Ph 406: Elementary Particle Physics
Problem Set 3

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1. Deduce the nonrelativistic form factors,

$$F(q^2) = \int \rho(r)\, e^{i\mathbf{q}\cdot\mathbf{r}}\, d^3\mathbf{r}, \tag{1}$$

for the spherically symmetric charge densities with characteristic radius R,

$$\rho_a(r) = \begin{cases} 3Q/4\pi R^3 & (r < R), \\ 0 & (r > R), \end{cases} \tag{2}$$

$$\rho_b(r) = \frac{Q}{4\pi R^2}\delta(r - R), \tag{3}$$

and

$$\rho_c(r) = \frac{Q}{2\pi\sqrt{2\pi}R^3} e^{-r^2/2R^2}, \tag{4}$$

all of which have total charge Q. Expand these form factors to order $(qR)^2$.

A neutral particle might have charge distributions ρ_+ and ρ_- with the above forms, but with different values of the characteristic radii R_+ and R_-.

The data are often fit to the form,[1]

$$F_n(q^2) = \frac{Q}{[1 + (qR)^2]^n}, \tag{5}$$

with $n = 2$. What are the corresponding forms of the charge distributions $\rho_n(r)$ for $n = 1$, 2 and 3?

2. **Arbitrary 2×2 Unitary Matrices and Pauli Spin Matrices**

This problem concerns operators that act on 2-component spinors. Such operators can be expressed as 2×2 matrices. Operators that preserve the normalization of a state are called unitary.

Two of the simplest unitary operators on 2-component spinors are the identify matrix $\mathbf{I}_2 = \mathbf{I}$, and the spin-flip operator X (called the NOT operator in quantum computation),

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathsf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6}$$

[1]For a review of nucleon form factors, see C.F. Perdrisat *et al.*, *Nucleon electromagnetic form factors*, Prog. Part. Nucl. Phys. **59**, 694 (2007), http://kirkmcd.princeton.edu/examples/EP/perdrisat_ppnp_59_694_07.pdf.

An arbitrary 2×2 unitary matrix U can be written as

$$\mathsf{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{7}$$

where a, b, c and d are complex numbers such that $\mathsf{U}\mathsf{U}^\dagger = \mathbf{I}$. The decomposition (7) is somewhat trivial. Express the general unitary matrix U as the sum of four unitary matrices, times complex coefficients, of which two are the classical unitary matrices $\mathbf{I}$ and X given above. Denote the "partner" of $\mathbf{I}$ by Z and the "partner" of X by Y such that

$$\mathsf{XY} = i\mathsf{Z}, \qquad \mathsf{YZ} = i\mathsf{X}, \qquad \mathsf{ZX} = i\mathsf{Y}. \tag{8}$$

You have, of course, rediscovered the so-called Pauli spin matrices,[2,3]

$$\boldsymbol{\sigma}_x\ (= \boldsymbol{\sigma}_1) = \mathsf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_y\ (= \boldsymbol{\sigma}_2) = \mathsf{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_z\ (= \boldsymbol{\sigma}_3) = \mathsf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

As usual, we define the Pauli "vector" $\boldsymbol{\sigma}$ as the triplet of matrices

$$\boldsymbol{\sigma} = (\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y, \boldsymbol{\sigma}_z). \tag{10}$$

Show that for ordinary 3-vectors $\mathbf{a}$ and $\mathbf{b}$,

$$(\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma}) = (\mathbf{a} \cdot \mathbf{b})\ \mathbf{I} + i\ \boldsymbol{\sigma} \cdot \mathbf{a} \times \mathbf{b}. \tag{11}$$

With this, show that a general 2×2 unitary matrix can be written as

$$\mathsf{U} = e^{i\delta} \left(\cos\frac{\theta}{2}\ \mathbf{I} + i \sin\frac{\theta}{2}\ \hat{\mathbf{u}} \cdot \boldsymbol{\sigma} \right) = e^{i\delta} e^{i\frac{\theta}{2}\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}}, \tag{12}$$

where δ and θ are real numbers and $\hat{\mathbf{u}}$ is a real unit vector.[4] By the exponential e^{O} of an operator O we mean the Taylor series $\sum_n \mathsf{O}^n/n!$ where $\mathsf{O}^0 = \mathbf{I}$.

