Ph 406: Elementary Particle Physics
Problem Set 9

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1. **Double-Beta Decay**

   a. In 1935, Goeppert-Mayer made a calculation[1] of the process $A \to A'ee\overline{\nu}\overline{\nu}$, so-called **2-neutrino double-beta decay**, which was first definitively observed in 1987.[2] Make a quick order-of-magnitude **estimate** (not a Fermi-theory calculation) of the lifetime of a nucleus against double-beta decay, starting from the fact that the lifetime of the neutron is 880 s. *What is the characteristic time scale/frequency for the collision of two quarks within a nucleon of radius ≈ 1 fermi? What fraction of these collisions result in a beta-decay of a neutron?*

   b. An important comment by Majorana[3] was that neutrinos might not be "Dirac" particles as considered in the Notes (Lectures 6, 7, 19, *etc.*),[4] but rather what are called the (Dirac) neutrino and antineutrino are combined in a single particle, **the** neutrino, which then is its own antiparticle (and which implies that lepton number is not conserved).

   It was realized by Furry[5] that, if neutrinos behave according to Majorana's view, there could exist the phenomenon of **neutrinoless double-beta decay**, $A \to A'ee$. Further, if the nuclear matrix elements were similar for 2-neutrino double-beta decay and neutrinoless double-beta decay, then the rate for the latter would be much larger (and the lifetime much shorter), because the 3-body phase space (Lecture 11 of the Notes) for the final state of the latter reaction is much larger than the 5-body phase space for the former.

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[1] M. Goeppert-Mayer, *Double Beta-Disintegration*, Phys. Rev. **48**, 512 (1935), `http://kirkmcd.princeton.edu/examples/EP/goeppert-mayer_pr_48_512_35.pdf`.

[2] S.R. Elliott *et al.*, *Direct Evidence for Two-Neutrino Double-Beta Decay in* ^{82}*Se*, Phys. Rev. Lett. **59**, 2020 (1987), `http://kirkmcd.princeton.edu/examples/EP/elliott_prl_59_2020_87.pdf`.

[3] E. Majorana, *Teoria simmetrica dell'elettrone e del positrone*, Nuovo Cim. **14**, 171 (1937), `http://kirkmcd.princeton.edu/examples/EP/majorana_nc_14_171_37.pdf`. `http://kirkmcd.princeton.edu/examples/EP/majorana_nc_14_171_37_english.pdf`.
F. Wilczek, *Majorana Returns*, Nature Phys. **5**, 614 (2009), `http://kirkmcd.princeton.edu/examples/neutrinos/wilczek_np_5_614_09.pdf`.

[4] Massless Dirac fermions are sometimes called Weyl fermions, and the theory of massless Dirac neutrino is often called the "two component" theory, following H. Weyl, *Elektron und Gravitation. I*, Z. Phys. **56**, 330 (1929), `http://kirkmcd.princeton.edu/examples/EP/weyl_zp_56_330_29.pdf`.
Gravitation and the Electron, Proc. Nat. Acad. Sci. **15**, 323 (1929), `http://kirkmcd.princeton.edu/examples/GR/weyl_pnas_15_323_29.pdf`.

[5] W.H. Furry, *On Transition Probabilities in Double Beta-Disintegration*, Phys. Rev. **56**, 1184 (1939), `http://kirkmcd.princeton.edu/examples/EP/furry_pr_56_1184_39.pdf`.

To date, no neutrinoless double-beta decay has been observed, with limits on the lifetime ($> 10^{25}$ yr) being longer than that for observed 2-neutrino decays.[6]

Draw a Feynman diagram for neutrinoless double-beta decay, as due to the decay of two d-quarks. Include arrows on the spin-1/2 particle lines indicating the nominal helicities expected for the V-A model of the weak interaction (with $1-\gamma_5$ coupling).[7] Then, recalling the spin-1/2 propagator (p. 28, Lecture 3 of the Notes), the matrix element includes a factor (between the final-state electron spinors),

$$(1-\gamma_5)\frac{q^\mu\gamma_\mu - m_\nu}{q^2 - m_\nu^2}(1-\gamma_5). \tag{1}$$

