Ph 406: Elementary Particle Physics
Problem Set 4

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1. The form,

$$\mathsf{U} = e^{i\delta}\left(\cos\frac{\theta}{2}\,\mathbf{I} + i\sin\frac{\theta}{2}\,\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}\right) = e^{i\delta}e^{i\frac{\theta}{2}\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}}, \tag{1}$$

of a general 2×2 unitary matrix [(Set 2, eq. (12)] suggests that these matrices have something to do with rotations. Certainly, a matrix that describes the rotation of a vector is a unitary transformation.

A general 2-component (spinor) state $|\psi\rangle = \psi_+|+\rangle + \psi_-|-\rangle$, where $\left|\psi_+\right|^2 + \left|\psi_-\right|^2 = 1$, can also be written as,

$$|\psi\rangle = e^{i\delta}\left(\cos\theta|+\rangle + e^{i\phi}\sin\theta|-\rangle\right). \tag{2}$$

The overall phase δ has no meaning to a measurement of $|\psi\rangle$. So, it is tempting to interpret parameters θ and ϕ as angles describing the orientation in a spherical coordinate system (r,θ,ϕ) of a unit 3-vector that is associated with the state $|\psi\rangle$. The state $|+\rangle$ might then correspond to the unit 3-vector $\hat{\mathbf{z}}$ that points up along the z-axis, while $|-\rangle \leftrightarrow -\hat{\mathbf{z}}$.

However, this doesn't work! The suggestion is that the state $|+\rangle$ corresponds to angles $\theta = 0$, $\phi = 0$ and state $|-\rangle$ to angles $\theta = \pi$, $\phi = 0$. With this hypothesis, eq. (2) gives a satisfactory representation of a spin-up state as $|+\rangle$, but it implies that the spin-down state would be $-|+\rangle = e^{i\pi}$ times the spin-up state, which is not really distinct from the spin-up state.

We fix up things be writing,

$$|\psi\rangle = e^{i\delta}\left[\cos\frac{\theta}{2}|+\rangle + e^{i\phi}\sin\frac{\theta}{2}|-\rangle\right], \tag{3}$$

and identifying angles θ and ϕ with the polar and azimuthal angles of a unit 3-vector in an abstract 3-space (sometimes called the Bloch sphere). That is, we associate the state $|\psi\rangle$ with the unit 3-vector whose components are $\psi_x = \sin\theta\cos\phi$, $\psi_y = \sin\theta\sin\phi$ and $\psi_z = \cos\theta$. Now, the associations,

$$\text{spin up} \leftrightarrow (\theta = 0, \phi = 0) \leftrightarrow |+\rangle, \qquad \text{spin down} \leftrightarrow (\theta = \pi, \phi = 0) \leftrightarrow |-\rangle, \tag{4}$$

given by eq. (3) are satisfactory.

We then infer from eq. (3) that the spin-up and spin-down states in the direction (θ, ϕ) are, to within an overall phase factor,

$$|+(\theta,\phi)\rangle \propto \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}, \qquad |-(\theta,\phi)\rangle \propto |+(\pi-\theta,\phi+\pi)\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}. \quad (5)$$

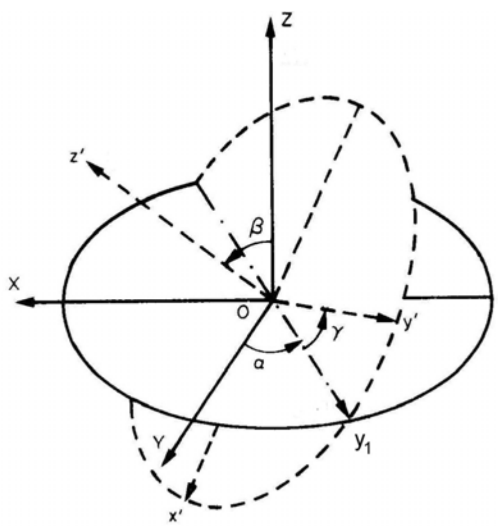
The standard form of the spin-up/down states is,

$$|+(\theta,\phi)\rangle = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\phi/2} \\ \sin\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix}, \qquad |-(\theta,\phi)\rangle = \begin{pmatrix} \sin\frac{\theta}{2}\, e^{-i\phi/2} \\ -\cos\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix}, \quad (6)$$

which is consistent with eq. (5), but perhaps does not obviously follow from it.

The Problem: Deduce the up and down 2-component spinor states along direction (θ, ϕ) in a spherical coordinate system via rotation matrices (where first a rotation is made by angle θ and then by angle ϕ).

Rotation Matrices



A general rotation in 3-space is characterized by 3 angles. We follow Euler in naming these angles as in the figure above.[1] The rotation takes the axis (x, y, z) into the axes (x', y', z') in 3 steps:

(a) A rotation by angle α about the z-axis, which brings the y-axis to the y_1 axis.

(b) A rotation by angle β about the y_1-axis, which brings the z-axis to the z'-axis.

(c) A rotation by angle γ about the z'-axis, which brings the y_1-axis to the y'-axis (and the x-axis to the x'-axis).

[1] From sec. 58 of Landau and Lifshitz, *Quantum Mechanics*, 2nd ed. (Pergamon, 1965), http://kirkmcd.princeton.edu/examples/QM/landau_qm_65.pdf

The 2×2 unitary matrix that corresponds to this rotation (of coordinate axes) is,

$$\begin{aligned}
\mathsf{R}(\alpha,\beta,\gamma) &= \begin{pmatrix} \cos\frac{\beta}{2} e^{i(\alpha+\gamma)/2} & \sin\frac{\beta}{2} e^{i(-\alpha+\gamma)/2} \\ -\sin\frac{\beta}{2} e^{i(\alpha-\gamma)/2} & \cos\frac{\beta}{2} e^{-i(\alpha+\gamma)/2} \end{pmatrix} \\
&= \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \\
&= \mathsf{R}_{z'}(\gamma)\mathsf{R}_{y_1}(\beta)\mathsf{R}_z(\alpha), \qquad (7)
\end{aligned}$$

where the decomposition into the product of 3 rotation matrices[2] follows from the particular rules,

$$\mathsf{R}_x(\phi) = \begin{pmatrix} \cos\frac{\phi}{2} & i\sin\frac{\phi}{2} \\ i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}, \qquad (8)$$

$$\mathsf{R}_y(\phi) = \begin{pmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}, \qquad (9)$$

$$\mathsf{R}_z(\phi) = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}. \qquad (10)$$

Convince yourself that the combined rotation (7) could also be achieved if first a rotation is made by angle γ about the z axis, then a rotation is made by angle β about the original y axis, and finally a rotation is made by angle α about the original z axis.

There is unfortunately little consistency among various authors as to the conventions used to describe rotations. I follow the notation of Barenco et al.,[3] *who appear to write eq. (7) simply as,*

$$\mathsf{R}(\alpha,\beta,\gamma) = \mathsf{R}_z(\gamma)\mathsf{R}_y(\beta)\mathsf{R}_z(\alpha). \qquad (11)$$

Occasionally one needs to remember that in eq. (11) the axes of the second and third rotations are the results of the previous rotation(s).