What is the determinant of the matrix representation of U? The subset of 2×2 unitary matrices with unit determinant is called the special unitary group SU(2). What is the version of eq. (12) that describes 2×2 special unitary operators?

You may wish to convince yourself of a factoid related to eq. (12), namely that if A *is a square matrix of any order such that* $\mathsf{A}^2 = \mathbf{I}$, *then* $e^{i\theta\mathsf{A}} = \cos\theta\ \mathbf{I} + i\sin\theta\ \mathsf{A}$, *provided that* θ *is a real number. It follows that* A *can also be written in the exponential form*

$$\mathsf{A} = e^{i\pi/2} e^{-i\frac{\pi}{2}\mathsf{A}} = e^{-i\pi/2} e^{i\frac{\pi}{2}\mathsf{A}}. \tag{13}$$

[2] W. Pauli, *Zur Quantenmechanik des magnetischen Elektrons*, Z. Phys. **43**, 601 (1927), http://kirkmcd.princeton.edu/examples/QM/pauli_zp_43_601_27.pdf.

[3] The Pauli spin matrices (and the unit matrix $\mathbf{I}$) are not only unitary, they are also hermitian, meaning that they are identical to their adjoints: $\boldsymbol{\sigma}_j^\dagger = \boldsymbol{\sigma}_j$.

[4] Note that if make the replacements $\theta \to -\theta$ and $\hat{\mathbf{u}} \to -\hat{\mathbf{u}}$ we obtain another valid representation of U, since the physical operation of a rotation by angle θ about an axis $\hat{\mathbf{u}}$ is identical to a rotation by $-\theta$ about the axis $-\hat{\mathbf{u}}$.

There are several unitary operators of interest, such as the Pauli matrices, that are their own inverse. If we call such an operator V, *then its exponential representation of* V *can be written in multiple ways,*

$$\mathsf{V} = e^{i\delta} e^{i\frac{\theta}{2}\hat{\mathbf{v}}\cdot\boldsymbol{\sigma}} = \mathsf{V}^{-1} = e^{-i\delta} e^{-i\frac{\theta}{2}\hat{\mathbf{v}}\cdot\boldsymbol{\sigma}}. \tag{14}$$

3. Give the explicit 4×4 matrix form of the four Dirac matrices γ_μ,[5] as well as that for $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$, in their representation via the 2×2 Pauli matrices $\mathbf{I}$ and $\boldsymbol{\sigma}_i$, $i = 1, 2, 3$,

$$\gamma_0 = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ -\boldsymbol{\sigma}_i & 0 \end{pmatrix}, \tag{15}$$

It should be then evident that $\mathsf{tr}(\gamma_\mu) = 0 = \mathsf{tr}(\gamma_5)$, where tr is the trace operator. Then, it immediately follows that $\mathsf{tr}(\not{a}) = 0$, where $\not{a} \equiv a^\mu \gamma_\mu$ and a_μ is an arbitrary 4-vector.

Show that

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}\mathbf{I}_4, \tag{16}$$

where $\eta_{\mu\nu}$ has diagonal elements $1, -1, -1, -1$ and $\mathbf{I}_4$ is the 4×4 unit matrix,[6] and hence that

$$\mathsf{tr}(\gamma_\mu\gamma_\nu) = 4\eta_{\mu\nu}, \qquad \text{and} \qquad \mathsf{tr}(\not{a}\not{b}) = 4a_\mu b^\mu \equiv 4ab. \tag{17}$$

Show also that

$$\mathsf{tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 4(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}), \tag{18}$$

and hence that

$$\mathsf{tr}(\not{a}\not{b}\not{c}\not{d}) = 4[(ab)(cd) - (ac)(bd) + (ad)(bc)]. \tag{19}$$

A factoid which you need not demonstrate is that the Dirac equivalent of eq. (11) is

$$\not{a}\not{b} = ab\mathbf{I}_4 + \frac{a^\mu b^\nu}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu). \tag{20}$$

If you think that maxtrix manipulation is the key to physics, then you might enjoy my course, **Physics of Quantum Computation**,
http://kirkmcd.princeton.edu/examples/ph410problems.pdf.

