Ph 406: Elementary Particle Physics
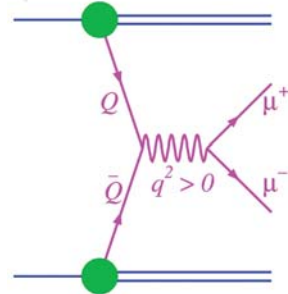# Problem Set 2

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1. The reactions $\pi^\pm p \to \mu^+\mu^-$ are thought to proceed via single-photon exchange according to the so-called Drell-Yan diagram. Use the quark model to predict the cross-section ratio

$$\frac{\sigma_{\pi^- p \to \mu^+\mu^-}}{\sigma_{\pi^+ p \to \mu^+\mu^-}} \, .$$

2. Discuss the motion of an electron of charge $-e$ and rest mass m that is at rest on average inside a plane electromagnetic wave which propagates in the $+z$ direction of a rectangular coordinate system. Suppose the wave is linearly polarized along $\mathbf{x}$,

$$\mathbf{E}_{\text{wave}} = \hat{\mathbf{x}} E_0 \cos(kz - \omega t), \qquad \mathbf{B}_{\text{wave}} = \hat{\mathbf{y}} E_0 \cos(kz - \omega t), \qquad (1)$$

where $\omega = kc$ is the angular frequency of the wave, $k = 2\pi/\lambda$ is the wave number, c is the speed of light in vacuum, and $\hat{\mathbf{x}}$ is a unit vector in the x direction.

Consider only weak fields, for which the dimensionless field-strength parameter $\eta \ll 1$, where

$$\eta = \frac{eE_0}{m\omega c} \, . \qquad (2)$$

First, ignore the longitudinal motion, and deduce the transverse motion, expressing its amplitude in terms of η and λ. Then, in a "macroscopic" view which averages over the "microscopic" motion, the time-average total energy of the electron can be regarded as $\overline{m}c^2$, where $\overline{m} > m$ is the effective mass of the electron (considered as a quasiparticle in the quantum view). That is, the "background" electromagnetic field has "given" mass to the electron beyond that in zero field.

This is an electromagnetic version of the Higgs (Kibble) mechanism.[1]

Also, deduce the form of the longitudinal motion for $\eta \ll 1$.

Not for credit: Where does the energy and (transverse) momentum of the electron come from?

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[1] For an insightful discussion by Kibble that elaborates on the present problem, see [1].

3. **The Weizsäcker-Williams Approximation**

Following an earlier discussion by Fermi [2], Weizsäcker [3] and Williams [4] noted that the electromagnetic fields of an electron in uniform relativistic motion are predominantly transverse, with $\mathbf{E} \approx \mathbf{B}$ (in Gaussian units). This is very much like the fields of a plane wave, so one is led to regard a fast electron as carrying with it a cloud of virtual photons that it can shed (radiate) if perturbed.

The key features of the frequency spectrum of the fields can be estimated as follows. To an observer at rest at distance b from the electron's trajectory, the peak electric field is $E = \gamma e / b^2$, and the field remains above half this strength for time $b/\gamma c$, so the frequency spectrum of this pulse extends up to $\omega_{\max} \approx \gamma c / b$. The total energy of the pulse (relevant to this observer) is $U \approx E^2 \text{Vol} \approx \gamma^2 e^2 / b^4 \cdot b^2 \cdot b/\gamma \approx \gamma e^2 / b$.

If the electron radiates all of this energy, the energy spectrum would be

$$\frac{dU(\omega)}{d\omega} \approx \frac{U}{\omega_{\max}} \approx \frac{e^2}{c}. \tag{3}$$

This result does not depend on the choice of impact parameter b, and is indeed of general validity (to within a factor of $\ln \gamma$). The number of photons n_ω of frequency ω is thus

$$dn_\omega = \frac{dU(\omega)}{\hbar \omega} \approx \frac{e^2}{\hbar c} \frac{d\omega}{\omega} = \alpha \frac{d\omega}{\omega}, \tag{4}$$

where $\alpha = e^2 / \hbar c \approx 1/137$ is the fine structure constant.

The quick approximation (3)-(4) is not accurate at high frequencies. In general, additional physical arguments are needed to identify the maximum frequency of its validity, called the characteristic or critical frequency ω_C, or equivalently, the minimum relevant impact parameter $b_{\min}$. A more detailed evaluation of the high-frequency tail of the virtual photon spectrum shows it to be [2, 3, 4, 5]

$$dn_\omega \approx \alpha \frac{d\omega}{\omega} e^{-2\omega b_{\min}/\gamma c} \qquad \text{(high frequency)}. \tag{5}$$

From this, we see the general relation between the critical frequency and the minimum impact parameter is

$$\omega_C \approx \gamma \frac{c}{b_{\min}}, \qquad b_{\min} \approx \gamma \lambda_C. \tag{6}$$

The characteristic angular spread θ_C of the radiation pattern near the critical frequency can be estimated from eq. (6) by noting that the radiation is much like that of a beam of light with waist $b_{\min}$. Then, from the laws of diffraction we conclude that

$$\theta_C \approx \frac{\lambda_C}{b_{\min}} \approx \frac{1}{\gamma}. \tag{7}$$

This behavior is also expected in that a ray of light emitted in the electron's rest frame at 90° appears at angle $1/\gamma$ to the laboratory direction of the electron.

