Ph 206 Lecture 13

WAVES IN CONDUCTING MEDIA

If the conductivity σ of the medium is non-zero then energy will be lost to the Joule heating caused by the induced currents ($\bar{J} = \sigma \bar{E}$). This energy must come from the wave — so the wave dies out inside a conducting medium. — 'Conductors' do not conduct electromagnetic waves!

We begin to study this effect for a plane wave in the conductor, say

$$\bar{E} = \bar{E}_0 \, e^{i(kz - \omega t)}$$

Plugging this into (p. 132)

$$\nabla^2 \bar{E} = \frac{\epsilon\mu}{c^2} \frac{\partial^2 \bar{E}}{\partial t^2} + \frac{4\pi\mu\sigma}{c^2} \frac{\partial \bar{E}}{\partial t}$$

we find

$$k^2 = \epsilon\mu \frac{\omega^2}{c^2} + \frac{4\pi i \mu\sigma\omega}{c^2}$$

$$= \epsilon\mu \frac{\omega^2}{c^2} \left(1 + \frac{4\pi i \sigma}{\epsilon\omega} \right)$$

so

$$k = \sqrt{\epsilon\mu} \frac{\omega}{c} \left\{ \sqrt{\frac{\sqrt{1 + \left(\frac{4\pi\sigma}{\epsilon\omega}\right)^2} + 1}{2}} + i \sqrt{\frac{\sqrt{1 + \left(\frac{4\pi\sigma}{\epsilon\omega}\right)^2} - 1}{2}} \right\}$$

If we write $k = \alpha + i\beta$ with α, β real

then $\bar{E} = E_0 \, e^{-\beta z} \, e^{i(\alpha z - \omega t)}$

So indeed the wave dies out as it moves thru the medium

In good conductors (large σ) $\beta \sim \sqrt{\epsilon\mu} \frac{\omega}{c} \sqrt{\frac{2\pi\sigma}{\epsilon\omega}} = \frac{\sqrt{2\pi\sigma\mu\omega}}{c}$

So the wave dies out like $e^{-\beta z} = e^{-\frac{z}{d}}$

where $d = \frac{c}{\sqrt{2\pi\sigma\mu\omega}} \equiv$ <u>skin depth</u> or penetration depth

High frequencies waves can exist only near the surface of conductors.

In the wave equation above, we found the ratio of the conduction-current term $\left(\sim \frac{\partial \bar{E}}{\partial t}\right)$ to the displacement-current term $\left(\sim \frac{\partial^2 \bar{E}}{\partial t^2}\right)$ to be $\frac{4\pi\sigma}{\epsilon\omega}$

In Lecture 7 we identified $\frac{\epsilon}{4\pi\sigma}$ as the characteristic time for motion of charges inside a conductor so as to cancel any applied field. We called this the 'relaxation time'.

THUS $\frac{4\pi\sigma}{\epsilon\omega} \sim \frac{\text{PERIOD OF WAVE}}{\text{RELAXATION TIME OF CONDUCTOR}}$

IF THIS QUANTITY IS LARGE, THEN CHARGES HAVE ENOUGH TIME TO MOVE SO AS TO CANCEL THE WAVE DURING THE FIRST PERIOD – AND THE WAVE CANNOT PENETRATE INTO THE CONDUCTOR.

IF THIS QUANTITY IS SMALL THEN THE WAVE DIES AWAY SLOWLY OVER MANY PERIODS AND WE CAN SPEAK OF WAVE PROPAGATION INSIDE THE CONDUCTOR.

THE QUANTITY $\frac{4\pi\sigma}{\epsilon\omega}$ MAY BE INTERPRETED YET ANOTHER WAY. THE VELOCITY OF THE WAVE IS RELATED TO ITS FREQUENCY BY

$$v = f\lambda = \omega \lambda\!\!\!- = \frac{\omega}{k} \qquad \text{WHERE } \lambda\!\!\!- = \frac{1}{k} = \frac{\lambda}{2\pi}$$

THEN $\frac{4\pi\sigma}{\epsilon\omega} = \frac{4\pi\sigma\lambda\!\!\!-}{\epsilon v} = \frac{4\pi}{c}\sigma\lambda\!\!\!-\sqrt{\frac{\mu}{\epsilon}}$

NOW $\sigma\lambda\!\!\!- = \frac{\sigma\lambda\!\!\!-^2}{\lambda\!\!\!-} = \frac{1}{\text{RESISTANCE OF A CUBE OF EDGE } \lambda\!\!\!-}$

THE QUANTITY $\frac{4\pi}{c}$ HAS THE DIMENSIONS OF RESISTANCE IN C.G.S UNITS

TO CONVERT TO ORDINARY OHMS, MULTIPLY BY 9×10^{11} (BECKER p 431)

SO OHMS $= \frac{36\pi \times 10^{11}}{3 \times 10^{10}} = 120\pi \sim \underline{377 \text{ OHMS}}$

THIS QUANTITY IS SOMETIMES CALLED THE <u>RESISTANCE OF THE VACUUM</u> AND WILL APPEAR AGAIN.