[5]The matrices γ_μ were introduced by Dirac in the form used here, but with his γ_4 being our γ_0, in sec. 3 of *The Quantum Theory of the Electron*, Proc. Roy. Soc. London A **117**, 610 (1928),
http://kirkmcd.princeton.edu/examples/QED/dirac_prsla_117_610_28.pdf.

[6]The matrix $\mathbf{I}_4$ is typically denoted by 1.

Solutions

1. **Form Factors.** We take the z-axis along the direction of the 3-momentum vector $\mathbf{q}$, such that the form factor associated with charge density a spherically symmetry charge density $\rho(r)$ is

$$\begin{aligned} F(q^2) &= \int_0^\infty 2\pi r^2\,dr\,\rho(r)\int_{-1}^{1} d\cos\theta\, e^{iqr\cos\theta} = \int_0^\infty 2\pi r^2\,dr\,\rho(r)\frac{e^{iqr}-e^{-iqr}}{iqr} \\ &= \frac{4\pi}{q}\int_0^\infty r\,dr\,\rho(r)\sin qr. \end{aligned} \tag{21}$$

Then, the form factor associated with charge density (2) is,

$$\rho_a(r) = \begin{cases} 3Q/4\pi R^3 & (r < R), \\ 0 & (r > R), \end{cases} \tag{2}$$

$$F_a(q^2) = \frac{3Q}{4\pi R^3}\frac{4\pi}{q^3}\int_0^{qR} qr\,d\,qr\,\sin qr = \frac{3Q}{(qR)^3}(\sin qR - qR\cos qR) \approx Q\left(1-\frac{(qR)^2}{10}\right). \tag{22}$$

The first zero of F_a is for $qR = \tan qR$. This can be found by going to Wolfram Alpha, http://www.wolframalpha.com/, and entering x = tan x. The result is that $q = 4.493..../R$. For $R = 1$ fermi, we need $q = 4.5\cdot 197 = 877$ MeV to detect the first zero.

The form factor associated with charge density (3) is,

$$\rho_b(r) = \frac{Q}{4\pi R^2}\delta(r-R), \tag{3}$$

$$F_b(q^2) = \frac{Q}{4\pi R^2}\frac{4\pi}{q}\int_0^\infty r\,dr\,\delta(r-R)\sin qr = Q\frac{\sin qR}{qR} \approx Q\left(1-\frac{(qR)^2}{6}\right). \tag{23}$$

The first zero of F_b is at $q = \pi/R$.

The form factor associated with charge density (4) is,

$$\rho_c(r) = \frac{Q}{2\pi\sqrt{2\pi}R^3}e^{-r^2/2R^2}, \tag{4}$$

$$F_c(q^2) = \frac{Q}{2\pi\sqrt{2\pi}R^3}\frac{4\pi}{q}\int_0^\infty r\,dr\,e^{-r^2/2R^2}\sin qr = Q\,e^{-q^2R^2/2} \approx Q\left(1-\frac{(qR)^2}{2}\right). \tag{24}$$

F_c has no zeroes, but its characteristic width in q is $1/R$.

To recover a (spherically symmetric) charge distribution $\rho(r)$ from a form factor $F(q^2)$ we use the inverse transform,

$$\begin{aligned} \rho(r) &= \frac{1}{(2\pi)^3}\int_0^\infty 2\pi q^2\,dq\,F(q^2)\int_{-1}^{1} d\cos\theta\, e^{-iqr\cos\theta} = \frac{1}{(2\pi)^2}\int_0^\infty q^2\,dq\,F(q^2)\frac{e^{-iqr}-e^{iqr}}{-iqr} \\ &= \frac{1}{2\pi^2 r}\int_0^\infty q\,dq\,F(q^2)\sin qr. \end{aligned} \tag{25}$$

Hence, the form factor (4) corresponds to

$$\rho_n(r) = \frac{Q}{2\pi^2 r R^{2n}} \int_0^\infty q\, dq\, \frac{\sin qr}{[1/R^2 + q^2]^n} . \qquad (26)$$

The general form of this integral is given in Gradsthetyn and Ryzhik, 3.737.2. For $n = 1$, G&R 3.723.3 gives

$$\rho_1(r) = \frac{Q\, e^{-r/R}}{4\pi r R^2} , \qquad (27)$$

which is not well behaved at the origin, although $\int \rho_1\, d\text{Vol} = Q$. Note that the form factor $F_1(q^2)$ is hardly distinguishable at small q from the forms (22), (23) and (24). The divergence of the charge density (27) at small r could only be revealed in the form factor/Fourier transform by measurements at large q (large energy of the probe), which reinforces that high energies are needed to reveal phenomena at small distances.

