}$$

where the magnetic flux linked by the amp clamp is

$$\Phi_M = \sum_{\text{small loops}} \int \mathbf{B} \cdot d\mathbf{A} \approx \oint_{\text{large loop}} \frac{dN}{dl} dl\ \mathbf{B} \cdot \hat{\mathbf{l}} A \approx \frac{NA}{L} \oint_{\text{large loop}} \mathbf{B} \cdot d\mathbf{l} = \frac{4\pi}{c}\frac{NA}{L} I(t). \tag{4}$$

We have used Ampere's law in the last step of eq. (4), assuming it to hold in its static form for low-frequency currents as well. For current $I = I_0 \cos\omega t$ in the wire, the voltage in the amp clamp is therefore

$$V(t) = \frac{4\pi NA\omega I_0 \sin\omega t}{c^2 L} \left(= \frac{\mu_0 NA\omega I_0 \sin\omega t}{L} \text{ in SI units} \right). \tag{5}$$

Note that we do not have to include a term in Φ_M due to flux linked by the large loop – because of the return wire down the center of the small loops the amp clamp does not link any flux due to magnetic field lines perpendicular to the plane of the clamp. Rather, the use of Ampere's law in eq. (4) shows that the clamp links flux only for wires that pass through the clamp, and that the amount of this flux linkage is independent of the position of the wire relative to the clamp. Further, the amount of flux linked is independent of any possible tilt of the wire with respect to the plane of the clamp.

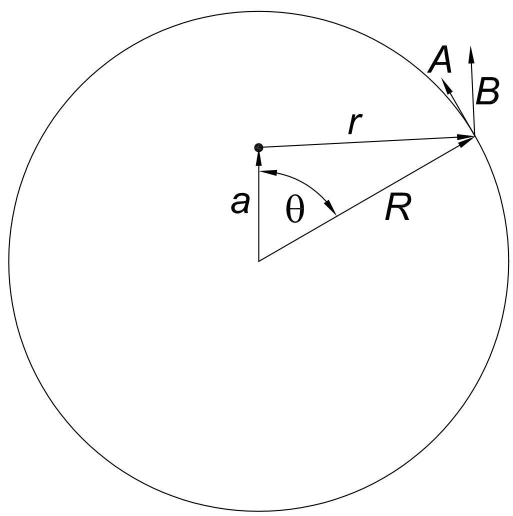
Another way to think about this problem recalls the concept of mutual inductance. Labelling the amp clamp as loop 1 and the test wire (which we presume is part of a loop rather than an antenna) as loop 2, then

$$V_1 = -M_{12}\dot{I}_2, \tag{6}$$

where $M_{12} = \Phi_1/cI_2$ is the mutual inductance between loops 1 and 2. But $M_{12} = M_{21} = \Phi_2/cI_1$. Clearly the flux Φ_2 in loop 2, the loop that contains the test wire, due to a current I_1 in the amp clamp is independent of the exact position of the test wire – since the flux is entirely inside the winding of the amp clamp. Hence, M_{12}, and also V_1 in the amp clamp, is independent of the position of the test wire, so long as it passes through the loop of the amp clamp.

Note that the voltage found above arises between the terminals of the amp clamp even if the coil is open circuit. In practice, a high-impedance voltmeter would be connected across the terminals to measure the voltage. In this case, only a neglible current flows in the amp clamp, and we need not worry about any effect due to the magnetic field of this tiny current.

Mathematical footnote: It may be instructive to make an explicit calculation of the flux linked by the amp clamp due to a wire perpendicular to the plane of the clamp at distance a from its center. The clamp has radius R. We first consider a small loop of area $\mathbf{A}$ such that the radius vector $\mathbf{R}$ to this loop makes angle θ to the vector $\mathbf{a}$ that points from the center of the loop to the wire.