THEN $\frac{4\pi\sigma}{\epsilon\omega} \sim \frac{\text{RESISTANCE OF VACUUM}}{\text{RESISTANCE OF A CUBE OF EDGE } \lambda\!\!\!- \text{ OF THE MEDIUM}}$

AGAIN, IF THIS RATIO IS LARGE THE WAVE WILL NOT PROPAGATE BUT IS 'SHORT CIRCUITED' BY THE GOOD CONDUCTOR ....

For GOOD CONDUCTORS, $\frac{4\pi\sigma}{\epsilon\omega} \gg 1$ AND $k \to \sqrt{\epsilon\mu}\,\frac{\omega}{c}\sqrt{\frac{2\pi\sigma}{\epsilon\omega}}\,(1+i)$

AGAIN WE INTRODUCE THE SKIN DEPTH $d = \frac{c}{\sqrt{2\pi\sigma\omega\mu}} = \frac{c}{\omega}\sqrt{\frac{\omega}{2\pi\sigma\mu}} \ll \frac{c}{\omega}$

SO $k \to \frac{1}{d}(1+i)$

AND $\bar{E} = \bar{E}_0\, e^{-\frac{z}{d}}\, e^{i\left(\frac{z}{d}-\omega t\right)}$

DIES OUT

WAVE VELOCITY $= \omega d = c\sqrt{\frac{\omega}{2\pi\sigma\mu}} \ll c$

$\Rightarrow$ 'INDEX OF REFRACTION' $= \sqrt{\frac{2\pi\sigma\mu}{\omega}} \gg 1$

WE CAN OBTAIN THE MAGNETIC FIELD FROM $\bar{\nabla}\times\bar{E} = -\frac{1}{c}\frac{\partial\bar{B}}{\partial t} = -\frac{\mu}{c}\frac{\partial\bar{H}}{\partial t}$

OR $i\bar{k}\times\bar{E} = i\,\frac{\mu\omega}{c}\bar{H}$. So WITH $\bar{k} = \frac{\hat{k}}{d}(1+i)$

$$\bar{H} = \frac{c}{\mu\omega d}(1+i)\,\hat{k}\times\bar{E} = \sqrt{\frac{2\pi\sigma}{\mu\omega}}\,(1+i)\,\hat{k}\times\bar{E} \quad\Rightarrow\quad |\bar{H}| \gg |\bar{E}|$$

INSIDE THE CONDUCTOR WE STILL HAVE $\bar{E} \perp$ TO $\bar{H}$, AND BOTH $\perp$ TO $\hat{k}$, THE DIRECTION OF WAVE PROPAGATION. BUT THE FACTOR $1+i$ MEANS THAT $\bar{E}$ AND $\bar{H}$ ARE OUT OF PHASE – THEY DO NOT BOTH REACH THEIR MAXIMA AT THE SAME PLACE AND TIME. THIS IS UNLIKE THE CASE IN NON-CONDUCTORS WHERE $\bar{E}$ AND $\bar{H}$ ARE ALWAYS IN PHASE. (REMEMBER THAT $\bar{E}$ CAN BE BROKEN INTO 2 COMPONENTS SAY $\bar{E}_x$ AND $\bar{E}_y$ IF $\hat{k}=\hat{z}$, AND THESE 2 COMPONENTS NEED NOT BE IN PHASE...)

ANOTHER STRIKING FEATURE CONCERNS THE ENERGY DENSITY.

RECALL $U_E = \frac{\epsilon E^2}{8\pi}$ WHILE $U_H = \frac{\mu H^2}{8\pi}$

$$= \frac{\mu}{8\pi}\,\frac{4\pi\sigma}{\mu\omega}\,E^2 = \frac{\epsilon E^2}{8\pi}\cdot\frac{4\pi\sigma}{\epsilon\omega}$$

SO $\frac{U_H}{U_E} = \frac{4\pi\sigma}{\epsilon\omega} \gg 1$. THE ELECTRIC FIELD NEARLY VANISHES INSIDE THE CONDUCTOR — WHICH CANNOT SUPPORT A STATIC E FIELD AT ALL.

