Princeton University

Ph304 Midterm Examination
Electrodynamics

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(7-9 pm, Wed. Mar. 12, 2003)

Do all work you wish graded in the exam booklets provided.

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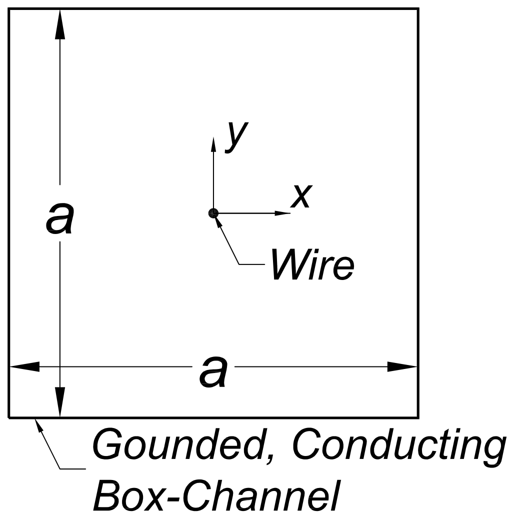
Please do all work in the exam booklets provided.

You may use either Gaussian or SI units on this exam.

1. (10 pts.) What is the electric potential in cylindrical coordinates $V(r, \theta, z)$ when a charge q is located at $(r_0, z_0 > 0)$ and there is a grounded conducting plane at $z = 0$ that has a (conducting) hemispherical boss of radius $a < b = \sqrt{r_0^2 + z_0^2}$ whose center is at the origin. What is the electrostatic force on the charge q for the case that $r_0 = 0$?

2. (10 pts.) An Iarocci tube is a low-cost descendent of a Geiger counter whose walls form a square prism of edge a, with a wire along its center. In the basic configuration, the walls are conducting, and grounded, as shown below



Give a series expansion the electrostatic potential $V(x, y)$ inside the Iarocci tube supposing the wire carries charge q per unit length.

The potential has a logarithmic divergence at the wire, so we specify the charge per unit length on the wire rather than its potential. Then, the presence of (nonsingular) surfaces at a specified potential permits a "simple" series expansion at points not on the wire.

A physical device with this geometry will have a wire of nonzero radius $r_0 \ll a$.

For points (x, y) close to the wire, the series can be summed. Do so, to relate the potential of a wire of radius r_0 to the charge q.

3. (10 pts.) A cylinder of relative dielectric constant ε_r rotates with constant angular velocity ω about its axis. A uniform magnetic field $\mathbf{B}$ is parallel to the axis, in the same sense as $\vec{\omega}$. Find the resulting dielectric polarization $\mathbf{P}$ in the cylinder and the surface and volume charge densities σ and ρ, neglecting terms of order $(\omega a/c)^2$, where a is the radius of the cylinder.

This problem can be conveniently analyzed by starting in the rotating frame, in which $\mathbf{P}' = \mathbf{P}$ and $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$, when $(v/c)^2$ corrections are neglected. Consider also the electric displacement $\mathbf{D}$.

Solutions (in Gaussian units)

1. We use the image method.

   First, we bring the hemispherical boss to zero potential by imagining that a charge $q' = -qa/b$ is placed at distance a^2/b along the line from the origin to charge q. The cylindrical coordinates of charge q' are

$$\frac{a^2}{b^2}(r_0, 0, z_0). \tag{1}$$

   Next, to bring the plane $z = 0$ to zero potential, we add images charges for both q and q'. Namely, we imagine charge $q'' = -q$ at

$$(r_0, 0, -z_0), \tag{2}$$

   and $q''' = -q' = qa/b$ at

$$\frac{a^2}{b^2}(r_0, 0, -z_0). \tag{3}$$

   Then, both the plane $z = 0$ and the spherical shell of radius a about the origin are at zero potential.