Show that this reduces to

$$\frac{-2m_\nu}{q^2 - m_\nu^2}(1-\gamma_5) \approx \frac{-2m_\nu}{q^2}(1-\gamma_5). \tag{2}$$

Thus, the matrix element is proportional to the mass m_ν of the neutrino, and the decay rate is proportional to the square of the mass, so the rate vanishes if the neutrino were massless.[8]

In neutrinoless double-beta decay the four momentum q^μ of the virtual neutrino is small, but we expect that $q^2 \gg m_\nu^2$ while also $q^2 \ll m_e^2$. This suggests that the rate for neutrinoless double-beta decay is suppressed relative to that for 2-neutrino double-beta decay by a factor $(m_\nu/E)^2$ where E is very roughly of order 1 keV. This suppression may be sufficient to overcompensate for the larger phase space for neutrinoless double-beta decay, and make its rate smaller than that for 2-neutrino double-beta decay.

In searches for neutrinoless double-beta decay (negative to date), a factor of two improvement in the limit on the neutrino mass requires a factor of four improvement in the limit on the lifetime, *etc.*

While Majorana neutrinos could lead to neutrinoless double-beta decay, this need not be the only explanation for the nonconservation of lepton number in the decay. If neutrinoless double-beta decay were detected, additional effort would still be required to identify its cause.

The virtual neutrino in neutrinoless double-beta decay can be a mixture of the three mass eigenstates ν_i. *This implies that the* m_ν *written above is actually* $\sum_i U_{ei}^2 m_i$, *where* $U_{\alpha i}$ *is the CKMNS mixing matrix for neutrinos (p. 356, Lecture 19 of the Notes). Hence, it may be that neutrinoless double-beta decay is further suppressed by the numerical values of the neutrino-mixing matrix.*

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[6] See, for example, R. Henning, *Current status of neutrinoless double-beta decay searches*, Rep. Phys. **1**, 29 (2016), `http://kirkmcd.princeton.edu/examples/neutrinos/henning_rp_1_29_16.pdf`.

[7] R.P. Feynman and M. Gell-Mann, *Theory of the Fermi Interaction*, Phys. Rev. **109**, 193 (1958), `http://kirkmcd.princeton.edu/examples/EP/feynman_pr_109_193_58.pdf`.

[8] This result was perhaps first obtained, by a different argument, on p. 314 of K.M. Case, *Reformulation of the Majorana Theory of the Neutrino*, Phys. Rev. **107**, 307 (1957), some 9 months before the emergence of the V-A theory, `http://kirkmcd.princeton.edu/examples/neutrinos/case_pr_107_307_57.pdf`.

2. **Helicity of Neutrinos.** (M. Goldhaber *et al.*, Phys. Rev. **109**, 1015 (1958), `http://kirkmcd.princeton.edu/examples/EP/goldhaber_pr_109_1015_58.pdf`; see also p. 1423 ff in *Alpha-, Beta-, and Gamma-Ray Spectroscopy*, ed. by K. Siegbahn.)

Suppose a spin-0 nucleus A decays via electron capture, $A + e^- \to B^\star + \nu_e$, to a spin-1 excited state $B^\star$ of nucleus B along with emission of a neutrino (Gamow-Teller transition). Then, suppose the spin-1 state $B^\star$ decays via emission of an electric-dipole photon to the spin-0 ground state of B. We wish to infer the helicity of the neutrino by observation of the angular distribution of the photon. To do this, the direction of the neutrino in the laboratory must be singled out by some feature of the apparatus. This is possible by an ingenious argument of Goldhaber *et al.*

Take the z axis along the direction of the neutrino in the decay $A \to B^\star$, such that the neutrino has $S_z^{(\nu)} = \pm\frac{1}{2}$ (with $S_z^{(\nu)} = -\frac{1}{2}$ in the $V - A$ theory of the weak interaction, which is to be confirmed). The nucleus $B^\star$ could have $S_z^{(B^\star)} = -1,\ 0$ or 1, leading to J_z for the final state of $-\frac{3}{2},\ -\frac{1}{2},\ \frac{1}{2}$ or $\frac{3}{2}$. (Recall that $L_z = 0$ for any two-body state where the z axis is along the direction of momentum in the rest frame.) But, the initial state has only spin $\frac{1}{2}$ from the electron, assuming electron capture from an S-wave orbital (K capture), so if $S_z^{(\nu)} = -\frac{1}{2}$ only $S_z^{(B^\star)} = 0$ or 1 are possible for nucleus $B^\star$, while if $S_z^{(\nu)} = \frac{1}{2}$ only $S_z^{(B^\star)} = -1$ or 0 are possible