Note that according to eqs. (8)-(10),

$$\boldsymbol{\sigma}_x = \boldsymbol{\sigma}_1 = -i\mathsf{R}_x(180^\circ), \qquad \boldsymbol{\sigma}_y = \boldsymbol{\sigma}_2 = -i\mathsf{R}_y(180^\circ), \qquad \boldsymbol{\sigma}_z = \boldsymbol{\sigma}_3 = -i\mathsf{R}_z(180^\circ), \qquad (12)$$

and also,

$$\boldsymbol{\sigma}_x = i\mathsf{R}_x(-180^\circ), \qquad \boldsymbol{\sigma}_y = i\mathsf{R}_y(-180^\circ), \qquad \boldsymbol{\sigma}_z = i\mathsf{R}_z(-180^\circ), \qquad (13)$$

so that the Pauli spin matrices are equivalent to the formal matrices for 180° rotations only up to a phase factor i.

[2]The order of operations is that the rightmost rotation in eq. (7) is to be performed first.
[3]http://kirkmcd.princeton.edu/examples/QM/barenco_pra_52_3457_95.pdf

Show that a more systematic relation between the Pauli spin matrices and the rotation matrices is that eqs. (8)-(10) can be written as,

$$\mathsf{R}_u(\phi) = e^{i\frac{\phi}{2}\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}}, \tag{14}$$

which describes a rotation of the coordinate axes in Bloch space by angle ϕ about the $\hat{\mathbf{u}}$ axis (in a right-handed convention).

Rather than rotating the coordinate axes, we may wish to rotate vectors in Bloch space by an angle ϕ about a given axis $\hat{\mathbf{u}}$, while leaving the coordinate axes fixed. The operator,

$$\mathsf{R}_u(-\phi) = e^{-i\frac{\phi}{2}\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}} \tag{15}$$

performs this type of rotation. *With this in mind, you can finally solve the main problem posed on p. 2.*

2. **Helicity Conservation in High-Energy Electromagnetic Interactions** of point-like spin-1/2 particles.

Recalling pp. 86 and 88 of Lecture 6 of the Notes, general (spin-1/2) particle 4-spinors u for plane-wave states,

$$\psi = u\, e^{-ipx} = u\, e^{-ip_\mu x^\mu}, \tag{16}$$

with rest mass m, 3-momentum $\mathbf{p}$ and energy $E = \sqrt{p^2 + m^2}$, can be written as,

$$u = \sqrt{E+m}\begin{pmatrix} \chi \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\chi \end{pmatrix} = \begin{pmatrix} \sqrt{E+m}\,\chi \\ \frac{p}{\sqrt{E+m}}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\chi \end{pmatrix} = \begin{pmatrix} \sqrt{E+m}\,\chi \\ \sqrt{E+m}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\chi \end{pmatrix}, \tag{17}$$

where the 2-spinor χ obeys $\chi^\dagger\chi = 1$. Similarly, antiparticle 4-spinors v are associated with plane-wave states,[4,5]

$$\tilde{\psi} = v\, e^{ipx}, \tag{18}$$

(note the sign change with respect to the form (16)), that can be written as,

$$v = \sqrt{E+m}\begin{pmatrix} \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\tilde{\chi} \\ \tilde{\chi} \end{pmatrix} = \begin{pmatrix} \frac{p}{\sqrt{E+m}}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\tilde{\chi} \\ \sqrt{E+m}\,\tilde{\chi} \end{pmatrix} = \begin{pmatrix} \sqrt{E-m}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\tilde{\chi} \\ \sqrt{E+m}\,\tilde{\chi} \end{pmatrix}, \tag{19}$$

where $\tilde{\chi}$ is a 2-spinor with $\tilde{\chi}^\dagger\tilde{\chi} = 1$.

These states obey the Dirac equations $i\partial^\mu\gamma_\mu\psi = \not{p}\psi = m\psi$ and $i\partial^\mu\gamma_\mu\tilde{\psi} = -\not{p}\tilde{\psi} = m\tilde{\psi}$, which imply the 4-spinor equations $\not{p}u = mu$ and $-\not{p}v = mv$.

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[4]The antiparticle of particle a is often denoted as $\bar{a}$, but as $\bar{u}$ is the adjoint of a Dirac 4-spinor u, we write $\tilde{a}$ for the antiparticle of state a.

[5]Dirac interpreted his negative-energy solutions as related to "anti-electrons" on p. 52 of *Quantised Singularities in the Electromagnetic Field*, Proc. Roy. Soc. London A **133**, 60 (1931), http://kirkmcd.princeton.edu/examples/QED/dirac_prsla_133_60_31.pdf.
That paper is also noteworthy for relating the possible existence of a magnetic monopole of pole strength p to the electric charge e by $ep = \hbar/2$.

The positive and negative helicity spinor states for a particle with 3-momentum $\mathbf{p}$ in direction (θ, ϕ) are $\chi_+ = | + (\theta, \phi)\rangle$ and $\chi_- = | - (\theta, \phi)\rangle$, respectively, recalling eq. (6), while the helicity states of an antiparticle are $\tilde{\chi}_+ = | - (\theta, \phi)\rangle = \chi_-$ and $\tilde{\chi}_- = -| + (\theta, \phi)\rangle = -\chi_+$. In all cases, positive helicity means spin in the direction of momentum $\mathbf{p}$.

In the high-energy limit, these 4-spinors simplify to,

$$u \to \sqrt{E} \begin{pmatrix} \chi \\ \hat{\mathbf{p}} \cdot \boldsymbol{\sigma}\, \chi \end{pmatrix}, \qquad v \to \sqrt{E} \begin{pmatrix} \hat{\mathbf{p}} \cdot \boldsymbol{\sigma}\, \tilde{\chi} \\ \tilde{\chi} \end{pmatrix}, \tag{20}$$

Give explicit forms of the helicity spinors $u_+(\theta, \phi)$, $u_-(\theta, \phi)$, $v_+(\theta, \phi)$ and $v_-(\theta, \phi)$for (anti)particles moving and at angles (θ, ϕ) to the $+z$-axis, and also their simplification to $u_+(0)$, $u_-(0)$, $v_+(0)$ and $v_-(0)$ for motion along the z-axis in the high-energy limit.