The flux $d\Phi$ through this loop due to current I in the wire is

$$d\Phi = \mathbf{B} \cdot \mathbf{A} = \frac{2IA}{cr} \hat{\mathbf{r}} \cdot \hat{\mathbf{R}}, \tag{7}$$

where the distance from the wire to the loop is

$$r = \sqrt{R^2 - 2aR\cos\theta + a^2}. \tag{8}$$

Since $\mathbf{r} = \mathbf{R} - \mathbf{a}$, we have

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{R}} = \frac{\mathbf{R} - \mathbf{a}}{r} \cdot \hat{\mathbf{R}} = \frac{R - a\cos\theta}{r}, \tag{9}$$

And the flux linked by the small loop is

$$d\Phi = \frac{2IA}{c} \frac{R - a\cos\theta}{R^2 - 2aR\cos\theta + a^2}. \tag{10}$$

The total flux linked by the amp clamp is obtained by integration over θ, noting that the number of turns in interval $d\theta$ is $Nd\theta/2\pi$:

$$\begin{aligned}\Phi &= \int d\Phi = \frac{NIA}{\pi c}\int_0^{2\pi} d\theta \frac{R - a\cos\theta}{R^2 - 2aR\cos\theta + a^2} \\ &= \frac{NIA}{\pi c R}\int_0^{2\pi} d\theta \frac{1 - \frac{a}{R}\cos\theta}{1 - 2\frac{a}{R}\cos\theta + \frac{a^2}{R^2}} = \frac{NIA}{\pi c a}\int_0^{2\pi} d\theta \frac{\frac{R}{a} - \cos\theta}{1 - 2\frac{R}{a}\cos\theta + \frac{R^2}{a^2}}. \end{aligned} \tag{11}$$

According to Gradshteyn and Ryzhik 3.792,

$$\int_0^{2\pi} d\theta \frac{1}{1 - 2a\cos\theta + a^2} = \frac{2\pi}{1 - a^2} \qquad (a^2 < 1). \tag{12}$$

Hence, the identity

$$\begin{aligned} 2\pi &= \int_0^{2\pi} d\theta \frac{1 - 2a\cos\theta + a^2}{1 - 2a\cos\theta + a^2} \\ &= (1 + a^2)\int_0^{2\pi} d\theta \frac{1}{1 - 2a\cos\theta + a^2} - 2a\int_0^{2\pi} d\theta \frac{\cos\theta}{1 - 2a\cos\theta + a^2} \end{aligned} \tag{13}$$

tells us that

$$\int_0^{2\pi} d\theta \frac{\cos\theta}{1 - 2a\cos\theta + a^2} = \frac{2\pi a}{1 - a^2} \qquad (a^2 < 1). \tag{14}$$

Using the integrals (12) and (14), we find that when $a < R$ the next to last form of eq. (11) yields

$$\Phi = \frac{NIA}{\pi c R}\int_0^{2\pi} d\theta \frac{1 - \frac{a}{R}\cos\theta}{1 - 2\frac{a}{R}\cos\theta + \frac{a^2}{R^2}} = \frac{NIA}{\pi c R}\left(\frac{2\pi}{1 - \frac{a^2}{R^2}} - \frac{2\pi\frac{a^2}{R^2}}{1 - \frac{a^2}{R^2}}\right) = \frac{2NIA}{cR} = \frac{4\pi NIA}{cL}, \tag{15}$$

which is independent of a (so long as $a < R$). Likewise, the last form of eq. (11) yields

$$\Phi = \frac{NIA}{\pi c a}\int_0^{2\pi} d\theta \frac{\frac{R}{a} - \cos\theta}{1 - 2\frac{R}{a}\cos\theta + \frac{R^2}{a^2}} = \frac{NIA}{\pi c a}\left(\frac{2\pi\frac{R}{a}}{1 - \frac{R^2}{a^2}} - \frac{2\pi\frac{R}{a}}{1 - \frac{R^2}{a^2}}\right) = 0, \tag{16}$$

for any $a > R$.

2. **Whistlers.** We solve this problem by relating the index of refraction to the dielectric constant, which is related to the polarization of the medium, which is due to the generation of dipole moments of the individual electrons by the incident wave. So, we start the analysis by considering the motion of a single electron, as acted on by the wave and by the static magnetic field, neglecting the force between electrons, and that between electrons and ions, as the plasma is dilute.

   Before adding up the dipole moments of the individual electrons, we pause to reflect on the relation between the radiation of an individual oscillating electron and the incident wave, which gives us a clue that only one of the two polarizations of the incident wave can propagate through the ionosphere. The usual analysis of this problem (see sec. 7.6 of the textbook of Jackson) makes no such pause, which is not strictly necessary although it may give a more physical reason for the behavior that the equations reveal.