Let θ' be the angle of emission of the photon with respect to the $-z$ axis in the rest frame of state $B^\star$. (In the lab frame, $B^\star$ moves along the $-z$ axis.) Use the spin-1 rotation matrix, `http://pdg.lbl.gov/2015/reviews/rpp2015-rev-clebsch-gordan-coefs.pdf`, to show that the angular distribution of the $E1$ photons is $\sin^2\theta'$ if $S_z^{(B^\star)} = 0$, and $(1+\cos^2\theta')/2$ when $S_z^{(B^\star)} = 1$. Remember that the photon can only have $S_{z'}^{(\gamma)} = \pm 1$ along the z' axis, which is along the photon's direction in the $B^\star$ rest frame.

In particular, show that photons emitted at $\theta' = 0$ or 180° can only have $S_z^{(\gamma)} = +1$ if $S_z^{(\nu)} = -\frac{1}{2}$, and only $S_z^{(\gamma)} = -1$ if $S_z^{(\nu)} = \frac{1}{2}$.

Let E_K be the energy of the neutrino emitted in the (K-capture) decay of A, and E_0 be the excitation energy of $B^\star$ with respect to ground state B ($M_{B^\star} = M_B + E_0 \gg E_0,\ E_K$). Deduce the energy E'_γ of the photon in the rest frame of the $B^\star$, and transform this to the lab frame, expressing E_γ as a function of θ', E_K and E_0. What are the minimum and maximum values for E_γ? You should find that the highest photon energy occurs for $\theta' = 0$, for which these photons have $S_z = 1$ (and negative helicity as these photons are moving along the $-z$ axis) if the neutrino has negative helicity; and that if the neutrino had positive helicity, the highest-energy photons would have positive helicity also.

So if we can measure the helicity of the highest-energy photons, we determine the helicity of the neutrino.

The helicity of photons can be determined by passing them through a filter consisting of magnetized iron, which attenuates photons of +1 and −1 helicity by different amounts. The reaction here is just Compton scattering of polarized electrons and photons. Because the electron has spin $\frac{1}{2}$, an electron can only absorb a photon whose spin is opposite, which flips the spin of the intermediate electron prior to the radiation (scattering) of the final photon.

We want to determine the helicity of only the highest-energy photons, so a final trick is needed. Suppose the photons from the $B^\star$ decay impinge upon other ground-state B nuclei that are at rest in the lab. Calculate the energy of the photons such that a nucleus B at rest can be excited to the level $B^\star$. The latter states decay back to the ground state B by photon emission, in effect scattering only a certain subset of the photons from the first $B^\star$ decay into the detector.

For the historical experiment, $A = \mathrm{Eu}^{152}$, and $B = \mathrm{Sm}^{152}$, for which $E_K = 840$ keV, while $E_0 = 961$ keV. Due to recoil effects, you should have found that even the highest-energy photons from the first $B^\star$ decay have insufficient energy to re-excite B nuclei at rest. However, the lifetime of the spin-1 Sm^{152} excited state was measured to be 7×10^{-14} s. Convert this to a width in eV. What fraction of the Breit-Wigner mass/energy distribution (recall p. 11, Lecture 1 of the Notes) of the short-lived $B^\star$ in the reaction $\gamma + B \to B^\star$ lies with the energy distribution of the photons from the first $B^\star \to B + \gamma$ decay (assuming that distribution to be flat between its min and max energies)?

If the lifetime of the $B^\star$ level had been too long, the overlap would be too small for the experiment to work. Also, if the lifetime were long, the $B^\star$ atom might have collided with another atom and changed its momentum prior to the photon decay. Then, the correlation between the decay-photon helicity and the neutrino helicity would have been lost.

So it's a small miracle that any system exists in nature which permits this measurement!