If these are pointlike particles of charge e, their electromagnetic interaction is described by the 4-current $j_\mu = e\,\gamma_\mu$. Verify that the matrix elements $\langle \bar{u}_-(\theta)|\gamma_\mu|u_+(0)\rangle$ vanish for $\mu = 0, 1, 2, 3$, and similarly that $\langle \bar{v}_+(\theta)|\gamma_\mu|u_+(0)\rangle = 0$. Remember that $\bar{v} = v^\dagger \gamma_0$, *etc.*

Digression: Electric Charge Conjugation. The above claim that the antiparticle helicity 2-spinors $\tilde{\chi}_\pm$ are related to the particle helicity 2-spinors $\chi_\pm$ by $\tilde{\chi}_\pm = \pm\chi_\mp$ can be justified by considerations of a transformation, called electric charge conjugation with symbol C, between particles and their antiparticles (with respect to their electromagnetic interactions), such that $\tilde{\psi} = C\psi^\star$ is the antiparticle state of a spin-1/2 particle ψ.[6]

[6]That $\tilde{\psi} = C\psi^\star$ and not $\tilde{\psi} = C\psi$ follows from the sign change in the spacetime waveform between eqs. (16) and (18).

Charge conjugation leaves mass unchanged, such that a particle and its antiparticle have the same rest mass m. This was not initially understood by Dirac, who first speculated that the antiparticle of an electron is a proton, *A Theory of Electrons and Protons*, Proc. Roy. Soc. London A **126**, 360 (1930), http://kirkmcd.princeton.edu/examples/QED/dirac_prsla_126_360_30.pdf.
The charge-conjugation operator C was discussed (in a different representation, and not given a name) on p. 130 of W. Pauli, *Contributions mathématique à la théorie des matrices de Dirac*, Ann. Inst. H. Poincaré **6**, 109 (1936), http://kirkmcd.princeton.edu/examples/QED/pauli_aihp_6_109_36.pdf.
The term "charge conjugation" (but with the symbol L) may have been first used in H.A. Kramers, *The use of charge conjugated wavefunctions in the hole theory of the electron*, Proc. Roy. Neder. Acad. Sci. **40**, 814 (1937), http://kirkmcd.princeton.edu/examples/neutrinos/kramers_pknaw_40_814_37.pdf.
The term antimatter was introduced by Schuster in 1898, but in his vision antimatter had negative mass; *Potential Matter—A Holiday Dream*, Nature **58**, 367, 618 (1898),
http://kirkmcd.princeton.edu/examples/GR/schuster_nature_58_367_98.pdf
http://kirkmcd.princeton.edu/examples/GR/schuster_nature_58_618_98.pdf.
The present vision of antiparticles via electric charge conjugation of particles is perhaps closer to Kelvin's image method for a planar conductor, p. 288 of W. Thomson, *Effects of Electrical Influence on Internal Spherical and on Plane Conducting Surfaces*, Camb. Dublin Math. J. **4**, 276 (1849),
http://kirkmcd.princeton.edu/examples/EM/thomson_cdmj_4_276_49.pdf.

One way to do this starts with the Dirac equation for a spin-1/2 particle state ψ,[7]

$$i\partial^\mu \gamma_\mu \psi = m\psi. \tag{21}$$

We expect that the antiparticle state $\tilde{\psi}$ also satisfies the Dirac equation,

$$i\partial^\mu \gamma_\mu \tilde{\psi} = m\tilde{\psi}. \tag{22}$$

A clever step is to take the complex conjugate of eq. (21),

$$-i\partial^\mu \gamma_\mu^* \psi^* = m\psi^*. \tag{23}$$

Applying the desired charge-conjugation operator C to this, we have,

$$-i\partial^\mu C\gamma_\mu^* \psi^* = mC\psi^* = m\tilde{\psi}. \tag{24}$$

For this to be the Dirac equation (22),[8] we require that,

$$-C\gamma_\mu^* = \gamma_\mu C. \tag{25}$$

You can verify that this implies the electric-charge-conjugation matrix operator to be,[9]

$$C = i\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}. \tag{26}$$

Then, applying the electric-charge-conjugation transformation to the particle 4-spinor u of eq. (17), we obtain (on suppression of the overall factor $\sqrt{E+m}$) the antiparticle spinor,

$$\tilde{u} = i\gamma_2 \begin{pmatrix} \chi^\star \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}^\star}{E+m}\chi^\star \end{pmatrix} = \begin{pmatrix} i\sigma_2 \frac{\mathbf{p}\cdot\boldsymbol{\sigma}^\star}{E+m}\chi^\star \\ -i\sigma_2\chi^\star \end{pmatrix} = \begin{pmatrix} \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}(-i\sigma_2\chi^\star) \\ -i\sigma_2\chi^\star \end{pmatrix} = \begin{pmatrix} \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\tilde{\chi} \\ \tilde{\chi} \end{pmatrix} = v, \tag{27}$$

using that fact (verify it!) that $\sigma_2\boldsymbol{\sigma}^* = -\boldsymbol{\sigma}\sigma_2$. Hence, the antiparticle 2-spinor $\tilde{\chi}$ is related to its corresponding particle 2-spinor χ by,

$$\tilde{\chi} = -i\sigma_2\chi^\star, \qquad \chi = i\sigma_2\tilde{\chi}^*. \tag{28}$$

[7]This argument follows sec. 5.4, p. 107 of F. Halzen and A.D. Martin, *Quarks and Leptons* (Wiley, 1984), http://kirkmcd.princeton.edu/examples/EP/halzen_martin_84.pdf.

[8]For $\tilde{\psi} = v\,e^{ipx}$, eqs. (24)-(25) lead to the spinor form of the Dirac equation for antiparticles, $-\not{p}v = mv$.

[9]Warning: Many people write $C\gamma_0$ for the matrix C of eq. (26).

In particular, the helicity 2-spinors of eq. (6) transform under electric-charge conjugation as,

$$\chi_+ = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\phi/2} \\ \sin\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix} \quad \to \quad \tilde\chi_+ = -i\sigma_2 \chi_+^\star = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\phi/2} \\ \cos\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix} = \chi_-, \qquad (29)$$

$$\chi_- = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\phi/2} \\ \cos\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix} \quad \to \quad \tilde\chi_- = -i\sigma_2 \chi_-^\star = \begin{pmatrix} -\cos\frac{\theta}{2}\, e^{-i\phi/2} \\ -\sin\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix} = -\chi_+, \qquad (30)$$

as claimed above.