The equation of motion of an ionized electron in a circularly polarized plane wave is (in Gaussian units)

$$m\ddot{\mathbf{r}} - (-e)\frac{\dot{\mathbf{r}}}{c} \times B_E\hat{\mathbf{z}} = -eE_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})e^{i(kz-\omega t)} \approx -eE_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})e^{i(kz_{\rm ave}-\omega t)}, \tag{17}$$

where for motion of small amplitude the phase factor e^{ikz} can be approximated by the constant value $e^{ikz_{\rm ave}}$. Since the magnetic field of the wave is much smaller than that of the Earth, we ignore the magnetic part of the Lorentz force due to the wave. In the steady state, we expect that the oscillatory motion of the charge follows that of the driving wave (with a possible phase shift), so we try the form:

$$\mathbf{r}_\pm = \mathbf{r}_{\rm ave} + r_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})e^{i(kz_{\rm ave}-\omega t)}. \tag{18}$$

Inserting eq. (18) into (17) we find

$$-m\omega^2 r_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) - i\frac{e\omega B_E}{c} r_0(-\hat{\mathbf{y}} \pm i\hat{\mathbf{x}}) = -eE_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}), \tag{19}$$

$$r_0\left(-m\omega^2 \pm \frac{e\omega B_E}{c}\right)(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) = -eE_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}), \tag{20}$$

and hence,

$$r_0 = \frac{eE_0}{m\omega(\omega \mp \omega_B)}\,, \tag{21}$$

where

$$\omega_B = \frac{eB_E}{mc}\left(= \frac{eB_E}{m} \text{ in SI units}\right). \tag{22}$$

[Interlude: Recalling the comment at the end of problem 3, we pause to consider the radiation of the oscillating charge, which depends on its acceleration. From eqs. (18) and (21), the acceleration can be written

$$\ddot{\mathbf{r}}_\pm = -\frac{e\omega}{m(\omega \mp \omega_B)}E_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})e^{-i\omega t} \approx \pm\frac{e\omega}{m\omega_B}E_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})e^{-i\omega t}, \tag{23}$$

where the approximation holds for the case of interest that $\omega \ll \omega_B$. The radiation electric field varies as $-(-e)\ddot{\mathbf{r}}_\pm$, which is in phase with the incident wave for the + case = left-handed circular polarization, and 180° out of phase for the − case = right-handed polarization. When the response of the medium is out of phase with the incident wave, we have a situation much like a metal in which the radiation of the medium cancels out the incident wave. So we may expect that the result of continuing our analysis to calculate the polarization, the dielectric constant, and the index of refraction is that the right-handed wave won't propagate.]

Since $\mathbf{r}_\pm - \mathbf{r}_{\rm ave}$ measures the separation of an electron from its average position, the resulting polarization density is

$$\mathbf{P}_\pm = N(-e)(\mathbf{r}_\pm - \mathbf{r}_{\rm ave}) = -\frac{Ne^2}{m\omega(\omega \mp \omega_B)}\mathbf{E} \equiv \chi_\pm \mathbf{E}, \tag{24}$$

and the (relative) dielectric "constant" is

$$\epsilon_\pm = 1 + 4\pi\chi_\pm = 1 - \frac{4\pi Ne^2}{m\omega(\omega \mp \omega_B)} = 1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_B)}\,, \tag{25}$$

where the square of the plasma frequency is given by

$$\omega_p^2 = \frac{4\pi Ne^2}{m}\left(= \frac{Ne^2}{\epsilon_0 m} \text{ in SI units}\right). \tag{26}$$

For radio waves with $\omega \ll \omega_B \approx \omega_p$,

$$\epsilon_\pm \approx \pm\frac{\omega_p^2}{\omega\omega_B}\,. \tag{27}$$

The phase velocity of the plane waves is related by

$$v_{\rm phase} = \frac{\omega}{k} = \frac{c}{n} = \frac{c}{\sqrt{\epsilon}}\,. \tag{28}$$

Comparing eqs. (27) and (28) we see that the phase velocity is imaginary for right-handed waves (ϵ_-), which means that these waves are attenuated rapidly. Only the left-handed radio waves (ϵ_+) propagate in the ionosphere, and their phase velocity is

$$v_{\rm phase} = c\frac{\sqrt{\omega\omega_B}}{\omega_p}\,. \tag{29}$$

For these waves, the wave vector is related to frequency by

$$k = \frac{\omega}{c}\sqrt{\epsilon_+} = \frac{\omega_p}{c}\sqrt{\frac{\omega}{\omega_B}}\,, \tag{30}$$

and so the group velocity is given by

$$v_{\rm group} = \frac{d\omega}{dk} = \frac{1}{dk/d\omega} = 2c\frac{\sqrt{\omega\omega_B}}{\omega_p} = 2v_{\rm phase}\,. \tag{31}$$

Higher frequencies travel faster.