The Formation Length

To complete an application of the Weizsäcker-Williams method, we must also know over what interval the virtual photon cloud is shaken off the electron to become the radiation detected in the laboratory. Intense (and hence, physically interesting) radiation processes are those in which the entire cloud of virtual photons is emitted as rapidly as possible. This is usefully described by the so-called formation time t_0 and the corresponding formation length $L_0 = vt_0$ where $v \approx c$ is the velocity of the relativistic electron.

The formation length (time) is the distance (time) the electron travels while a radiated wave advances one wavelength λ ahead of the projection of the electron's motion onto the direction of observation. The wave takes on the character of radiation that is no longer tied to its source only after the formation time has elapsed. That is,

$$\lambda = ct_0 - vt_0 \cos\theta \approx L_0(1 - \beta\cos\theta) \approx L_0\left(\frac{1}{2\gamma^2} + \frac{\theta^2}{2}\right), \tag{8}$$

for radiation observed at angle θ to the electron's trajectory. Thus, the formation length is given by

$$L_0 \approx \frac{2\lambda}{\theta^2 + 1/\gamma^2} \tag{9}$$

If the frequency of the radiation is near the critical frequency (6), then the radiated intensity is significant only for $\theta \lesssim \theta_C \approx 1/\gamma$, and the formation length is

$$L_0 \approx \gamma^2\lambda \qquad (\lambda \approx \lambda_C). \tag{10}$$

A good discussion of the formation length in both classical and quantum contexts has been given in ref. [6].

Summary of the Method

A relativistic electron carries with it a virtual photon spectrum of α photons per unit frequency interval. When radiation occurs, for whatever reason, the observed frequency spectrum will closely follow this virtual spectrum. In cases where the driving force for the radiation extends over many formation lengths, the spectrum of radiated photons per unit path length for intense processes is given by expressions (4)-(5), which describe the radiation emitted over one formation length, divided by the formation length (9):

$$\frac{dn_\omega}{dl} \approx \frac{\alpha}{L_0(\omega)}\frac{d\omega}{\omega} \times \begin{cases} 1 & (\omega < \omega_C), \\ e^{-\omega/\omega_C} & (\omega \geq \omega_C). \end{cases} \tag{11}$$

Synchrotron radiation, undulator radiation, transition radiation, and Čerenkov radiation are examples of processes which can be described within the context of classical electromagnetism, but for which the Weizsäcker-Williams approximation is also suitable. Čerenkov radiation and transition radiation are often thought of as rather weak processes, but the Weizsäcker-Williams viewpoint indicates that they are actually as

intense as is possible for radiation by a single charge, in the sense that the entire virtual photon cloud is liberated over a formation length.

In this problem, we emphasize a simplified version of the Weizsäcker-Williams method with the goal of illustrating the main qualitative features of synchrotron radiation.[2]

The Problem: Synchrotron Radiation

Synchrotron radiation arises when a charge, usually an electron, is deflected by a magnetic field.[3] For a large enough region of uniform magnetic field, the electron's trajectory would be a complete circle. However, synchrotron radiation as considered here occurs whenever the magnetic-field region is longer than a formation length.[4]

Consider an electron of Lorentz factor $\gamma \gg 1$ that moves in a circle of radius R. First, deduce the characteristic/critical frequency ω_C of the radiation by estimating the time width of a pulse of radiation detected by a fixed observer as the "searchlight" beam of synchrotron radiation sweeps past him/her. Then, reconsider the meaning of the formation length L_0 noting that the electron moves on a circle but photons move on a chord to find a cubic relation between the wavelength λ and the "formation angle" $\theta_0 = L_0/R$. Consider separately the limiting cases $\lambda \gg \lambda_C$ and $\lambda \ll \lambda_C$, and estimate the frequency spectra dn_ω/dl for these cases, as well as the energy spectra $dU(\omega)/dl$, where l is path length along the electron's trajectory. Compare the integral over frequency of your energy spectrum with the Larmor formula (*using the Hint at the end of prob. 5*).