3. The cross section for inelastic scattering of electrons off some target can be expressed in terms of two generalized structure functions $W_{1,2}(q^2,\nu)$ where $q = p_{ei} - p_{ef}$ and $\nu = q_0 = E_i - E_f$, as on p. 131, Lecture 8 of the Notes. If the inelastic scattering is due to the interaction of the virtual photon emitted by the incident electron with a spin-1/2, charge Q, mass m constituent of the target, such that the rest of the target is a "spectator" to this interaction, then the cross section is that given on p. 99, Lecture 6 of the Notes, and we infer that,[10]

$$W_1(q*2,\nu) = \frac{-q^2}{4m^2} Q^2\, \delta\left(\nu + \frac{q^2}{2m}\right), \qquad W_2(q^2,\nu) = Q^2\, \delta\left(\nu + \frac{q^2}{2m}\right). \qquad (31)$$

An argument of Bjorken[11] is that the lab-frame energy difference between the initial and final electron can be written as,

$$E_i - E_f = \nu = q_0 = \frac{qP}{M}, \qquad (32)$$

where P is the energy-momentum 4-vector of the target (of rest mass M), which is just $P = (M,0,0,0)$ in the lab frame. Then, in a frame in which the target has very high momentum, the 4-vector p of a constituent which carries (scalar) fraction x of the target's 3-momentum can be written approximately as $p \approx xP$. A consequence of this approximation is that the constituent mass m is related by $m^2 = p^2 \approx x^2P^2 = x^2M^2$, *i.e.*, that $m \approx xM$ (as appropriate for consideration of very high-energy scattering). This permits us to rewrite eq. (31) as[12]

$$W_1 = \frac{-q^2}{4M^2x^2} Q^2\, \delta\left(\nu + \frac{q^2}{2Mx}\right), \qquad W_2 = Q^2\, \delta\left(\nu + \frac{q^2}{2Mx}\right). \qquad (33)$$

Supposing the constituents are distributed with the target (as viewed from a frame in which the target has high speed) with probability $f(x)\, dx$, give expressions for the generalized structure functions W_1 and W_2 in terms of a single variable x.

[10] C.G. Callan, Jr and D.J. Gross, *High-Energy Electroproduction and the Constitution of the Electric Current*, Phys. Rev. Lett. **22**, 156 (1969), http://kirkmcd.princeton.edu/examples/EP/callan_prl_22_156_69.pdf.

[11] J.D. Bjorken and E.A. Paschos, *Inelastic Electron-Proton and γ-Proton Scattering and the Structure of the Nucleon*, Phys. Rev. **185**, 1975 (1969), http://kirkmcd.princeton.edu/examples/EP/bjorken_pr_185_1975_69.pdf.

[12] A different version of this argument is given on p. 139, Lecture 8 of the Notes, where a Breit frame is used.

Solutions

1. **Rotation and Pauli Spin Matrices.**

   The rotations (8)-(10) are readily seen to be exponentials of the Pauli matrices,

$$\mathsf{R}_x(\phi) = \begin{pmatrix} \cos\frac{\phi}{2} & i\sin\frac{\phi}{2} \\ i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix} = \cos\frac{\phi}{2}\mathbf{I} + i\sin\frac{\phi}{2}\boldsymbol{\sigma}_x = e^{i\frac{\phi}{2}\boldsymbol{\sigma}_x}, \tag{34}$$

$$\mathsf{R}_y(\phi) = \begin{pmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix} = \cos\frac{\phi}{2}\mathbf{I} + i\sin\frac{\phi}{2}\boldsymbol{\sigma}_y = e^{i\frac{\phi}{2}\boldsymbol{\sigma}_y}, \tag{35}$$

$$\mathsf{R}_z(\phi) = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} = \cos\frac{\phi}{2}\mathbf{I} + i\sin\frac{\phi}{2}\boldsymbol{\sigma}_z = e^{i\frac{\phi}{2}\boldsymbol{\sigma}_z}, \tag{36}$$

   recalling eq. (1).

   Digression: Pauli Spin Matrices and Rotations.

   The NOT operation, $\mathsf{X} = \boldsymbol{\sigma}_x$, that "flips" a bit can be interpreted as a rotation by 180° of the Bloch-sphere state vector about the x-axis. Thus,

$$\boldsymbol{\sigma}_x \begin{pmatrix} \cos\frac{\alpha}{2} \\ e^{i\beta}\sin\frac{\alpha}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\alpha}{2} \\ e^{i\beta}\sin\frac{\alpha}{2} \end{pmatrix} = \begin{pmatrix} e^{i\beta}\sin\frac{\alpha}{2} \\ \cos\frac{\alpha}{2} \end{pmatrix}, \tag{37}$$

   while a rotation $\mathbf{R}_x(180°)$ by 180° about the x-axis in our abstract spherical coordinate system takes α to $\pi - \alpha$ and β to $-\beta$,

$$\mathbf{R}_x(180°) \begin{pmatrix} \cos\frac{\alpha}{2} \\ e^{i\beta}\sin\frac{\alpha}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi-\alpha}{2} \\ e^{-i\beta}\sin\frac{\pi-\alpha}{2} \end{pmatrix} = e^{-i\beta} \begin{pmatrix} e^{i\beta}\sin\frac{\alpha}{2} \\ \cos\frac{\alpha}{2} \end{pmatrix}. \tag{38}$$

   Since the overall phase of a state does not affect its meaning, our prescription can be considered satisfactory thus far.

   Can we interpret the operation $\boldsymbol{\sigma}_y$ as a rotation by 180° about the y-axis? On one hand,

$$\boldsymbol{\sigma}_y \begin{pmatrix} \cos\frac{\alpha}{2} \\ e^{i\beta}\sin\frac{\alpha}{2} \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\alpha}{2} \\ e^{i\beta}\sin\frac{\alpha}{2} \end{pmatrix} = \begin{pmatrix} -ie^{i\beta}\sin\frac{\alpha}{2} \\ i\cos\frac{\alpha}{2} \end{pmatrix}, \tag{39}$$

   while a rotation $\mathbf{R}_y(180°)$ by 180° about the y-axis in our abstract spherical coordinate system takes α to $\pi - \alpha$ and β to $\pi - \beta$,

$$\mathbf{R}_y(180°) \begin{pmatrix} \cos\frac{\alpha}{2} \\ e^{i\beta}\sin\frac{\alpha}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi-\alpha}{2} \\ e^{i(\pi-\beta)}\sin\frac{\pi-\alpha}{2} \end{pmatrix} = ie^{-i\beta} \begin{pmatrix} -ie^{i\beta}\sin\frac{\alpha}{2} \\ i\cos\frac{\alpha}{2} \end{pmatrix}. \tag{40}$$

   Similarly, we interpret the operation $\boldsymbol{\sigma}_z$ as a rotation by 180° about the z-axis:

$$\boldsymbol{\sigma}_z \begin{pmatrix} \cos\frac{\alpha}{2} \\ e^{i\beta}\sin\frac{\alpha}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\frac{\alpha}{2} \\ e^{i\beta}\sin\frac{\alpha}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\alpha}{2} \\ -e^{i\beta}\sin\frac{\alpha}{2} \end{pmatrix}, \tag{41}$$

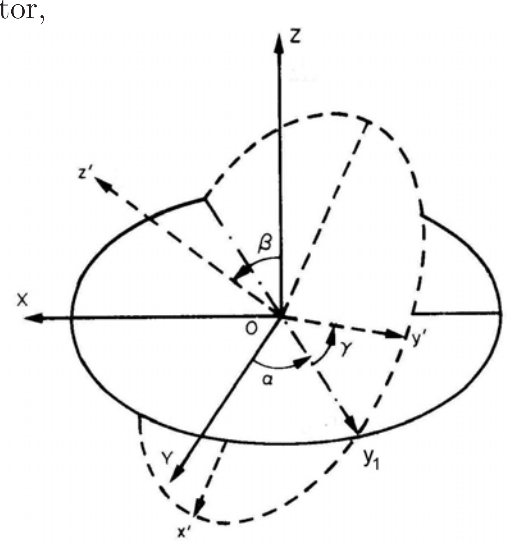
while a rotation $\mathbf{R}_z(180^\circ)$ by 180° about the z-axis in our abstract spherical coordinate system takes α to α and β to $\pi + \beta$,

$$\mathbf{R}_z(180^\circ)\begin{pmatrix} \cos\frac{\alpha}{2} \\ e^{i\beta}\sin\frac{\alpha}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\alpha}{2} \\ e^{i(\pi+\beta)}\sin\frac{\alpha}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\alpha}{2} \\ -e^{i\beta}\sin\frac{\alpha}{2} \end{pmatrix}. \tag{42}$$

Solution to the Main Problem.