For waves with $\omega \approx 10^5$ Hz and $\omega_B \approx \omega_p \approx 10^7$ Hz, $v_{\rm phase} \approx c/10$.

The difference in arrival times for pulses centered on frequencies $\omega_1 = 10^5$ and $\omega_2 = 2 \times 10^5$ Hz from the opposite side of the Earth ($d = 2 \times 10^9$ cm, which used to be the definition of a centimeter) is

$$\begin{aligned}\Delta t &= \frac{d}{v_{g,1}} - \frac{d}{v_{g,2}} = \frac{d}{2c}\frac{\omega_p}{\sqrt{\omega_2\omega_B}}\left(\sqrt{\frac{\omega_2}{\omega_1}} - 1\right) \\ &= \frac{2\times 10^9}{2\cdot 3\times 10^{10}}\frac{10^7}{\sqrt{2\times 10^5\cdot 10^7}}\left(\sqrt{2}-1\right) \\ &\approx 0.1 \text{ s.}\end{aligned} \tag{32}$$

3. **Thomson Scattering.** The incident plane wave $\mathbf{E}_0 e^{i(kz-\omega t)}$ shakes the electron, which emits primarily electric dipole radiation if the motion in nonrelativistic. The emitted radiation is at the same frequency ω as the incident wave, and is often interpreted as due to scattering of the incident wave.

   The total radiated power follows quickly from the Larmor formula:

$$\langle P_{\text{radiated}} \rangle = \frac{1}{4\pi\epsilon_0} \frac{2e^2 \langle a^2 \rangle}{3c^3} = \frac{1}{4\pi\epsilon_0} \frac{e^2 (eE_0/m)^2}{3c^3} = 4\pi\epsilon_0 c \frac{r_e^2 E_0^2}{3}, \tag{33}$$

   since $F_{\max} = ma_{\max} = eE_0$, $\langle a^2 \rangle = a_{\max}^2/2$, and the classical electron radius is defined by $r_e = e^2/4\pi\epsilon_0 mc^2$. The incident power per unit area is given by the Poynting vector:

$$\langle P_{\text{incident}} \rangle = \langle S \rangle = \frac{1}{2\mu_0} E_0 B_0 = \frac{1}{2\mu_0 c} E_0^2 = \frac{\epsilon_c c}{2} E_0^2, \tag{34}$$

   since $E_0 = cB_0$ for a plane wave in vacuum, and $\epsilon_0\mu_0 = c^2$. Hence, the total cross section is given by

$$\sigma_{\text{total}} = \frac{\langle P_{\text{scattered}} \rangle}{\langle P_{\text{incident}} \rangle} = \frac{8\pi}{3} r_e^2 \left(= \frac{\mu_0^2 e^4}{6\pi m^2} = \frac{e^4}{6\pi\epsilon_0^2 m^2 c^2} \right). \tag{35}$$

   which is known as the Thomson scattering cross section. The cross section has dimension (length)^2 = an area.

   To obtain the angular distribution, we recall that dipole radiation varies as $\sin^2\alpha$, where α is the angle between the direction of the acceleration and the direction to the observer. That is,

$$\frac{d\langle P_{\text{radiated}} \rangle}{d\Omega} = A \sin^2\alpha, \tag{36}$$

   so that

$$\begin{aligned} \langle P_{\text{radiated}} \rangle &= \int \frac{d\langle P_{\text{radiated}} \rangle}{d\Omega} d\Omega = 2\pi A \int \sin^2\alpha \; d\cos\alpha = 2\pi A \int_{-1}^{1} (1 - \cos^2\alpha) \; d\cos\alpha \\ &= 2\pi A \left(2 - \frac{2}{3}\right) = \frac{8\pi}{3} A. \end{aligned} \tag{37}$$