   The potential at an arbitrary point (r, θ, z) outside the conductor is therefore

$$V(r, \theta, z) = \frac{q}{r_1} - \frac{q}{r_2} - \frac{qab}{r_3} + \frac{qab}{r_4}\,, \tag{4}$$

   where

$$r_{1,2} = \sqrt{r^2 - 2rr_0\cos\theta + r_0^2 + (z \mp z_0)^2}, \tag{5}$$

   and

$$r_{3,4} = \sqrt{b^4r^2 - 2a^2b^2rr_0\cos\theta + a^4r_0^2 + (b^2z \mp a^2z_0)^2}. \tag{6}$$

   When $r_0 = 0$, the force on charge q is in the $-z$ direction, with magnitude

$$F = \frac{q^2}{4z_0^2} + \frac{q^2a/z_0}{(z_0 - a^2/z_0)^2} - \frac{q^2a/z_0}{(z_0 + a^2/z_0)^2} = \frac{q^2}{4z_0^2} + \frac{4q^2a^3z_0^3}{(z_0^4 - a^4)^2}\,. \tag{7}$$

2. To use techniques for solving Laplace's equation, $\nabla^2 V = 0$, for the potential V, we subdivide the cell into rectangular regions that have no charge in their interior. We work in a rectangular coordinate system with origin at the center of a cell.

   For the basic Iarocci tube shown on p. 1, we solve separately in the regions $x < 0$ and $x > 0$, and match solutions at the "boundary" $x = 0$. In each region, we know the potential on three of the four bounding surfaces, and we know the charge distribution $\sigma \propto \partial V/\partial n$ on the fourth:

$$V(x, -a/2) = V(x, a/2) = V(-a/2, y) = V(a/2, y) = 0, \qquad \frac{\partial V(0^+, y)}{\partial x} = -2\pi q\delta(y), \tag{8}$$

in Gaussian units, where the symmetry of the potential and the antisymmetry of E_x about $x = 0$ imply that $E_x(-\epsilon, y) = -E_x(\epsilon, y) = \partial V(\epsilon, y)/\partial x = -2\pi\sigma(0, y) = -2\pi q\delta(y)$.

The potential $V(x, y)$ for this problem is symmetric in x and y separately, and should also be symmetric with respect to the interchange of x and y. However, the method of separation of variables in two rectangular coordinates leads to oscillatory functions in one coordinate, and exponential functions in the other. Our solutions is built up of sums of products of these oscillatory and exponential functions. Any particular term of this series will not be invariant under exchange of x and y; yet we can have confidence that the sum of the series will have this symmetry. If you are not satisfied with the apparent noninvariance of the potential $V(x, y)$ so obtained, you could declare your solution to be $U(x, y) = [V(x, y) + V(y, x)]/2$, since if $V(x, y)$ is a valid expression for the potential, then $V(y, x)$ is also.

We chose to use oscillatory functions of y, and therefore exponetial functions of x. Symmetric oscillatory functions are cosines, so we write $\cos k_n y$ for the y functions. The exponential functions must vanish at $x = \pm a/2$, so these must involve hyperbolic sines, for which $\sinh k_n(a/2 - |x|)$ displays the required symmetry in x. Thus, a suitable form of the solution to Laplace's equation for a potential that vanishes on the outer boundaries and is symmetric in both x and y is

$$V(x, y) = \sum_n A_n \sinh k_n(a/2 - |x|) \cos k_n y. \tag{9}$$

The boundary condition at $y = \pm a/2$ requires that $\cos k_n a/2 = 0$, and hence that $k_n = (2n + 1)\pi/a$. The boundary condition at $x = 0$ can now be written

$$-2\pi q\delta(y) = \frac{\partial V(0^+, y)}{\partial x} = -\sum_n \frac{(2n + 1)\pi}{a} A_n \cosh \frac{(2n + 1)\pi}{2} \cos \frac{(2n + 1)\pi y}{a}. \tag{10}$$