We desire to rotate states in Bloch space, rather than the coordinate axes thereof, so we must heed eq. (15).

Referring to the figure on p. 3 (and reproduced below), rotation of the z-axis to the direction (θ, ϕ), keeping the x-axis in the original x-y plane, could be accomplished with rotation angles $\alpha = 0$, $\beta = \theta$, $\gamma = \phi$ in the general rotation (7). Hence, rotation of a Bloch vector along the z-axis to one along the direction (θ, ϕ) can be accomplished by the rotation operator,



$$\mathsf{R}(0, -\theta, -\phi) \quad = \quad \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\phi/2} & -\sin\frac{\theta}{2}\, e^{-i\phi/2} \\ \sin\frac{\theta}{2}\, e^{i\phi/2} & \cos\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix}. \tag{43}$$

Then,

$$|+(\theta,\phi)\rangle = \mathsf{R}(0, -\theta, -\phi)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\phi/2} \\ \sin\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix}, \tag{44}$$

$$|-(\theta,\phi)\rangle = \mathsf{R}(0, -\theta, -\phi)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\phi/2} \\ \cos\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix}, \tag{45}$$

as on p. 113, Lecture 7 of the Notes, and in agreement with eq. (6).

2. From eq. (6), Prob. 1, the helicity 2-spinors for a particle are,

$$\chi_+ = |+(\theta,\phi)\rangle = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\phi/2} \\ \sin\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix}, \qquad \chi_- = |-(\theta,\phi)\rangle = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\phi/2} \\ \cos\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix}. \quad (46)$$

The operator $\hat{\mathbf{p}}(\theta,\phi)\cdot\boldsymbol{\sigma}$ has the form,

$$\hat{\mathbf{p}}(\theta,\phi)\cdot\boldsymbol{\sigma} = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}, \qquad \text{such that} \qquad \hat{\mathbf{p}}(\theta,\phi)\cdot\boldsymbol{\sigma}\,\chi_\pm = \pm\chi_\pm. \quad (47)$$

Digression: Helicity Projection Operator for 2-Spinors. *If a general 2-spinor is written as $\chi = a_+\chi_+ + a_-\chi_-$, in terms of helicity spinors for the (θ,ϕ) direction, then $[\mathbf{I} \pm \hat{\mathbf{p}}(\theta,\phi)\cdot\boldsymbol{\sigma}]\chi = 2a_\pm\chi_\pm$. Hence,*

$$\frac{\mathbf{I} \pm \hat{\mathbf{p}}(\theta,\phi)\cdot\boldsymbol{\sigma}}{2} \quad (48)$$

are 2-spinor helicity projection operators *for the direction (θ,ϕ).*

The particle and antiparticle helicity 4-spinors $u_\pm$ and $v_\pm$ are, recalling eqs. (17)-(19), with $\tilde{\chi}_\pm = \pm\chi_\mp$, and defining $\sqrt{E_+} = \sqrt{E+m}$ and $\sqrt{E_-} = \sqrt{E-m} = p/\sqrt{E+m}$,

$$u_+(\theta,\phi) = \begin{pmatrix} \sqrt{E+m}\,\chi_+ \\ \frac{p}{\sqrt{E+m}}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\chi_+ \end{pmatrix} = \begin{pmatrix} \sqrt{E_+}\,\chi_+ \\ \sqrt{E_-}\,\chi_+ \end{pmatrix} = \begin{pmatrix} \sqrt{E_+}\,\cos\frac{\theta}{2}\, e^{-i\phi/2} \\ \sqrt{E_+}\,\sin\frac{\theta}{2}\, e^{i\phi/2} \\ \sqrt{E_-}\,\cos\frac{\theta}{2}\, e^{-i\phi/2} \\ \sqrt{E_-}\,\sin\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix}, \quad (49)$$

$$u_-(\theta,\phi) = \begin{pmatrix} \sqrt{E+m}\,\chi_- \\ \frac{p}{\sqrt{E+m}}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\chi_- \end{pmatrix} = \begin{pmatrix} \sqrt{E_+}\,\chi_- \\ -\sqrt{E_-}\,\chi_- \end{pmatrix} = \begin{pmatrix} -\sqrt{E_+}\,\sin\frac{\theta}{2}\, e^{-i\phi/2} \\ \sqrt{E_+}\,\cos\frac{\theta}{2}\, e^{i\phi/2} \\ \sqrt{E_-}\,\sin\frac{\theta}{2}\, e^{-i\phi/2} \\ -\sqrt{E_-}\,\cos\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix} \quad (50)$$

$$v_+(\theta,\phi) = \begin{pmatrix} \frac{p}{\sqrt{E+m}}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\tilde{\chi}_+ \\ \sqrt{E+m}\,\tilde{\chi}_+ \end{pmatrix} = \begin{pmatrix} -\sqrt{E_-}\,\chi_- \\ \sqrt{E_+}\,\chi_- \end{pmatrix} = \begin{pmatrix} \sqrt{E_-}\,\sin\frac{\theta}{2}\, e^{-i\phi/2} \\ -\sqrt{E_-}\,\cos\frac{\theta}{2}\, e^{i\phi/2} \\ -\sqrt{E_+}\,\sin\frac{\theta}{2}\, e^{-i\phi/2} \\ \sqrt{E_+}\,\cos\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix} \quad (51)$$

$$v_-(\theta,\phi) = \begin{pmatrix} \frac{p}{\sqrt{E+m}}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\tilde{\chi}_- \\ \sqrt{E+m}\,\tilde{\chi}_- \end{pmatrix} = \begin{pmatrix} -\sqrt{E_-}\,\chi_+ \\ -\sqrt{E_+}\,\chi_+ \end{pmatrix} = \begin{pmatrix} -\sqrt{E_-}\,\cos\frac{\theta}{2}\, e^{-i\phi/2} \\ -\sqrt{E_-}\,\sin\frac{\theta}{2}\, e^{i\phi/2} \\ -\sqrt{E_+}\,\cos\frac{\theta}{2}\, e^{-i\phi/2} \\ -\sqrt{E_+}\,\sin\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix} \quad (52)$$

Note that $v_\pm = C u_\pm^*$, *using the electric-charge-conjugation operator* $C = i\gamma_2$ *found in eq. (26).*