For $n = 2$, G&R 3.729.2 gives

$$\rho_2(r) = \frac{Q\, e^{-r/R}}{8\pi R^3} , \qquad (28)$$

a simple exponential falloff. *The case $n = 2$ is often called the* dipole *form factor, for reasons obscure to me.* For $n = 3$, G&R 3.737.4 gives

$$\rho_3(r) = \frac{Q\, e^{-r/R}}{32\pi R^3} \left(1 + \frac{r}{R}\right) , \qquad (29)$$

which distribution has slightly more charge at large radii than the case of $n = 2$.

2. **Arbitrary 2×2 Unitary Matrix**

A straightforward expansion of a general 2×2 unitary matrix U that involves the unit matrix $\mathbf{I}$ and the NOT matrix X is

$$\begin{aligned} \mathsf{U} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \frac{a+d}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{a-d}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{b+c}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{-b+c}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \frac{a+d}{2} \mathbf{I} + \frac{a-d}{2} \mathsf{Z} + \frac{b+c}{2} \mathsf{X} + \frac{-b+c}{2} \tilde{\mathsf{Y}}. \end{aligned} \qquad (30)$$

The unitary matrices $\tilde{\mathsf{Y}}$ and Z have real matrix elements, which seems desirable at first glance. However, when multiplying the unitary matrices based on expansion (30), we find the products

$$\mathsf{X}\tilde{\mathsf{Y}} = \mathsf{Z}, \qquad \tilde{\mathsf{Y}}\mathsf{Z} = \mathsf{X}, \qquad \mathsf{Z}\mathsf{X} = -\tilde{\mathsf{Y}}. \qquad (31)$$

A symmetric pattern of products is obtained, following Pauli, if we use the unitary matrix $\mathsf{Y} = i\tilde{\mathsf{Y}}$. Then,

$$\boldsymbol{\sigma}_x = \mathsf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_y = \mathsf{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_z = \mathsf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (32)$$

and

$$\mathsf{XY} = i\mathsf{Z}, \qquad \mathsf{YZ} = i\mathsf{X}, \qquad \mathsf{ZX} = i\mathsf{Y}. \tag{33}$$

We can now write our expansion of a general 2×2 unitary matrix as

$$\mathsf{U} = a\,\mathbf{I} + \mathbf{b} \cdot \boldsymbol{\sigma}, \tag{34}$$

where a is a complex number (in general different from the a of eq. (30)), $\mathbf{b}$ is a triplet of complex numbers, and $\boldsymbol{\sigma}$ is the triplet $(\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y, \boldsymbol{\sigma}_z)$ of Pauli matrices.

The (hermitian) Pauli matrices $\boldsymbol{\sigma}_j$ obey

$$\boldsymbol{\sigma}_j^\dagger = \boldsymbol{\sigma}_j, \qquad \boldsymbol{\sigma}_j^2 = \mathbf{I}, \qquad \text{and} \qquad \boldsymbol{\sigma}_j\boldsymbol{\sigma}_k = i\epsilon_{jkl}\,\boldsymbol{\sigma}_l \quad \text{when } j \neq k, \tag{35}$$

where $\epsilon_{jkl} = 1$ for an even permutation of xyz, -1 for an odd permutation, and 0 otherwise.[7] Thus,

$$\begin{aligned}
(\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma}) &= \sum_j a_j\, \boldsymbol{\sigma}_j \sum_k b_k\, \boldsymbol{\sigma}_k = \sum_{j=k} a_j b_k\, \boldsymbol{\sigma}_j\boldsymbol{\sigma}_k + \sum_{j\neq k} a_j b_k\, \boldsymbol{\sigma}_j\boldsymbol{\sigma}_k \\
&= (\mathbf{a} \cdot \mathbf{b})\, \mathbf{I} + i \sum_{j\neq k} a_j b_k \epsilon_{jkl}\, \boldsymbol{\sigma}_l \\
&= (\mathbf{a} \cdot \mathbf{b})\, \mathbf{I} + i\, \boldsymbol{\sigma} \cdot \mathbf{a} \times \mathbf{b}.
\end{aligned} \tag{36}$$