On multiplying eq. (10) by $\sin(n\pi y/a)$ and integrating from 0 to a we find that

$$A_n = \frac{4q}{(2n + 1) \cosh \frac{(2n+1)\pi}{2}}. \tag{11}$$

Hence, the potential for a basic Iarocci tube with a wire at its center can be written as

$$\begin{aligned} V(x, y) &= 4q \sum_n \frac{\sinh \frac{(2n+1)\pi(a/2-|x|)}{a}}{(2n + 1) \cosh \frac{(2n+1)\pi}{2}} \cos \frac{(2n + 1)\pi y}{a} \qquad \text{(wire, origin at center)} \\ &= 4q \sum_n \frac{\tanh \frac{(2n+1)\pi}{2} \cosh \frac{(2n+1)\pi|x|}{a} - \sinh \frac{(2n+1)\pi|x|}{a}}{(2n + 1)} \cos \frac{(2n + 1)\pi y}{a}. \end{aligned} \tag{12}$$

I claim that eq. (12) has the symmetry $V(x, y) = V(y, x)$, but that would only be obvious if we were able to sum the series. I don't know how to do this in general, but we can sum the series for small x and y, where we will find the potential depends only on $r = \sqrt{x^2 + y^2}$, which has the desired exchange symmetry.

The image method can be used to generate another solution to this problem. A doubly infinite set of charges $(-1)^{m+n}q$ at positions (ma, na), where m and n are any integer (positive or negative), is consistent with all four bounding planes of the box-channel being at ground potential. Hence, we can write

$$\begin{aligned} V(x,y) &= 2q\sum_m\sum_n(-1)^{m+n}\ln\frac{1}{\sqrt{(x-ma)^2+(y-na)^2}}+C \\ &= -q\sum_m\sum_n(-1)^{m+n}\ln\left[(x-ma)^2+(y-na)^2\right]+C. \end{aligned} \tag{13}$$

We require that the potential be zero on the boundary, which leads to an infinite set of representations of constant C. For example, forcing $V(a/2, a/2) = 0$, we can write

$$V(x,y) = -q\sum_m\sum_n(-1)^{m+n}\ln\frac{(m-x/a)^2+(n-y/a)^2}{(m-1/2)^2+(n-1/2)^2}\,. \tag{14}$$

This form is "obviously" invariant under the exchange of x and y, since it is invariant under the exchange of indices m and n. For (x, y) near the origin, we can suppose that the series is dominated by the term with $m = n = 0$, which implies that $V \approx 2q\ln(a/\sqrt{2}r)$. Note that the potential of a wire on the axis of a grounded cylinder of radius a is $V_{\rm cyl} = 2q\ln(a/r)$, where q is the linear charge density on the wire.

The potential (12) at the origin diverges. But, of course, a physical realization of an Iarocci tube involves a wire of finite radius r_0. We can estimate the potential at the surface of the wire, where $x^2 + y^2 = r_0^2 \ll a$, using the second form of eq. (12):

$$\begin{aligned} V_{\rm wire} &= 4q\sum_{n=0}\frac{\tanh\frac{(2n+1)\pi}{2}\cosh\frac{(2n+1)\pi|x|}{a}-\sinh\frac{(2n+1)\pi|x|}{a}}{2n+1}\cos\frac{(2n+1)\pi y}{a} \\ &\approx 4qRe\sum_{n=0}\frac{\cosh\frac{(2n+1)\pi|x|}{a}-\sinh\frac{(2n+1)\pi|x|}{a}}{2n+1}e^{(2n+1)\pi iy/a} \\ &= 4qRe\sum_{n=0}\frac{e^{-(2n+1)\pi|x|/a}e^{(2n+1)\pi iy/a}}{2n+1} = 4qRe\sum_{n=1}\frac{\left[e^{\pi(-|x|+iy)/a}\right]^{2n+1}}{2n+1} \\ &= 2qRe\ln\frac{1+e^{\pi(-|x|+iy)/a}}{1-e^{\pi(-|x|+iy)/a}} = 2qRe\ln\frac{\sinh\frac{\pi|x|}{a}+i\sin\frac{\pi y}{a}}{\cosh\frac{\pi|x|}{a}-\cos\frac{\pi y}{a}}\,. \end{aligned} \tag{15}$$