In case of high-speed motion ($E_+ \approx E_- \approx E$) along the $+z$-axis the 4-spinors are,

$$u_+(0) \to \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_-(0) \to \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_+(0) \to \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad v_-(0) \to \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \tag{53}$$

Recalling that,

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$
$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{54}$$

we have that,

$$\gamma_0 u_+(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \gamma_1 u_+(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix},$$
$$\gamma_2 u_+(0) = \begin{pmatrix} 0 \\ i \\ 0 \\ -i \end{pmatrix}, \quad \gamma_3 u_+(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \tag{55}$$

To evaluate matrix elements such as $\langle \bar{u}_f | \gamma_\mu | u_i \rangle$ we recall that this equals $u_f^\dagger \gamma_0 \gamma_\mu u_i$, so we multiply eq. (55) by γ_0 to obtain,

$$\gamma_0\gamma_0 u_+(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \gamma_0\gamma_1 u_+(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

$$\gamma_0\gamma_2 u_+(0) = \begin{pmatrix} 0 \\ i \\ 0 \\ i \end{pmatrix}, \quad \gamma_0\gamma_3 u_+(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{56}$$

Now, we can use eqs. (50)-(52) to see that,

$$\langle \bar{u}_-(\theta,\phi)|\gamma_\mu|u_+(0)\rangle = u_-^\dagger(\theta,\phi)\gamma_0\gamma_\mu u_+(0) = 0 = \langle \bar{v}_+(\theta,\phi)|\gamma_\mu|u_+(0)\rangle. \tag{57}$$

Hence, a high-energy pointlike spin-1/2 particle of a given helicity cannot couple a particle of the opposite helicity via the electromagnetic interaction, nor can it annhilate with an antiparticle of the same helicity. It is possible for a high-energy spin-1/2 particle of a given helicity to scatter into a particle of the same helicity, or annihilate with an antiparticle of opposite helicity, via single-photon emission.

Examples where helicity conservation in the high-energy limit is useful in providing a simplified understanding include e^+e^- annihilation into a pair of spin-0 or spin-1/2 particles, as well as elastic scattering of electrons off spin-0 and spin-1/2 particles, as discussed on p. 118 ff, Lecture 7 of the Notes.

Digression: Orthogonality. If we label the four components of a general spinor ψ as ψ_i, $i = 1, 4$, then,

$$\phi^\dagger\psi = \phi_1^*\phi_1 + \phi_2^*\phi_2 + \phi_3^*\phi_3 + \phi_4^*\phi_4, \tag{58}$$

while,

$$\bar{\phi}\psi = \phi^\dagger\gamma_0\psi = \phi_1^*\phi_1 + \phi_2^*\phi_2 - \phi_3^*\phi_3 - \phi_4^*\phi_4. \tag{59}$$

Then, from eqs. (49)-(52),

$$u_+^\dagger u_+ = u_-^\dagger u_- = v_+^\dagger v_+ = v_-^\dagger v_- = 2E, \tag{60}$$
$$u_+^\dagger u_- = u_+^\dagger v_+ = u_-^\dagger v_- = v_+^\dagger v_- = 0, \tag{61}$$
$$u_+^\dagger v_- = -2p = -u_-^\dagger v_+, \tag{62}$$

while,

$$\bar{u}_+ u_+ = \bar{u}_- u_- = \bar{v}_+ v_+ = \bar{v}_- v_- = 2m, \tag{63}$$
$$\bar{u}_+ u_- = \bar{u}_+ v_+ = \bar{u}_- v_- = \bar{v}_+ v_- = \bar{u}_+ v_- = -\bar{u}_- v_+ = 0. \tag{64}$$

That is, the u and v spinors are orthogonal with respect to the scalar product $\bar{\phi}\psi$, but not with respect to $\phi^\dagger\psi$.

Digression: Helicity Projection Operator for 4-Spinors. The generalization to 4-spinors of the 2-spinor helicity projection operators (48) is,[13]

$$\frac{1}{2}\begin{pmatrix} \mathbf{I} \pm \hat{\mathbf{p}}(\theta,\phi)\cdot\boldsymbol{\sigma} & 0 \\ 0 & \mathbf{I} \pm \hat{\mathbf{p}}(\theta,\phi)\cdot\boldsymbol{\sigma} \end{pmatrix} = \frac{1 \pm \gamma_0\,\hat{\mathbf{p}}(\theta,\phi)\cdot\boldsymbol{\gamma}\,\gamma_5}{2}, \tag{65}$$

recalling from eq. (68) of Set 3 that,

$$\gamma_0\gamma_i\gamma_5 = \begin{pmatrix} \boldsymbol{\sigma}_i & 0 \\ 0 & \boldsymbol{\sigma}_i \end{pmatrix}. \tag{66}$$

Then, recalling eqs. (49)-(52), the operator $1 + \gamma_0\,\hat{\mathbf{p}}(\theta,\phi)\cdot\boldsymbol{\gamma}\,\gamma_5/2$ projects positive helicity for particles, but negative helicity for antiparticles, while $1 - \gamma_0\,\hat{\mathbf{p}}(\theta,\phi)\cdot\boldsymbol{\gamma}\,\gamma_5/2$ projects negative helicity for particles, but positive helicity for antiparticles,

Note that $\gamma_\mu\gamma_0\hat{\mathbf{p}}\cdot\boldsymbol{\gamma}\,\gamma_5 = \gamma_0\hat{\mathbf{p}}\cdot\boldsymbol{\gamma}\,\gamma_5\gamma_\mu$, which has the consequence that the helicity 4-spinor states $\psi_\pm$ satisfy the Dirac equation, $i\partial^\mu\gamma_\mu\psi_\pm = m\,\psi_\pm$,

$$\frac{1 \pm \gamma_0\,\hat{\mathbf{p}}\cdot\boldsymbol{\gamma}\,\gamma_5}{2} i\partial^\mu\gamma_\mu\psi = i\partial^\mu\gamma_\mu\frac{1 \pm \gamma_0\,\hat{\mathbf{p}}\cdot\boldsymbol{\gamma}\,\gamma_5}{2}\psi = i\partial^\mu\gamma_\mu\psi_\pm = m\frac{1 \pm \gamma_0\,\hat{\mathbf{p}}\cdot\boldsymbol{\gamma}\,\gamma_5}{2}\psi = m\,\psi_\pm. \tag{67}$$