   Comparing with eq. (33) we see that

$$A = \frac{\epsilon_0 c r_e^2 E_0^2}{2}, \tag{38}$$

   so that

$$\frac{d\langle P_{\text{radiated}} \rangle}{d\Omega} = \frac{\epsilon_0 c r_e^2 E_0^2}{2} \sin^2\alpha. \tag{39}$$

   If the incident wave is polarized in the x direction, then the acceleration is along the x axis, and the angle between the acceleration and the observer is $\alpha = 90^\circ - \theta$. Thus,

$$\frac{d\langle P_{x\text{ pol}} \rangle}{d\Omega} = \frac{\epsilon_0 c r_e^2 E_0^2}{2} \cos^2\theta, \tag{40}$$

and

$$\frac{d\sigma_{x\text{ pol}}}{d\Omega} = \frac{d\left\langle P_{x\text{ pol}}\right\rangle/d\Omega}{\left\langle P_{\text{incident}}\right\rangle} = r_e^2\cos^2\theta. \tag{41}$$

However, if the incident wave is polarized in the y direction, then the angle between the acceleration and the observer is always 90°, so we have

$$\frac{d\sigma_{y\text{ pol}}}{d\Omega} = \frac{d\left\langle P_{y\text{ pol}}\right\rangle/d\Omega}{\left\langle P_{\text{incident}}\right\rangle} = r_e^2. \tag{42}$$

The differential cross section for unpolarized light is the average of eqs. (41) and (42):

$$\frac{d\sigma_{\text{unpol}}}{d\Omega} = r_e^2\frac{1+\cos^2\theta}{2}. \tag{43}$$

As a check, we recalculate the total cross section:

$$\sigma_{\text{unpol}} = \int\frac{d\sigma_{\text{unpol}}}{d\Omega}d\Omega = 2\pi r_e^2\int_{-1}^{1}\frac{1+\cos^2\theta}{2}d\cos\theta = \pi r_e^2\left(2+\frac{2}{3}\right) = \frac{8\pi}{3}r_e^2. \tag{44}$$

To discuss scattering of a circularly polarized wave, we need a bit more detail. But it suffices to recall that the scattered intensity depends on the square of the electric (or magnetic) field, and that the radiation electric field varies as the perpendicular component of the acceleration. For $\mathbf{E}_\pm = E_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})e^{i(kz-\omega t)}$, we again have that the acceleration is $\mathbf{a}_\pm \propto (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})$. According to the observer in the x-z plane, $\mathbf{a}_{\pm,\perp} \propto (\cos\theta\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})$. Hence,

$$\frac{d\sigma_\pm}{d\Omega} \propto \left|\mathbf{E}_{\text{rad}}\right|^2 \propto \left|\mathbf{a}_{\pm,\perp}\right|^2 \propto \cos^2\theta + 1, \tag{45}$$

which is the same for both left- and right-handed polarization. Again, the total cross section must be given by eq. (35), so the normalization follows from eqs. (43) and (44):

$$\frac{d\sigma_\pm}{d\Omega} = r_e^2\frac{1+\cos^2\theta}{2}. \tag{46}$$

[Differences between the scattering of unpolarized, left- and right-handed circularly polarized waves arise only when we keep track of the polarization of the scattered wave as well.]

Supplement 1: The Origin of the Index of Refraction.

[Adapted from Vol I, secs. 30-7, 31-1,2 of the Feynman Lectures on Physics.]

We combine the results of probs. 2 and 3 to show how the sum of the incident wave plus the forward scattered wave in a medium gives a wave that has phase velocity c/n, where n is the index of refraction.

First, we note that while the usual interpretation of the index of refraction is that it changes the phase velocity of a wave, we can equivalently think of it as causing a (spatially varying) phase shift in the wave. To see this, consider a plane wave

$\mathbf{E}_0 e^{i(kz-\omega t)}$ that is incident on a dielectric medium that occupies the space $z > 0$. Inside the medium, the wave vector k' is relative by $k' = n\omega/c = nk$, so for $z > 0$ the wave propagation is described by

$$\mathbf{E}_0 e^{i(k'z-\omega t)} = \mathbf{E}_0 e^{i(kz-\omega t+(k'-k)z)} = \mathbf{E}_0 e^{i(kz-\omega t+(n-1)\omega z/c)}. \tag{47}$$

Thus, we can say that the effect of the index n is to induce the phase shift $(n-1)\omega z/c$ after the wave has propagated distance z.