The condition that matrix (34) be unitary can now be written

$$\begin{aligned}
\mathbf{I} &= \mathsf{U}\mathsf{U}^\dagger = (a\,\mathbf{I} + \mathbf{b} \cdot \boldsymbol{\sigma})(a^\star\,\mathbf{I} + \mathbf{b}^\star \cdot \boldsymbol{\sigma}^\dagger) = (a\,\mathbf{I} + \mathbf{b} \cdot \boldsymbol{\sigma})(a^\star\,\mathbf{I} + \mathbf{b}^\star \cdot \boldsymbol{\sigma}) \\
&= (|a|^2 + |\mathbf{b}|^2)\,\mathbf{I} + \boldsymbol{\sigma} \cdot [2Re(a\mathbf{b}^\star) + i\,\mathbf{b} \times \mathbf{b}^\star].
\end{aligned} \tag{37}$$

Hence, we need

$$(|a|^2 + |\mathbf{b}|^2) = 1, \tag{38}$$

$$0 = 2Re(a\mathbf{b}^\star) + i\,\mathbf{b} \times \mathbf{b}^\star = 2Re(a\mathbf{b}^\star) + 2Re(\mathbf{b}) \times Im(\mathbf{b}). \tag{39}$$

If $a \neq 0$, we write it as $a = a_0 e^{i\delta}$ where a_0 and δ are real. We also write $\mathbf{b} = e^{i\delta}(\mathbf{c} + i\mathbf{d})$ where $\mathbf{c}$ and $\mathbf{d}$ are real vectors. Then, we eq. (39) becomes

$$0 = Re(a\mathbf{b}^\star) + Re(\mathbf{b}) \times Im(\mathbf{b}) = a_0\,\mathbf{c} + \mathbf{c} \times \mathbf{d}, \tag{40}$$

which implies that $\mathbf{c} = 0$. Thus,

$$\mathbf{b} = ib_0 e^{i\delta}\,\hat{\mathbf{u}}, \tag{41}$$

where $b_0 = |d|$ and $\hat{\mathbf{u}} = \mathbf{d}/\,|d|$ is a real unit vector.

On the other hand, if $a = 0$ then eq. (39) requires that vector $Re(\mathbf{b})$ must be parallel to vector $Im(\mathbf{b})$, so the vector $\mathbf{b}$ can be written as

$$\mathbf{b} = Re(b)\,\hat{\mathbf{u}} + iIm(b)\,\hat{\mathbf{u}} = ib_0 e^{i\delta}\,\hat{\mathbf{u}}, \tag{42}$$

[7] **Digression**. If one defines $I = -i\sigma_1$, $J = -i\sigma_2$ and $K = -i\sigma_3$, then $I^2 = J^2 = K^2 = IJK = -\mathbf{I}$, such that $\{\mathbf{I}, I, J, K\}$ are a represention of Hamilton's quaterions (aka Clifford algebra).

where b_0 and δ are real, and $\hat{\mathbf{u}}$ is a real unit vector.

Hence, in any case the general 2×2 unitary matrix (34) can be written

$$\mathsf{U} = e^{i\delta}(a_0\ \mathbf{I} + i b_0\ \hat{\mathbf{u}} \cdot \boldsymbol{\sigma}), \tag{43}$$

where the real numbers a_0 and b_0 obey

$$a_0^2 + b_0^2 = 1, \tag{44}$$

so that condition (38) is satisfied. We can formally express a_0 and b_0 in terms of an angle θ such that

$$a_0 = \cos\frac{\theta}{2}, \qquad b_0 = \sin\frac{\theta}{2}. \tag{45}$$

Then,

$$\mathsf{U} = e^{i\delta}\left(\cos\frac{\theta}{2}\ \mathbf{I} + i\sin\frac{\theta}{2}\ \hat{\mathbf{u}} \cdot \boldsymbol{\sigma}\right) = e^{i\delta} e^{i\frac{\theta}{2}\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}}. \tag{46}$$