Helicity of Neutrinos*

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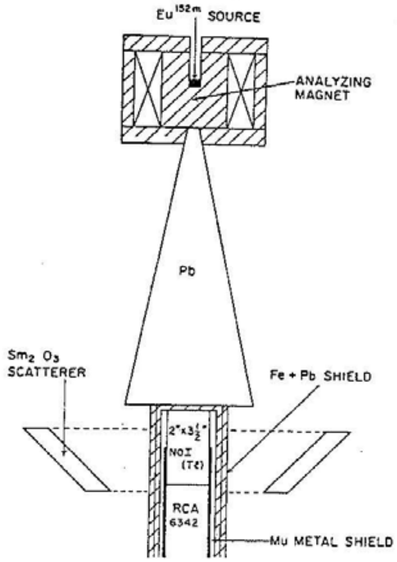


Fig. 6. Schematical arrangement of neutrino helicity experiment. (From Goldhaber *et al.*[1])

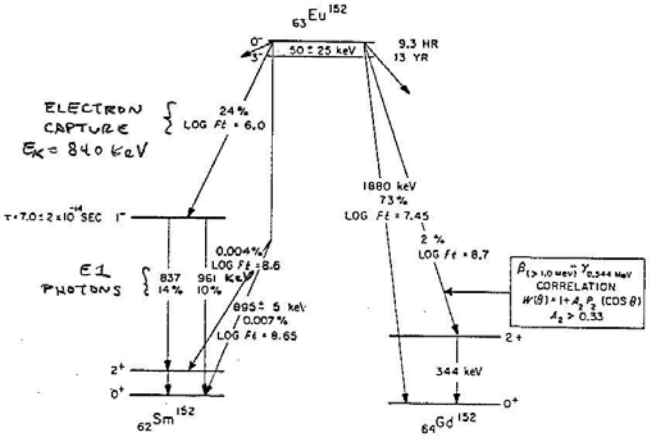


Fig. 4. Partial decay scheme of Eu^{152m}

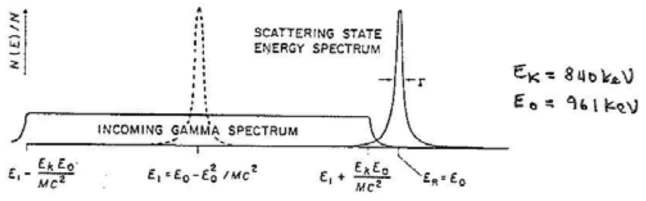


Fig. 5. Schematic diagram of incoming photon spectrum and resonance level width. (Not to scale)

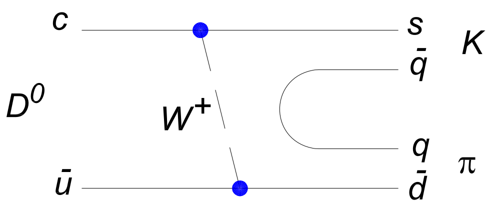
3. **Spurions**

   The spurion concept[9] is that in weak decays of an s quark the hadronic isospin character of the final state can be predicted by supposing the s quark absorbs a fictitious particle, the spurion, that is an isospin state $|I = \frac{1}{2}, I_3 = -\frac{1}{2}\rangle$ (and that an $\overline{s}$ quark absorbs an antispurion that is an isospin state $|I = \frac{1}{2}, I_3 = \frac{1}{2}\rangle$). Use this model to predict

$$\frac{\Gamma_{\Xi^0\to\Lambda\pi^0}}{\Gamma_{\Xi^-\to\Lambda\pi^-}}, \qquad \text{and} \qquad \frac{\Gamma_{\Omega^-\to\Xi^0\pi^-}}{\Gamma_{\Omega^-\to\Xi^-\pi^0}}. \tag{3}$$

   Compare with data reported at `http://pdg.lbl.gov/2013/tables/contents_tables_baryons.html`.

4. The charmed meson D^0 can decay to $K\pi$ via the Cabibbo-favored W-exchange diagram (with gluons not shown),



   If this were the only possible diagram, predict the ratio of branching ratios:

$$\frac{\Gamma(D^0 \to K^-\pi^+)}{\Gamma(D^0 \to \overline{K}^0\pi^0)}. \tag{4}$$

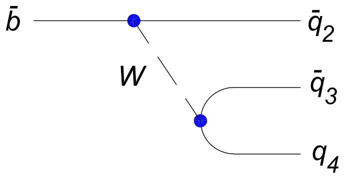
   Draw any other Cabibbo-favored diagrams for these decays.