THE MISSING ENERGY HAS 'DISAPPEARED' INTO JOULE HEATING.

DRUDE MODEL OF CONDUCTORS

WE HAVE SEEN ABOVE HOW NEAR RESONANCE DIELECTRICS ARE VERY MUCH LIKE CONDUCTORS. THIS INSIGHT IS CARRIED FURTHER BY ARGUMENTS ATTRIBUTED TO DRUDE.

I.e. HAVE ABSORPTIVE PART

IN CONDUCTORS SOME ELECTRONS ARE FREE TO MOVE WITH NO RESTORING FORCE. THIS IS EQUIVALENT TO A SPRING WITH NATURAL FREQUENCY $\omega_0 = \sqrt{K/M} = 0$.

IN REAL CONDUCTORS THERE STILL IS DAMPING – ENERGY TRANSFER DUE TO COLLISIONS. THE DAMPING CONSTANT γ_0 HAS DIMENSIONS 1/TIME. IN CONDUCTORS WE USUALLY WRITE

$$\gamma_0 = \frac{1}{\tau} \qquad \tau = \text{MEAN TIME BETWEEN COLLISIONS.}$$

FOR FREQUENCIES $\omega \ll \omega_i$ OF ANY SPECTRAL LINES, THE DIELECTRIC CONSTANT OF THE CONDUCTOR IS

$$\epsilon = 1 + \frac{4\pi N e^2 f_0/m}{-\omega^2 - i\gamma_0\omega} = 1 + i\,\frac{4\pi N e^2 f_0/m}{\omega(\gamma_0 + i\omega)} = 1 + i\,\frac{4\pi N f_0 e^2 \tau}{m\omega(1 - i\omega\tau)},$$

WHERE f_0 = FRACTION OF ELECTRONS WHICH ARE FREE.

RECALL FROM LECTURE 7 THAT THE D.C. CONDUCTIVITY COULD BE EXPRESSED AS

$$\sigma_0 = \frac{N f_0 e^2 \tau}{m},$$

SO

$$\epsilon = 1 + i\,\frac{4\pi\sigma_0}{\omega(1 - i\omega\tau)}.$$

WE CAN RELATE THIS TO THE TIME-DEPENDENT CONDUCTIVITY BY RECALLING THE DISPERSION RELATION FROM p 133, LECTURE 11

$$K^2 = \epsilon\frac{\omega^2}{c^2}\left(1 + i\,\frac{4\pi\sigma}{\epsilon\omega}\right)$$

FOR A CONDUCTOR WITH $\epsilon = 1$, $K^2 = \frac{\omega^2}{c^2}\left(1 + i\frac{4\pi\sigma}{\omega}\right) = \epsilon_{EFF}\frac{\omega^2}{c^2}$.

CLEARLY WE IDENTIFY $\underline{\sigma = \frac{\sigma_0}{1 - i\omega\tau}}$.

[$E = E_0 e^{-i\omega t} = E_0\cos\omega t - i E_0 \sin\omega t$

$J = \mathrm{Re}(\sigma E) = \mathrm{Re}\,\sigma\,\mathrm{Re}\,E - \mathrm{Im}\,\sigma\,\mathrm{Im}\,E$

$= \underbrace{\mathrm{Re}\,\sigma\, E_0\cos\omega t}_{\text{IN PHASE}} + \underbrace{\mathrm{Im}\,\sigma\, E_0 \sin\omega t}_{90^\circ \text{ OUT OF PHASE}}$]

[THIS COULD ALSO BE SEEN FROM OUR SOLUTION FOR THE ELECTRON'S MOTION:

$$\bar{x} = \frac{e\bar{E}}{m(-\omega^2 - i\gamma\omega)} \Rightarrow \dot{\bar{x}} = \frac{-i\omega e\bar{E}}{-m\omega(\omega + i\gamma)} = \frac{\tau e\bar{E}}{m(1 - i\omega\tau)}$$

SO

$$\bar{J} = N f_0 e\dot{\bar{x}} = \frac{N f_0 e^2\tau\bar{E}}{m(1 - i\omega\tau)} = \sigma\bar{E}$$]