By the exponential e^{A} of an operator A we, of course, mean the Taylor series

$$e^{\mathsf{A}} = \sum_{n=0}^{\infty} \frac{\mathsf{A}^n}{n!}\,. \tag{47}$$

For two $\mathsf{noncommuting}$ operators A and B, in general $e^{\mathsf{A}+\mathsf{B}} = e^{\mathsf{B}+\mathsf{A}} \neq e^{\mathsf{A}} e^{\mathsf{B}} \neq e^{\mathsf{B}} e^{\mathsf{A}}$.

The validity of the exponential form in eq. (46) is confirmed by noting that

$$\begin{aligned} e^{i\frac{\theta}{2}\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}} &= \sum_j \frac{(i\frac{\theta}{2}\hat{\mathbf{u}}\cdot\boldsymbol{\sigma})^j}{j!} = \left[\mathbf{I} - \frac{(\frac{\theta}{2})^2(\hat{\mathbf{u}}\cdot\boldsymbol{\sigma})^2}{2} + \ldots\right] + i\left[\frac{\theta}{2}\hat{\mathbf{u}}\cdot\boldsymbol{\sigma} - \frac{(\frac{\theta}{2})^3(\hat{\mathbf{u}}\cdot\boldsymbol{\sigma})^3}{6} + \ldots\right] \\ &= \left[1 - \frac{(\frac{\theta}{2})^2}{2} + \ldots\right]\mathbf{I} + i\left[\frac{\theta}{2} - \frac{(\frac{\theta}{2})^3}{6} + \ldots\right]\hat{\mathbf{u}}\cdot\boldsymbol{\sigma} = \cos\frac{\theta}{2}\ \mathbf{I} + i\sin\frac{\theta}{2}\ \hat{\mathbf{u}}\cdot\boldsymbol{\sigma}, \end{aligned} \tag{48}$$

via repeated uses of eq. (36) with $\mathbf{a} = \mathbf{b} = \hat{\mathbf{u}}$.

While the Pauli operators $\boldsymbol{\sigma}_j$ do not commute with one another, we see from eq. (48) that $e^{a\boldsymbol{\sigma}_j + b\boldsymbol{\sigma}_k} = e^{b\boldsymbol{\sigma}_k + a\boldsymbol{\sigma}_j}$. However, $e^{a\boldsymbol{\sigma}_j + b\boldsymbol{\sigma}_k} \neq e^{a\boldsymbol{\sigma}_j} e^{b\boldsymbol{\sigma}_k} \neq e^{b\boldsymbol{\sigma}_k} e^{a\boldsymbol{\sigma}_j}$ when $j \neq k$. In particular, $e^{i\frac{\theta}{2}\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}} \neq e^{i\frac{\theta}{2}u_x\boldsymbol{\sigma}_x} e^{i\frac{\theta}{2}u_y\boldsymbol{\sigma}_y} e^{i\frac{\theta}{2}u_z\boldsymbol{\sigma}_z}$.

The matrix form of eq. (46) is

$$\mathsf{U} = e^{i\delta}\begin{pmatrix} \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\ u_x & i\sin\frac{\theta}{2}(u_x - iu_y) \\ i\sin\frac{\theta}{2}(u_x + iu_y) & \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\ u_x \end{pmatrix}, \tag{49}$$

so the determinant of U is

$$\Delta_{\mathsf{U}} = e^{2i\delta}\left[\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\left(u_x^2 + u_y^2 + u_z^2\right)\right] = e^{2i\delta}. \tag{50}$$

Hence, the 2×2 special unitary operators (those for which $\Delta_{\mathsf{U}} = 1$) are those with $\delta = 0$ or π,

$$\mathsf{U} = \pm\left(\cos\frac{\theta}{2}\,\mathbf{I} + i\sin\frac{\theta}{2}\,\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}\right) = \pm e^{i\frac{\theta}{2}\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}}, \qquad \mathsf{U} \in \mathrm{SU}(2). \tag{51}$$