Then, writing $\ln\left[\left(\sinh\frac{\pi|x|}{a}+i\sin\frac{\pi y}{a}\right)/\left(\cosh\frac{\pi|x|}{a}-\cos\frac{\pi y}{a}\right)\right] = u + iv$ we have

$$e^{u+iv} = e^u\cos v + ie^u\sin v = 1 - e^{-2\pi(x-iy)/a} = \frac{\sinh\frac{\pi|x|}{a}+i\sin\frac{\pi y}{a}}{\cosh\frac{\pi|x|}{a}-\cos\frac{\pi y}{a}} \tag{16}$$

$$\begin{aligned} e^{2u} &= \frac{\sinh^2\frac{\pi|x|}{a}+\sin^2\frac{\pi y}{a}}{\left(\cosh\frac{\pi|x|}{a}-\cos\frac{\pi y}{a}\right)^2} = \frac{\cosh\frac{\pi|x|}{a}+\cos\frac{\pi y}{a}}{\cosh\frac{\pi|x|}{a}-\cos\frac{\pi y}{a}} \\ &\approx \frac{2}{\frac{1}{2}[(\frac{\pi x}{a})^2+(\frac{\pi y}{a})^2]} = \left(\frac{2a}{\pi r_0}\right)^2, \end{aligned} \tag{17}$$

$$u \approx \ln \frac{2a}{\pi r_0} , \tag{18}$$

and we finally have

$$V_{\rm wire} \approx 2q \ln \frac{2a}{\pi r_0} = 2q \ln \frac{0.64a}{r_0} . \tag{19}$$

Which is the better approximation to $V_{\rm wire}$, the value (19) or the value $2q \ln(a/\sqrt{2}r_0) = 2q \ln(0.71a/r_0)$ that was inferred from the image method?

As a possible guide, we now find a problem for which eq. (19) is the "exact" solution.

Suppose the box channel had width b in x, but still has height a in y. Then, the form (9) would still apply, with the replacement of the x-functions by $\sinh(2n+1)\pi(b/2-|x|)/a$. The boundary condition (10) leads to Fourier coefficients A_n in which the hyperbolic cosine is now $\cosh(2n+1)\pi b/2a$, so that the potential can now be written

$$V(x,y) = 4q \sum_n \frac{\tanh \frac{(2n+1)\pi b}{2a} \cosh \frac{(2n+1)\pi |x|}{a} - \sinh \frac{(2n+1)\pi |x|}{a}}{(2n+1)} \cos \frac{(2n+1)\pi y}{a} . \tag{20}$$

For $b \gg a$, this becomes exactly the second line of eq. (15). Using the first line of eq. (17), we see that this potential can also be written

$$V(x,y) \quad = \quad q \ln \frac{\cosh \frac{\pi |x|}{a} + \cos \frac{\pi y}{a}}{\cosh \frac{\pi |x|}{a} - \cos \frac{\pi y}{a}} , \tag{21}$$

which vanishes at $y = \pm a/2$. For points near the origin, this becomes eq. (19). Hence, the approximation (19) is "exact" for points near a wire that is halfway between a pair of grounded conducting planes. This configuration seems to me to be farther from the case of a wire in square box channel than is a wire inside a circular tube. Hence, I infer that the potential on the wire in a square box channel is closer to $2q \ln(a/\sqrt{2}r_0)$ than $2q \ln(2a/\pi r_0)$.

3. The $\mathbf{v} \times \mathbf{B}$ force on an atom in the rotating cylinder is radially outwards, and increasing linearly with radius, so we expect a positive radial polarization $\mathbf{P} = P\hat{\mathbf{r}}$.

   There will be an electric field $\mathbf{E}$ inside the dielectric associated with this polarization. We now have a "chicken-and-egg" problem: the magnetic field induces some polarization in the rotating cylinder, which induces some electric field, which induces some more polarization, ...

   One way to proceed is to follow this line of thought to develop an iterative solution for the polarization. This is done somewhat later in the solution. Or, we can avoid the iterative approach by going to the rotating frame, where there is no interaction between the medium and the magnetic field, but where there is an effective electric field $\mathbf{E}'$.

   Solution via the Rotating Frame

   However, we must be cautious when using the rotating frame as to what part of the lore of nonrotating frames still applies.