4. A betatron[5] is a circular device of radius R designed to accelerate electrons (charge e, mass m) via a changing magnetic flux $\dot{\Phi} = \pi R^2 \dot{B}_{\rm ave}$ through the circle. Side views of early versions are shown below.
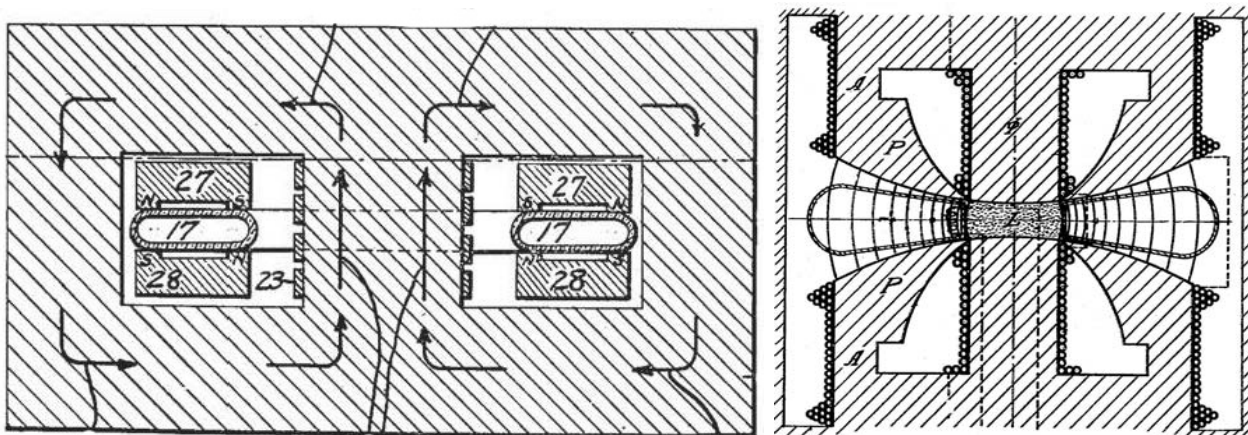
Deduce the relation between the magnetic field B at radius R and the magnetic field $B_{\rm ave}$ averaged over the area of the circle needed for a betatron to function. Also deduce

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[2]A more detailed analysis can reproduce the complete forms of the classical radiation, as has been demonstrated for synchrotron radiation in [8].

[3]Synchrotron radiation is sometimes called magnetic Bremsstrahlung. See, for example, sec. 74 of [7].

[4]The radiation observed when the magnetic field extends for less than a formation length has been discussed in [8, 9, 10].

[5]The name betatron was given by Kerst [14] to the circular induction accelerator after its first successful demonstration [15, 16], which followed a long conceptual history [17, 18, 19, 20, 21, 22, 23].

the maximum energy $\mathcal{E}$ to which an electron could be accelerated by a betatron in terms of B, $\dot{B}_{\rm ave} \equiv B/\tau$, R and the QED critical field strength $B_{\rm crit}$ of prob. 4, set. 1.

Hints: The electrons in this problem are relativistic, so it is useful to introduce the factor $\gamma = \mathcal{E}/mc^2$ where c is the speed of light. Recall that Newton's second law has the same form for nonrelativistic and relativistic electrons except that in the latter case the effective mass is γm. Recall also that for circular motion the rest frame acceleration is γ^2 times that in the lab frame, and that the power radiated by an accelerated charge is a relativistic invariant (in that both energy and time are time-components of 4-vectors; for elaborations, see, `http://kirkmcd.princeton.edu/examples/moving_far.pdf`*).*

For a problem on the stability of orbits in a betatron, see
`http://kirkmcd.princeton.edu/examples/betatron_osc.pdf`*.*
This example is an introduction to the issue of stability of beams in particle accelerators.

5. A magnetic monopole of magnetic charge g, mass m and velocity $\beta = v/c$ passes through a material with electron density N/cm^3 and unit relative permeability. Estimate the energy loss, dE/dx, of the monopole assuming $\beta \approx 1$. *Hint: What is the Lorentz force on a magnetic monopole?*

Compare your result to that for particles of electrical charge e supposing that the monopole satisfies Dirac's quantum condition $eg = \hbar c/2$, where $-e$ is the charge on an electron [28].[6]

For magnetic monopoles with $\beta \approx 10^{-3}$, as might be consistent with extragalactic origin, your result for dE/dx should be modified. Consider energy loss in a metal plate of resistivity ρ due to Joule heating by the Eddy currents induced by the changing magnetic flux as the onopole passes through with normal incidence. *You may assume that β remains constant. An example of an Eddy-current calculation is at*
`http://kirkmcd.princeton.edu/examples/pennies.pdf`
Note that the macroscopic concept of Joule heating does not apply for arbitrarily small length scales.

There exists an unresolved puzzle that if both electric charges and magnetic charges (monopoles) exist then energy may not be conserved,
`http://kirkmcd.princeton.edu/examples/comay.pdf`*.*

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[6]See, for example, sec. 6.12 of [5].

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