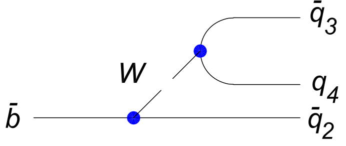
   Assuming that the decay $D^0 \to K^+\pi^-$ proceeds via a diagram similar to the above, predict the ratio of branching ratios:

$$\frac{\Gamma(D^0 \to K^+\pi^-)}{\Gamma(D^0 \to K^-\pi^-)}. \tag{5}$$

   Compare with data reported at `http://pdg.lbl.gov/2013/tables/rpp2013-tab-mesons-charm.pdf`.

5. There are four spin-0 mesons that contain one bottom quark: $B_u^+ = u\bar{b}$, $B_d^0 = d\bar{b}$, $B_s^0 = s\bar{b}$, and $B_c^+ = c\bar{b}$. These decay via the weak interaction by two graphs with roughly equal strength, as sketched below (the "spectator" model):





   Here we consider only nonleptonic final states. Suppose the four final-state quarks form exactly two mesons (as happens a few percent of the time). List the two dominant two-body decays for each of the four bottom mesons.

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[9] Attributed to G. Wentzel, *Heavy-Meson Decays and the Selection Rule* $\Delta I \le 1/2$, Phys. Rev. **101**, 1215 (1956), `http://kirkmcd.princeton.edu/examples/EP/wentzel_pr_101_1215_56.pdf`.

A complication arises for the B_c meson. The charm quark has a slightly shorter lifetime than the bottom quark. Hence, there are two more prominent two-body decays of the B_c involving $c \to Wq$ rather than $b \to Wq$ transitions. List these.

According to the measured values of the C-K-M matrix elements

$$\frac{V_{ub}}{V_{cb}} \approx \frac{V_{us}}{V_{ud}} \approx \frac{V_{cd}}{V_{cs}} \approx \lambda = \text{Cabibbo angle.}$$

List (or indicate on diagrams) the prominent two-body nonleptonic decays of the four bottom mesons that are suppressed by one power of λ in the matrix element (and hence by $\lambda^2 \approx 1/25$ in rate).

Note that $D^+ = c\bar{d}$, $D^0 = c\bar{u}$, and $D_s^0 = c\bar{s}$. If a meson is produced from, say, a $d\bar{d}$ state it could be a π^0, η, ρ^0, or ω^0. Here it is sufficient to list only the ρ^0 as empirically ρ produced seems favored...

Compare with data reported at `http://pdg.lbl.gov/2013/tables/rpp2013-tab-mesons-bottom.pdf`.

6. Both the $B_d^0 = d\bar{b}$ and $\bar{B}_d^0 = \bar{d}b$ can decay to common final states, such as $J/\psi K_S^0$ as you found in Prob. 5. Hence, there are transitions between B^0 and $\bar{B}^0$ and so these are not the states of definite mass and lifetime, which latter states can be written as

$$B_\pm^0 = pB^0 \pm q\overline{B}^0, \qquad \text{where} \qquad |p|^2 + |q|^2 = 1. \tag{6}$$

Taking into account the weak interactions, one writes the 2×2 Hamiltonian (in the $|B^0\rangle$-$|\overline{B}^0\rangle$ basis) as

$$\mathsf{H} = \mathsf{M} - \frac{i}{2}\mathsf{\Gamma}, \tag{7}$$

where the mass matrix M and the decay matrix $\mathsf{\Gamma}$ are Hermitian. (Since neutral B's decay, H itself is not Hermitian.) CPT invariance implies that the diagonal components of H are equal, and if CP is conserved M and $\mathsf{\Gamma}$ are real. Allowing for the possibility of CP violation, the Hamiltonian can be written as

$$\mathsf{H} = \begin{bmatrix} m & M_{12} \\ M_{12}^* & m \end{bmatrix} - \frac{i}{2}\begin{bmatrix} \gamma & \Gamma_{12} \\ \Gamma_{12}^* & \gamma \end{bmatrix}. \tag{8}$$

Unlike the eigenstates K_L^0 and K_S^0 of the neutral K system, the eigenstates $B_\pm^0$ of the neutral B system have roughly equal lifetimes, such that we can approximate matrix element Γ_{12} as zero.