As to the factoid related to eq. (13), whenever $\mathsf{A}^2 = \mathbf{I}$ *we can make the Taylor expansion,*

$$\begin{aligned} e^{i\theta\mathsf{A}} &= \sum_{k\ \mathrm{even}}^{\infty}\frac{(i\theta\mathsf{A})^k}{k!} + \sum_{k\ \mathrm{odd}}^{\infty}\frac{(i\theta\mathsf{A})^k}{k!} = \sum_{k\ \mathrm{even}}^{\infty}\frac{(-1)^{k/2}\theta^k}{k!}\mathbf{I} + i\sum_{k\ \mathrm{odd}}^{\infty}\frac{(-1)^{(k-1)/2}\theta^k}{k!}\mathsf{A} \\ &= \cos\theta\ \mathbf{I} + i\sin\theta\ \mathsf{A}. \end{aligned} \tag{52}$$

3. In terms of the 2×2 Pauli matrices $\mathbf{I}$ and $\boldsymbol{\sigma}_i$, the Dirac matrices γ_μ can be written as[8]

$$\gamma_0 = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & \boldsymbol{\sigma}_1 \\ -\boldsymbol{\sigma}_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} 0 & \boldsymbol{\sigma}_2 \\ -\boldsymbol{\sigma}_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} 0 & \boldsymbol{\sigma}_3 \\ -\boldsymbol{\sigma}_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix}. \tag{53}$$

Note that

$$\gamma_0\gamma_0 = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}\begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{pmatrix} = \mathbf{I}_4,$$

[8]The Dirac matrix γ_5 may have been introduced on p. 126 of W. Pauli, *Contributions mathématique à la théorie des matrices de Dirac*, Ann. Inst. H. Poincaré **6**, 109 (1936),
http://kirkmcd.princeton.edu/examples/QED/pauli_aihp_6_109_36.pdf.
Pauli's original γ_5 was just $\gamma_0\gamma_1\gamma_2\gamma_3$, without the factor of i that is now conventional. The matrix $\gamma_0\gamma_1\gamma_2\gamma_3$ was identified as one of 16 linearly independent 4×4 Dirac matrices on p. 881 of J. von Neumann, *Einige Bemerkungen zur Diracschen Theorie des relativistischen Drehelektrons*, Z. Phys. **48** 868 (1928),
http://kirkmcd.princeton.edu/examples/QED/voneumann_zp_48_868_28.pdf.

$$\boldsymbol{\gamma}_i\boldsymbol{\gamma}_i = \begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ -\boldsymbol{\sigma}_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ -\boldsymbol{\sigma}_i & 0 \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i & 0 \\ 0 & -\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i \end{pmatrix} = -\mathbf{I}_4,$$

$$\gamma_5\gamma_5 = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathbf{I} \\ \mathsf{I} & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathsf{I} \end{pmatrix} = \mathbf{I}_4, \tag{54}$$

where $\mathbf{I}_4$ is the 4×4 unit matrix. Hence, (after a bit of algebra),

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}\mathbf{I}_4, \qquad \gamma_5\gamma_\mu = -\gamma_\mu\gamma_5, \tag{55}$$

where we don't regard $\eta_{\mu\nu}$ as a γ-matrix although it is a 4×4 collection of numbers. Taking the trace of this γ-matrix equation, we have that

$$\mathsf{tr}(\gamma_\mu\gamma_\nu) + \mathsf{tr}(\gamma_\nu\gamma_\mu) = 2\mathsf{tr}(\gamma_\mu\gamma_\nu) = 2\eta_{\mu\nu}\mathsf{tr}(\mathbf{I}_4) = 8\eta_{\mu\nu}, \tag{56}$$

$$\mathsf{tr}(\gamma_\mu\gamma_\nu) = 4\eta_{\mu\nu}. \tag{57}$$

Hence,

$$\mathsf{tr}(\not{a}\not{b}) = a^\mu b^\nu \mathsf{tr}(\gamma_\mu\gamma_\nu) = 4a^\mu b^\nu \eta_{\mu\nu} = 4a^\mu b_\nu = 4ab. \tag{58}$$