In the rotating frame, any polarization charge density is at rest, and so does not interact with the magnetic field. Individual molecules are polarized by the effective field $\mathbf{E}'$ according to $\mathbf{p}' = \alpha \mathbf{E}'$, where α is the (scalar) molecular polarizability, whose value is that same in any frame in which the molecules are at rest. Summing up the microscopic polarization, we obtain the macroscopic polarization density (in the rotating frame),

$$\mathbf{P}' = \chi \mathbf{E}', \tag{22}$$

where $\mathbf{E}'$ and $\mathbf{P}'$ are the electric field and dielectric polarization in the rotating frame, and χ is the (scalar) dielectric susceptibility. If $v = \omega r \ll c$, then the electric field in the rotating frame is related to lab frame quantities by

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \tag{23}$$

where $\mathbf{E}$ is the electric field due to the polarization that we have yet to find. Since polarization is charge times distance, in the nonrelativistic limit the polarization is the same in the lab frame and the rotating frame: $\mathbf{P}' = \mathbf{P}$.

We do NOT expect that $\mathbf{E}' = 0$ (as would hold in the rest frame of a conductor with no external emf's), since the polarization would vanish in this case.

The velocity has magnitude $v = \omega r$, and is in the azimuthal direction. Thus, $\mathbf{v} \times \mathbf{B} = \omega B \mathbf{r}$, so that

$$\mathbf{P} = \chi \left(\mathbf{E} + \frac{\omega B}{c} \mathbf{r} \right) . \tag{24}$$

We need an additional relation to proceed. The suggestion is to consider the electric displacement $\mathbf{D}$. But, in which frame? This is the trickiest point in the problem. In the rotating (rest) frame of the dielectric, we expect that $\mathbf{D}' = \varepsilon_r \mathbf{E}'$ and (naively) that $\nabla \cdot \mathbf{D}' = 4\pi \rho'_{\text{free}}$, where $\rho'_{\text{free}} = \rho_{\text{free}}$ in the nonrelativistic limit. Since $\rho_{\text{free}} = 0$ in the lab frame for this problem, the preceding arument would imply that $\mathbf{D}' = 0$, and hence that $\mathbf{E}' = 0$, which in turn implies that $\mathbf{P}' = \mathbf{P} = 0$, which is not the case!

It's safer to consider the displacement in the lab frame, where we know that $\rho_{\text{free}} = 0$, and hence that $\mathbf{D} = 0$ since it has no sources. But we do not necessarily expect that $\mathbf{D} = \varepsilon_r \mathbf{E}$ in the lab frame, because in this frame we consider that the magnetic field is causing some of the polarization. So, we invoke the basic relation between $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{P}$ to write

$$\mathbf{D} = 0 = \mathbf{E} + 4\pi \mathbf{P}. \tag{25}$$

Thus,

$$\mathbf{E} = -4\pi \mathbf{P} \tag{26}$$

is the additional relation that we need. Recalling that $\chi = (\varepsilon_r - 1)/4\pi$, (24) leads to

$$\mathbf{P} = \frac{\varepsilon_r - 1}{4\pi c \varepsilon_r} \omega B \mathbf{r}. \tag{27}$$

The surface charge density is

$$\sigma_{\text{pol}} = \mathbf{P}(a) \cdot \hat{\mathbf{r}} = \frac{\varepsilon_r - 1}{4\pi c \varepsilon_r} \omega B a, \tag{28}$$

where a is the radius of the cylinder. As well as this surface charge density, there is a volume charge density,

$$\rho_{\rm pol} = -\nabla \cdot \mathbf{P} = -\frac{1}{r}\frac{\partial r P_r}{\partial r} = -\frac{\varepsilon_r - 1}{2\pi c \varepsilon_r}\omega B, \tag{29}$$

so that the cylinder remains neutral over all.

Both the surface and volume charge densities are proportional to $v(r)/c$, and are moving at velocity $v(r)$. Hence, the magnetic field created by these charges is of order v^2/c^2, and we neglect it in this analysis.