We now wish to show that this phase shift is due to the effect of the radiation of the electrons in the medium, which are accelerated by the incident wave. Following the suggestion of Feynman, we restrict our calculation to the case of a thin slab of medium that extend from $z = 0$ to Δz, and we consider an observer at $(0, 0, z \gg \Delta z)$.

For a slab of thickness Δz, the total phase shift impressed on a passing wave is] $(n-1)\omega\Delta z/c$, so the observer detects the waveform

$$\mathbf{E}_0 e^{i(kz-\omega t+(n-1)\omega\Delta z/c)} \approx \mathbf{E}_0 e^{i(kz-\omega t)} + i(n-1)\frac{\omega}{c}\Delta z \mathbf{E}_0 e^{i(kz-\omega t)}. \tag{48}$$

This is the superposition of the incident wave, plus a term proportional to Δz that we wish to interpret as due to the scattered wave.

Therefore, we now add up the scattered wave amplitudes due to the many electrons in the thin slab, as seen been the observer. We recall that the radiation electric field can be written

$$\mathbf{E}_{\rm rad} = \frac{1}{4\pi\epsilon_0 c^2}\left(\left(\int \frac{[\dot{\mathbf{J}}]}{R} d\mathrm{Vol} \times \hat{\mathbf{n}}\right) \times \hat{\mathbf{n}}\right) = -\frac{1}{4\pi\epsilon_0 c^2}\int \frac{[\dot{\mathbf{J}}_\perp]}{R} d\mathrm{Vol}, \tag{49}$$

where R is the distance from the observer to an electron. The current $\mathbf{J}$ is due to the acceleration of the electrons of the medium, whose density is N per unit volume. Thus,

$$\dot{\mathbf{J}} = N(-e)\mathbf{a} \tag{50}$$

We will consider the specific example of problem 2, in which the medium consists of ionized electrons immersed in a constant magnetic field $\mathbf{B} = B_E\hat{z}$. The incident wave has either left- (+) or right-handed (−) circular polarization, as given by eq. (1). [This example is perhaps not ideal, in that the methods used below hold only when the index is near unity, which is not the case for radio wave propagation in the ionosphere.]

From eqs. (18) and (21), we find the acceleration of an electron to be

$$\mathbf{a}_\pm = -\omega^2 \frac{e}{m\omega(\omega \mp \omega_B)} E_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) e^{i(kz_{\rm ave}-\omega t)} \approx -\omega^2 \frac{e}{m\omega(\omega \mp \omega_B)} \mathbf{E}_0 e^{-i\omega t}, \tag{51}$$

where we ignore the acceleration of an electron due to the radiated fields of other electrons, where $\omega_B = eB_0/m$, and we suppose that $\Delta z \ll \lambda$ so that $e^{ikz_{\rm ave}} \approx 1$.

We will evaluate the integral (49) in cylindrical coordinates, where the position of an electron is $(r, \phi, z) \approx (r, \phi, 0)$, so that $R^2 = r^2 + z^2$ with $(0, 0, z)$ being the coordinate of the observer. The retarded time is $t' = t - R/c$. Thus,

$$[\dot{\mathbf{J}}] = \omega^2 \frac{Ne^2}{m\omega(\omega \mp \omega_B)} \mathbf{E}_0 e^{-i\omega t} e^{i\omega R/c} \tag{52}$$

To calculate the radiation field, we need $[\dot{\mathbf{J}}_\perp]$, which we then integrate over all electrons in the thin slab. The z component of $[\dot{\mathbf{J}}_\perp]$ thereby integrates to zero, so what matters is the projection of $[\dot{\mathbf{J}}_\perp]$ onto the x-y plane. Now, the magnitude of $[\dot{\mathbf{J}}_\perp]$ is z/R times the magnitude of $[\dot{\mathbf{J}}]$, and the magnitude of the projection of $[\dot{\mathbf{J}}_\perp]$ onto the x-y plane is also z/R times the magnitude of $[\dot{\mathbf{J}}_\perp]$. That is, the relevant factor is $[\dot{\mathbf{J}}]z^2/R^2$. The volume element is $2\pi r dr \Delta z$, so eq. (49) becomes