Next, we consider (following `http://en.wikipedia.org/wiki/Gamma_matrices`)

$$\mathsf{tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = \mathsf{tr}\left(\gamma_\mu\gamma_\nu(2\eta_{\rho\sigma} - \gamma_\sigma\gamma_\rho)\right) = 2\eta_{\rho\sigma}\mathsf{tr}\left(\gamma_\mu\gamma_\nu\right) - \mathsf{tr}\left(\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\rho\right), \tag{59}$$

using eq. (55). For the term on the right, we'll continue the pattern of swapping γ_σ with its neighbor to the left,

$$\mathsf{tr}\left(\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\rho\right) = \mathsf{tr}\left(\gamma_\mu(2\eta_{\nu\sigma} - \gamma_\sigma\gamma_\nu)\gamma_\rho\right) = 2\eta_{\nu\sigma}\mathsf{tr}\left(\gamma_\mu\gamma_\rho\right) - \mathsf{tr}\left(\gamma_\mu\gamma_\sigma\gamma_\nu\gamma_\rho\right). \tag{60}$$

Again, for the term on the right swap γ_σ with its neighbor to the left,

$$\mathsf{tr}\left(\gamma_\mu\gamma_\sigma\gamma_\nu\gamma_\rho\right) = \mathsf{tr}\left((2\eta_{\mu\sigma} - \gamma_\sigma\gamma_\mu)\gamma_\nu\gamma_\rho\right) = 2\eta_{\mu\sigma}\mathsf{tr}\left(\gamma_\nu\gamma_\rho\right) - \mathsf{tr}\left(\gamma_\sigma\gamma_\mu\gamma\nu\gamma_\rho\right). \tag{61}$$

Equation (61) is the term on the right of eq. (60), and eq. (60) is the term on the right of eq. (59). We also use identity (57) to simplify terms, such as,

$$2\eta_{\rho\sigma}\mathsf{tr}\left(\gamma_\mu\gamma_\nu\right) = 2\eta_{\rho\sigma}(4\eta_{\mu\nu}) = 8\eta_{\rho\sigma}\eta_{\mu\nu}. \tag{62}$$

So when you plug eqs. (60)-(62) into eq. (59) we have,

$$\mathsf{tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 8\eta_{\rho\sigma}\eta_{\mu\nu} - 8\eta_{\nu\sigma}\eta_{\mu\rho} + 8\eta_{\mu\sigma}\eta_{\nu\rho} - \mathsf{tr}\left(\gamma_\sigma\gamma_\mu\gamma_\nu\gamma_\rho\right). \tag{63}$$

The terms inside the trace can be cycled, so

$$\mathsf{tr}\left(\gamma_\sigma\gamma_\mu\gamma_\nu\gamma_\rho\right) = \mathsf{tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma). \tag{64}$$

So really eq. (62) is

$$2\mathsf{tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 8\eta_{\rho\sigma}\eta_{\mu\nu} - 8\eta_{\nu\sigma}\eta_{\mu\rho} + 8\eta_{\mu\sigma}\eta_{\nu\rho}, \tag{65}$$

or

$$\mathsf{tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 4\left(\eta_{\rho\sigma}\eta_{\mu\nu} - \eta_{\nu\sigma}\eta_{\mu\rho} + \eta_{\mu\sigma}\eta_{\nu\rho}\right). \tag{66}$$

Finally,

$$\begin{aligned}\mathsf{tr}(\not{a}\not{b}\not{c}\not{d}) &= a^\mu b^\nu c^\rho d^\sigma \mathsf{tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 4a^\mu b^\nu c^\rho d^\sigma \left(\eta_{\rho\sigma}\eta_{\mu\nu} - \eta_{\nu\sigma}\eta_{\mu\rho} + \eta_{\mu\sigma}\eta_{\nu\rho}\right) \\ &= 4[(ab)(cd) - (ac)(bd) + (ad)(bc)].\end{aligned} \tag{67}$$

Digression: *For future reference, we note that*

$$\gamma_0\gamma_i\gamma_5 = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} \begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ -\boldsymbol{\sigma}_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\sigma}_i & 0 \\ 0 & \boldsymbol{\sigma}_i \end{pmatrix}. \tag{68}$$