This example is perhaps noteworthy in that a nonvanishing, static volume charge density arises in a linear dielectric material (with no external charges). In pure electrostatics this cannot happen, since $\mathbf{P} = \chi\mathbf{E}$ together with $\nabla \cdot \mathbf{D} = 0 = \nabla \cdot \mathbf{E} + 4\pi \nabla \cdot \mathbf{P}$ imply that $\rho_{\rm pol} = -\nabla \cdot \mathbf{P} = 0$.

We can now go back and examine the fields $\mathbf{E}'$ and $\mathbf{D}' = \varepsilon_r \mathbf{E}'$ in the rotating frame. Combining eqs. (23), (26) and (27) we find

$$\mathbf{E} = -(\varepsilon_r - 1)\frac{\omega B}{c\varepsilon_r}\mathbf{r}, \qquad \mathbf{E}' = \frac{\omega B}{c\varepsilon_r}\mathbf{r}, \qquad \text{and hence} \qquad \mathbf{D}' = \frac{\omega B}{c}\mathbf{r}. \tag{30}$$

If the relative dielectric constant ε_r were unity (as if the cylinder were a vacuum), then eq. (30) tells us that the lab electric field would vanish, as expected. The result that $\mathbf{D}' = \omega B \mathbf{r}/c$ is independent of the dielectric constant, and holds even if the cylinder were empty. The fact that $\nabla \cdot \mathbf{D}' = 2\omega B/c \neq 0$ would imply that $\rho'_{\rm free} \neq 0$ IF $\nabla \cdot \mathbf{D}' = 4\pi\rho'_{\rm free}$. Since this cannot be, we must re-examine our assumptions.

A useful excerise is to transform the lab-frame Maxwell equation $\nabla \cdot \mathbf{D} = 4\pi\rho_{\rm free}$ into the rotating frame. For this, we note that $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ transforms (in the nonrelativistic limit) to $\mathbf{E}' - \mathbf{v}/c \times \mathbf{B}' + 4\pi\mathbf{P}'$, and that $\rho_{\rm free}$ transforms to $\rho'_{\rm free}$. Hence our transformed Maxwell equation is $\nabla \cdot (\mathbf{E}' - \mathbf{v}/c \times \mathbf{B}' + 4\pi\mathbf{P}') = 4\pi\rho'_{\rm free}$. If we suppose that the electric displacement in the rotating frame obeys the basic definition $\mathbf{D}' = \mathbf{E}' + 4\pi\mathbf{P}'$, then

$$\nabla \cdot \mathbf{D}' = 4\pi\rho'_{\rm free} + \nabla \cdot \frac{\mathbf{v}}{c} \times \mathbf{B}'. \tag{31}$$

In the present problem, $\mathbf{B}' = \mathbf{B}$ to first order, so $\mathbf{v}/c \times \mathbf{B}' = \omega B \mathbf{r}/c$, whose divergence is $2\omega B/c$, which is the value for $\nabla \cdot \mathbf{D}'$ found above. Hence, we retain consistency with $\rho'_{\rm free} = 0$ while having a nonzero displacement $\mathbf{D}'$ in the rotating frame.

Experts will note that the result $\mathbf{D}' = \omega B \mathbf{r}/c$ is consistent with the (nonrelativistic) field transformation[1]

$$\mathbf{D}' = \mathbf{D} + \frac{\mathbf{v}}{c} \times \mathbf{H}, \tag{32}$$

since in the present problem $\mathbf{D} = 0$ and $\mathbf{H} = \mathbf{B}$. Further, experts know that the lab frame relation of the displacement $\mathbf{D}$ to the field $\mathbf{E}$ involves the magnetic field $\mathbf{H}$ as well, according to (in the nonrelativistic limit)

$$\mathbf{D} = \varepsilon_r \mathbf{E} + (\varepsilon_r \mu_r - 1)\frac{\mathbf{v}}{c} \times \mathbf{H}. \tag{33}$$

[1] See chap. E III of R. Becker, *Electromagnetic Fields and Interactions* (Dover, New York, 1964).