Show that this implies that q/p is a pure phase (with magnitude 1), so that

$$B_\pm^0 \approx e^{i\phi_\pm}\frac{B^0 \pm \overline{B}^0}{\sqrt{2}}. \tag{9}$$

The time evolution of the eigenstates can then be written as $|B_\pm\rangle \to e^{-\Gamma t/2}e^{im_\pm t}|B_\pm\rangle$. Show that the time evolution of initially pure B^0 and $\bar{B}^0$ states can be written as

$$|B^0\rangle \to F_c(t)|B^0\rangle + i\frac{q}{p}F_s(t)|\overline{B}^0\rangle, \qquad |\overline{B}^0\rangle \to i\frac{p}{q}F_s(t)|B^0\rangle + F_c(t)|\overline{B}^0\rangle, \tag{10}$$

where F_c and F_s are proportional to the cosine and sine of $\Delta m\, t/2$, respectively, with $\Delta m \equiv m_+ - m_-$.

If a B^0-$\overline{B}^0$ pair is produced at an electron-positron collider via the reaction $e^+e^- \to B^0\overline{B}^0$, it is produced in the **entangled**[10] state $(|B_1^0\rangle|\overline{B}_2^0\rangle - |\overline{B}_1^0\rangle|B_2^0\rangle)/\sqrt{2}$ with negative charge conjugation. Later, B_1 decays at time t_1 and B_2 decays at time t_2 (where t_1 can be larger than t_2 if we suppose that B_1 travels to the "left" and B_2 travels to the "right"). Consider the time evolution of the entangled state from (0,0) to (t_1, t_2) to show if the two $B's$ happened to decay at the same time $t_1 = t_2$ then one must be a $\overline{B}^0$ and the other a B^0 (although if $t_1 \neq t_2$ they have nonzero amplitudes to be both B^0's or both $\overline{B}^0$'s).[11]

Such persistence of the initial quantum correlation for spacelike-separated events puzzled Einstein, Podolsky and Rosen,[12] and remains ever impressive.[13]

A subtle consequence of this "EPR" quantum correlation is that if the lab frame of the e^+e^- is the same as the center-of-mass frame, as typically the case, CP violation could not be detected in certain decay modes. To make such decay modes useful for the study of CP violation, so-called asymmetric B-factories were built in which the center of mass is in motion in the lab frame.

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[10] The term **entangled** was introduced in a quantum context by E. Schrödinger (1935) in sec. 14 of his famous "cat" paper, http://kirkmcd.princeton.edu/examples/QM/schroedinger_cat.pdf. Schrödinger lived with two women, one of whom was his legal wife. The concept of quantum-mechanical entanglement first appeared in J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, 1932), *Mathematical Foundations of Quantum Mechanics*, (Princeton U. Press, 1955). For commentary by the author, see Prob. 5 of http://kirkmcd.princeton.edu/examples/ph410problems.pdf.

[11] This effect also occurs for K^0-$\overline{K}^0$ produced in the reaction $e^+e^- \to \phi \to K^0\overline{K}^0$, as discussed by H.J. Lipkin, *CP Violation and Coherent Decays of Kaon Pairs*, Phys. Rev. **176**, 1715 (1968), http://kirkmcd.princeton.edu/examples/EP/lipkin_pr_176_1715_68.pdf.

[12] A. Einstein, B. Podolsky and N. Rosen, *Can Quantum Mechanical Description of Physical Reality Be Considered Complete?* Phys. Rev. **47**, 777 (1935), http://kirkmcd.princeton.edu/examples/QM/einstein_pr_47_777_35.pdf.

[13] Some people say things like the decay of one B instantaneously forces the "other" B to be its antiparticle, and the "other" B then subsequently evolves according to the single-B prescription (10). This is **not** a good way to think about the nature of quantum correlations in entangled states.