Digression: Chirality *vs.* **Helicity States.**

On p. 116, Lecture 7 of the Notes, it was argued that in the high-energy limit we can consider a different form of the helicity projection operator, more properly called the chirality projection operator,

$$\frac{1 \pm \gamma_5}{2} = \frac{1}{2}\begin{pmatrix} \mathbf{I} & \pm\mathbf{I} \\ \pm\mathbf{I} & \mathbf{I} \end{pmatrix}, \tag{68}$$

which leads, for momentum $\mathbf{p} = p\,\hat{\mathbf{z}}$ along the z-axis, to,

$$\frac{1+\gamma_5}{\sqrt{E+m}}u_\pm = (1+\gamma_5)\begin{pmatrix} \chi_\pm \\ \pm\frac{p}{E+m}\,\chi_\pm \end{pmatrix} = \left(1 \pm \frac{p}{E+m}\right)\begin{pmatrix} \chi_\pm \\ \chi_\pm \end{pmatrix} \tag{69}$$

$$\frac{1-\gamma_5}{\sqrt{E+m}}u_\pm = (1-\gamma_5)\begin{pmatrix} \chi_\pm \\ \pm\frac{p}{E+m}\,\chi_\pm \end{pmatrix} = \left(1 \mp \frac{p}{E+m}\right)\begin{pmatrix} \chi_\pm \\ -\chi_\pm \end{pmatrix} \tag{70}$$

$$\frac{1+\gamma_5}{\sqrt{E+m}}v_\pm = (1+\gamma_5)\begin{pmatrix} -\frac{p}{E+m}\,\chi_\mp \\ \pm\chi_\mp \end{pmatrix} = \left(1 \mp \frac{p}{E+m}\right)\begin{pmatrix} \pm\chi_\mp \\ \pm\chi_\mp \end{pmatrix} \tag{71}$$

$$\frac{1-\gamma_5}{\sqrt{E+m}}v_\pm = (1-\gamma_5)\begin{pmatrix} -\frac{p}{E+m}\,\chi_\mp \\ \pm\chi_\mp \end{pmatrix} = \left(1 \pm \frac{p}{E+m}\right)\begin{pmatrix} \mp\chi_\mp \\ \pm\chi_\mp \end{pmatrix} \tag{72}$$

[13] We now follow the usual convention in writing the unit 4×4 matrix as 1.

Only in the high-energy limit do the chirality and the helicity projection operators produce that same results.

The result of the positive-(negative-)chirality operator on a particle 4-spinor is called a righthanded-(lefthanded-)chirality spinor, and conversely for antiparticles,

$$\frac{1+\gamma_5}{2}u \equiv u_R, \qquad \frac{1-\gamma_5}{2}u \equiv u_L, \qquad \frac{1+\gamma_5}{2}v \equiv v_L, \qquad \frac{1-\gamma_5}{2}v \equiv v_R, \tag{73}$$

Then, eq. (70) reminds us the a lefthanded-chirality particle is not precisely a negative-helicity state, *etc.* That is, $u_L = (1-\gamma_5)u/2$ contains a positive-helicity component of amplitude $(E+m-p)/(E+m+p) \approx m/2E$ relative to the nominal negative-helicity component. A famous application of this in the decays $\pi \to \mu\overline{\nu}_\mu$ *vs.* $\pi \to e\overline{\nu}_e$ is discussed on p. 293, Lecture 16 of the Notes. See also Set 9, Prob. 1b.

Similarly, a righthanded-chirality antiparticle state, $(1-\gamma_5)v/2$, is nominally a positive-helicity state (with 2-spinor $\tilde{\chi}_+ = \chi_-$), but has a negative-helicity component of amplitude $\approx m/E$.

Suppose that the antiparticle of state u is $v = \tilde{u}$. We can decompose u and v into chirality states, $u = u_R + u_L$, $v = v_R + v_L$. Then, recalling the electric-charge-conjugation operator (26) and that $\gamma_2\gamma_5 = -\gamma_5\gamma_2$,

$$\tilde{u}_R = i\gamma_2 u_R^* = i\gamma_2\frac{1+\gamma_5}{2}u^* = \frac{1-\gamma_5}{2}(i\gamma_2 u^*) = \frac{1-\gamma_5}{2}\tilde{u} = \frac{1-\gamma_5}{2}v = v_R. \tag{74}$$

Similarly, $\tilde{u}_L = v_L$. Note that the antiparticle of u_R (in the sense of electric-charge conjugation) is v_R and not v_L.

A peculiarity is that right- and lefthanded-chirality states $u_{R,L}$ (and $v_{R,L}$) do not strictly satisfy the Dirac equation $i\partial^\mu\gamma_\mu u = m\,u$, but rather,

$$i\partial^\mu\gamma_\mu u_{R,L} = m\,u_{L,R}, \tag{75}$$

(and similarly $i\partial^\mu\gamma_\mu v_{R,L} = m\,v_{R,L}$), since $\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$:

$$\frac{1\pm\gamma_5}{2}i\partial^\mu\gamma_\mu u = i\partial^\mu\gamma_\mu\frac{1\mp\gamma_5}{2}u = i\partial^\mu\gamma_\mu u_{L,R} = \frac{1\pm\gamma_5}{2}mu = m\,u_{R,L}. \tag{76}$$

Digression. Helicity Conservation when Chirality Approximates Helicity.

For relativistic spin-1/2 particles, with $E \gg m$, their chilarity and helicity states are essentially identical, as noted in eqs. (69)-(72). Then, for example, a matrix element between helicity states such as eq. (57), $\langle\bar{u}_-|\gamma_\mu|u_+\rangle$ is well approximated by the matrix element of chirality states,

$$\begin{aligned}
\langle\bar{u}_-|\gamma_\mu|u_+\rangle &\approx \langle\bar{u}_L|\gamma_\mu|u_R\rangle \\
&= \frac{1}{4}\langle[(1-\gamma_5)u]^\dagger\gamma_0|\gamma_\mu|(1+\gamma_5)u\rangle = \frac{1}{4}\langle u^\dagger(1-\gamma_5)\gamma_0|\gamma_\mu|(1+\gamma_5)u\rangle \\
&= \frac{1}{4}\langle u^\dagger\gamma_0|(1+\gamma_5)\gamma_\mu(1+\gamma_5)|u\rangle = \frac{1}{4}\langle\bar{u}|\gamma_\mu(1-\gamma_5)(1+\gamma_5)|u\rangle \\
&= 0,
\end{aligned} \tag{77}$$

recalling that $\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$ and $\gamma_5^2 = 1$.

That is, only helicity-conserving matrix elements of the operator γ_μ are nonzero for relativistic spin-1/2 states.

Digression: Orthogonality of Chirality States.