Then, using $\mathbf{E}$ from eq. (30), plus $\mu_r = 1$ and $\mathbf{B} = \mathbf{H}$ again leads to the result that $\mathbf{D} = 0$ in the lab frame.

> For the record, we pursue the consequence of supposing that since there is no free charge in this problem, the displacement obeys $\mathbf{D}' = 0$ in the rotating frame. Then, since $\mathbf{D}' = \varepsilon \mathbf{E}'$ we have that $\mathbf{E}' = 0$, and eq. (22) implies that $\mathbf{P}' = \mathbf{P} = 0$ also. But, eq. (23) now tells us that $\mathbf{E} = -\omega B \mathbf{r}/c \neq 0$, so that $\mathbf{D} = \mathbf{E} + 4\pi \mathbf{P} = \mathbf{E}$, and $\nabla \cdot \mathbf{D} = -2\omega B/c \neq 0$, independent of the dielectric constant. The rotating cylinder could be imaginary, and the above analysis still should hold. This is implausible.

Additional discussion of Maxwell's equations in a rotating frame has been given by G.N. Pelligrini and A.R. Swift, Am. J. Phys. **63**, 694 (1995).

Iterative Solution in the Lab Frame

The preceding analysis via the rotating from was somewhat tricky, so it is desirable to confirm the results by another method. Hence, we consider an iterative solution.

The axial magnetic field acts on the rotating molecules to cause a $\mathbf{v} \times \mathbf{B}$ force radially outwards. This can be described by an effective electric field

$$\mathbf{E}_0 = \frac{\omega B}{c} \mathbf{r}. \tag{34}$$

This field causes polarization

$$\mathbf{P}_0 = \chi \mathbf{E}_0 = \chi \frac{\omega B}{c} \mathbf{r}. \tag{35}$$

Associated with this is the uniform volume charge density

$$\rho_0 = -\nabla \cdot \mathbf{P}_0 = -2\chi \omega B. \tag{36}$$

According to Gauss' Law, this charge density sets up a radial electric field

$$\mathbf{E}_1 = 2\pi \rho_0 \mathbf{r} = -4\pi \chi \omega B \mathbf{r}. \tag{37}$$

At the next iteration, the total polarization is

$$\mathbf{P}_1 = \chi(\mathbf{E}_0 + \mathbf{E}_1) = \chi(1 - 4\pi\chi) \frac{\omega B}{c} \mathbf{r}. \tag{38}$$

This polarization implies a bound charge density ρ_1, which leads to a correction to field $\mathbf{E}_0$ that we call $\mathbf{E}_2$, ...

At the nth iteration, the polarization will have the form

$$\mathbf{P}_n = k_n \frac{\omega B}{c} \mathbf{r}. \tag{39}$$

Then, the bound charge density is

$$\rho_n = -\nabla \cdot \mathbf{P}_n = -2k_n \omega B, \tag{40}$$

which implies that the correction to the electric field becomes

$$\mathbf{E}_{n+1} = 2\pi\rho_n\mathbf{r} = -4\pi k_n \omega B\mathbf{r}. \tag{41}$$

The effective electric field at iteration $n+1$ is the sum of $\mathbf{E}_0$ due to the $\mathbf{v}\times\mathbf{B}$ force and $\mathbf{E}_{n+1}$ due to the polarization charge. Thus,

$$\mathbf{P}_{n+1} = \chi(\mathbf{E}_0 + \mathbf{E}_{n+1}) = \chi(1 - \pi k_n)\frac{\omega B}{c}\mathbf{r}. \tag{42}$$

But by definition,

$$\mathbf{P}_{n+1} = k_{n+1}\frac{\omega B}{c}\mathbf{r}. \tag{43}$$

Hence,

$$k_{n+1} = \chi(1 - 4\pi k_n). \tag{44}$$

If this sequence converges to the value k, then we must have

$$k = \chi(1 - 4\pi k), \tag{45}$$

so that

$$k = \frac{\chi}{1 + 4\pi\chi} = \frac{\varepsilon_r - 1}{4\pi\varepsilon_r}\,, \tag{46}$$

which again gives (24) for the polarization.