$$\begin{aligned}
\mathbf{E}_{\rm rad} &= -\frac{2\pi}{4\pi\epsilon_0 c^2}\omega^2 \Delta z \frac{Ne^2}{m\omega(\omega \mp \omega_B)} \mathbf{E}_0 e^{-i\omega t} \int_0^\infty r\ dr \frac{e^{i\omega R/c}}{R}\frac{z^2}{R^2} \\
&= -\frac{\omega^2}{2c^2}\Delta z \frac{\omega_p^2}{\omega(\omega \mp \omega_B)} \mathbf{E}_0 e^{-i\omega t} \int_z^\infty dR\ e^{i\omega R/c}\frac{z^2}{R^2} \\
&\approx -\frac{\omega^2}{2c^2}\Delta z \frac{\omega_p^2}{\omega(\omega \mp \omega_B)} \mathbf{E}_0 e^{-i\omega t} \int_z^\infty dR\ e^{i\omega R/c} \\
&= i\frac{\omega}{2c}\Delta z \frac{\omega_p^2}{\omega(\omega \mp \omega_B)} \mathbf{E}_0 e^{-i\omega t}[e^{i\omega\infty/c} - e^{i\omega z/c}] \\
&= -i\frac{\omega}{2c}\Delta z \frac{\omega_p^2}{\omega(\omega \mp \omega_B)} \mathbf{E}_0 e^{i(kz-\omega t)}, \qquad (53)
\end{aligned}$$

where the plasma frequency is $\omega_p^2 = Ne^2/\epsilon_0 m$, and in the 3rd and 4rth lines above we have recognized that the main contribution to the integral comes from the region $r \lesssim z$ where we approximate z^2/R^2 by unity.

Comparing with eq. (48), we identify

$$n_\pm - 1 \approx -\frac{1}{2}\frac{\omega_p^2}{\omega(\omega \mp \omega_B)}\,. \qquad (54)$$

Thus, so long as $n-1$ is small, we can write

$$n_\pm^2 \approx \left(1 - \frac{1}{2}\frac{\omega_p^2}{\omega(\omega \mp \omega_B)}\right)^2 \approx 1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_B)}, \qquad (55)$$

which is the same as we found in prob. 2 via $n_\pm^2 = \epsilon_{r,\pm}$.

Note that if the observer had been at $(0,0,-z)$ the reflected would be detected. Since the acceleration is in the x-y plane, the strength of the radiation is the same in the $+z$ and $-z$ directions. The electric field radiated by a thin (dielectric) slab is $90°$ out of phase with the incident wave. Thus we can summarize:

$$\begin{aligned}
\mathbf{E}_{\rm transmitted} &= \mathbf{E}_0(1 + i\alpha), \qquad (56) \\
\mathbf{E}_{\rm reflected} &= \mathbf{E}_0(i\alpha), \qquad (57)
\end{aligned}$$

For a thin slab where α is small, the transmitted and reflected waves are very close to $90°$ out of phase.

Supplement 2: Compton Scattering

For incident radiation of very short wavelengths, Thomson scattering is modified by quantum effects, and the resulting behavior is called Compton scattering. The most notable feature is that the frequency of the scattered radiation is lower than that of the incident wave, according the the famous Compton formula:

$$\omega_{\rm scat} = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos\theta)}\,. \tag{58}$$

This follows from the hypothesis that electromagnetic waves consist of quanta (photons) with energy $U = \hbar\omega$, momentum $P = \hbar k$, and that energy and momentum are conserved in the scattering process symbolized by $\omega + e \to \omega_{\rm scat} + e'$.

It turns out that the classical Thomson scatterng formula is modified by the quantum effects only in that the "phase space" for a scattered photon of frequency $\omega_{\rm scat}$ is less than that for frequency ω by the square of the ratio $\omega_{\rm scat}/\omega$, as readily inferred from the Rayleigh-Jeans law. Thus, the quantum version of eq. (46) is

$$\frac{d\sigma_{\rm Compton,\pm}}{d\Omega} = r_e^2 \frac{\omega_{\rm scat}^2}{\omega^2} \frac{1 + \cos^2\theta}{2}\,. \tag{59}$$

This result was first deduced by Klein and Nishina (1929) via a lengthy calculation involving the Dirac equation.