For general particle and antiparticle 4-spinors,

$$u = \sqrt{E+m}\begin{pmatrix} \chi \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\chi \end{pmatrix}, \qquad v = \sqrt{E+m}\begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\tilde{\chi} \\ \tilde{\chi} \end{pmatrix}, \tag{78}$$

with 2-spinors χ and $\tilde{\chi} = -i\sigma_2\chi^*$ (from eq. (28)) that obey $\chi^\dagger\chi = 1 = \tilde{\chi}^\dagger\tilde{\chi}$, we have that $u^\dagger u = 2E = v^\dagger v$ and $\bar{u}u = 2m = -\bar{v}v$. The the right- and lefthanded 4-spinors of eqs. (73) and (78) are then,

$$u_{R,L} = \frac{\sqrt{E+m}}{2}\begin{pmatrix} \left(1 \pm \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\right)\chi \\ \left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} \pm 1\right)\chi \end{pmatrix}, \qquad v_{R,L} = \frac{\sqrt{E+m}}{2}\begin{pmatrix} \left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} \mp 1\right)\tilde{\chi} \\ \left(1 \mp \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\right)\tilde{\chi} \end{pmatrix}, \tag{79}$$

These 4-spinors have normalizations,

$$u_R^\dagger u_R = E + \chi^\dagger\boldsymbol{\sigma}\cdot\mathbf{p}\,\chi, \qquad u_L^\dagger u_L = E - \chi^\dagger\boldsymbol{\sigma}\cdot\mathbf{p}\,\chi, \tag{80}$$

$$v_R^\dagger v_R = E - \tilde{\chi}^\dagger\boldsymbol{\sigma}\cdot\mathbf{p}\,\tilde{\chi}, \qquad v_L^\dagger v_L = E + \tilde{\chi}^\dagger\boldsymbol{\sigma}\cdot\mathbf{p}\,\tilde{\chi}, \tag{81}$$

while,

$$\bar{u}_R u_R = \bar{u}_L u_L = \bar{v}_R v_R = \bar{v}_L v_L = 0. \tag{82}$$

They also satisfy the relations,

$$u_R^\dagger u_L = u_R^\dagger v_R = u_L^\dagger v_L = v_R^\dagger v_L = 0, \tag{83}$$

$$u_R^\dagger v_L = \chi^\dagger(E + \boldsymbol{\sigma}\cdot\mathbf{p})\,\tilde{\chi}, \qquad u_L^\dagger v_R = \chi^\dagger(E - \boldsymbol{\sigma}\cdot\mathbf{p})\,\tilde{\chi}, \tag{84}$$

while,

$$\bar{u}_R v_L = \bar{u}_L v_R = 0, \qquad \bar{u}_R u_L = -\bar{v}_R v_L = m, \qquad -\bar{u}_R v_R = \bar{u}_L v_L = m\,\chi^\dagger\tilde{\chi}. \tag{85}$$

Digression: Antiparticles of Chirality States. Also note that the antiparticles (in the sense of electric-charge conjugation) of the chirality states (79) are, recalling that $\sigma_2\boldsymbol{\sigma}^* = -\boldsymbol{\sigma}\sigma_2$,

$$\begin{aligned} \tilde{u}_{R,L} &= i\gamma_2 u_{R,L}^* = \frac{\sqrt{E_+}}{2}\begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} \left(1 \pm \frac{\boldsymbol{\sigma}^*\cdot\mathbf{p}}{E+m}\right)\chi^* \\ \left(\frac{\boldsymbol{\sigma}^*\cdot\mathbf{p}}{E+m} \pm 1\right)\chi^* \end{pmatrix} \\ &= \frac{\sqrt{E_+}}{2}\begin{pmatrix} \left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} \mp 1\right)(-i\sigma_2\chi^*) \\ \left(1 \mp \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\right)(-i\sigma_2\chi^*) \end{pmatrix} = v_{R,L}, \qquad \tilde{v}_{R,L} = u_{R,L}. \end{aligned} \tag{86}$$

Digression: Two-Component Theory of Massless Fermions.

Until relatively recently, experimental evidence was consistent with neutrinos being massless. The character of Dirac 4-spinors for massless spin-1/2 states was considered by Weyl,[14] who formulated a "two component" theory.

For example, the four helicity states (49)-(52) reduce to two independent state when $m = 0$, since then $v_\pm = -u_\mp$. Also, when $m = 0$ then $\frac{p}{E+m} = 1$, and eqs. (69)-(72) indicate that the helicity spinors and the chirality spinors are identical, $u_R = u_+ = -v_- = -v_L$ and $u_L = u_- = -v_+ = -v_R$.

As discussed in Lecture 16 of the Notes, in the so-called V-A theory, only lefthanded particle (righthanded antiparticle) states participate in the weak interaction. Since the neutrino has no strong or electromagnetic interaction (presuming that the neutrino has no magnetic moment as well as no electric charge), then a righthanded neutrino (lefthanded antineutrino) would have no interactions (except gravity) and could be called sterile.[15] While a massless, sterile neutrino is a somewhat trivial concept, the possibility of a sterile neutrino with mass has led to considerable discussion/controversy, despite lack of clear experimental evidence for such a particle.[16]

3. Convoluting the generalized structure functions (31) with a longitudinal-momentum distribution $f(x)$ of constituents of a target particle of mass M, we have,

$$W_2(x) = \int_0^1 f(x')\, dx'\, Q^2 \delta\left(\nu + \frac{q^2}{2Mx'}\right) = \frac{Q^2 f(x)}{-q^2/2Mx^2} = \frac{Q^2 x f(x)}{\nu}\,, \tag{87}$$

where we recall that,

$$\int f(x)\, dx\, \delta[y(x)] = \int f(x)\, \frac{dy}{y'(x)} \delta(y) = \frac{f(x)}{g'(x)}\,, \qquad \text{for} \qquad x = y^{-1}(0). \tag{88}$$

so that in the present case,

$$\nu = -\frac{q^2}{2Mx}\,. \tag{89}$$

The result (87) is usually recast as,

$$\nu W_2(x) = Q^2 x f(x) \equiv F_2(x). \tag{90}$$

[14] H. Weyl, *Elektron und Gravitation. I*, Z. Phys. **56**, 330 (1929),
http://kirkmcd.princeton.edu/examples/EP/weyl_zp_56_330_29.pdf.
Gravitation and the Electron, Proc. Nat. Acad. Sci. **15**, 323 (1929),
http://kirkmcd.princeton.edu/examples/GR/weyl_pnas_15_323_29.pdf.

[15] The notion of a sterile neutrino seems to have been introduced on p. 986 of B. Pontecorvo, *Neutrino Experiments and the Problem of Conservation of Leptonic Charge*, Sov. Phys. JETP **26**, 984 (1968),
http://kirkmcd.princeton.edu/examples/neutrinos/pontecorvo_spjetp_26_984_68.pdf.

[16] A recent experimental limit on the existence of sterile neutrinos is P. Adamson *et al.*, *Improved Constraints on Sterile Neutrino Mixing from Disappearance Searches in the MINOS, MINOS + , Daya Bay, and Bugey-3 Experiments*, http://kirkmcd.princeton.edu/examples/neutrinos/adamson_prl_125_071801_20.pdf.

Similarly,

$$W_1(x) = \int_0^1 f(x')\,dx'\,\frac{-q^2}{4M^2x'^2}Q^2\delta\left(\nu + \frac{q^2}{2Mx'}\right) = \frac{-q^2}{4M^2x'^2}\frac{Q^2f(x)}{-q^2/2Mx^2} = \frac{Q^2f(x)}{2M}\,, \quad (91)$$

which is usually recast as,

$$2MW_1(x) = Q^2f(x) \equiv 2F_1(x). \quad (92)$$