THE HIGH FREQUENCY LIMIT

SUPPOSE THAT THE WAVE FREQUENCY IS MUCH GREATER THAN ANY NATURAL FREQUENCY (OR COLLISION FREQUENCY) OF THE MEDIUM:

$$\omega \gg \omega_i \qquad \omega \gg \gamma_i$$

THEN

$$\epsilon = 1 - \frac{4\pi N e^2}{m\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} \qquad \left(\sum_i f_i = 1\right)$$

WHERE $\omega_p^2 = \frac{4\pi N e^2}{m}$ ≡ <u>PLASMA FREQUENCY</u>

THEN FOR $\omega > \omega_p$, $\epsilon > 0 \Rightarrow n = \sqrt{\epsilon}$ IS REAL

AND THE WAVES PROPAGATE FREELY! IF THE WAVE FREQUENCY IS HIGH ENOUGH THE MEDIUM CANNOT RESPOND QUICKLY ENOUGH TO CANCEL THE WAVE. EVEN METALS ARE TRANSPARENT TO VERY HIGH FREQUENCY WAVES. FOR COPPER, $\omega_p \sim 10^{16}/\text{sec}$ COMPARED TO $\omega \sim 10^{13}/\text{sec}$ FOR VISIBLE LIGHT. (UNFORTUNATELY FOR MAXWELL, BY THE TIME WE HAVE 'LIGHT' OF FREQUENCY 1000 TIMES THAT OF THE VISIBLE, WE HAVE X-RAYS, AND QUANTUM PROCESSES CAN'T BE IGNORED....)

OF COURSE, FOR $\omega < \omega_p$ THE WAVES ARE DAMPED, SINCE $\epsilon < 0$ $\Rightarrow n$ IS PURE IMAGINARY.

$\omega > \omega_p$:

$v_p = \frac{\omega}{k} = \frac{c}{\sqrt{\epsilon}} > 1$

v_g ?

$k^2 = \frac{\epsilon\omega^2}{c^2} = \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2}$

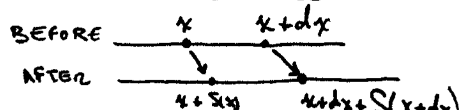
$\omega = \sqrt{c^2k^2 + \omega_p^2}$

$v_g = \frac{d\omega}{dk} = \frac{c^2 k}{\omega} = \frac{c^2}{v_p} < c$

THE NAME PLASMA <u>FREQUENCY</u> FOR THE QUANTITY $\frac{4\pi N e^2}{M}$ IS NOT ARBITRARY. IT IS THE NATURAL FREQUENCY OF A KIND OF MOTION OF CHARGES WITHIN A CONDUCTOR WHICH WE HAVE NOT YET CONSIDERED. NAMELY, THERE IS A SELF SUSTAINING OSCILLATORY MOTION POSSIBLE FOR THE FREE CHARGES.

WE CONSIDER A 1-DIMENSIONAL MOTION. THE POSITIVE IONS REMAIN FIXED, BUT THE ELECTRONS AT x ARE DISPLACED BY $S(x)$ IN THE x DIRECTION.

THE ELECTRONS INITIALLY IN A REGION dx ARE NOW BETWEEN $x + dx + S(x+dx)$ AND $x + S(x)$ — A REGION $\left(1 + \frac{dS}{dx}\right) \cdot dx$ LONG.



THEN CHARGE IN THIS REGION IS NOW $\underbrace{Ne\left(1 + \frac{dS}{dx}\right) dx}_{\text{IONS}} - \underbrace{Ne\,dx}_{\text{ELECTRONS}}$

WHERE N = # OF IONS OR ELECTRONS / VOLUME INITIALLY.

SO $\rho = Ne\frac{dS}{dx} \neq 0$ ($e > 0$ HERE)

THIS NON-ZERO CHARGE DENSITY CREATES AN ELECTRIC FIELD WHICH OBEYS $\bar{\nabla} \cdot \bar{E} = 4\pi\rho$. $\bar{E} = E\hat{x}$ IN OUR CASE, SO $\frac{dE}{dx} = 4\pi\rho$

$\Rightarrow E = 4\pi N e S(x)$

IN TURN THIS FIELD PUSHES ON THE ELECTRONS SO $M\ddot{S} = -eE = -4\pi N e^2 S$

HENCE OSCILLATORY MOTION IS POSSIBLE, WITH FREQUENCY $\underline{\omega_p^2 = \frac{4\pi N e^2}{M}}$

WHILE IT MAY TAKE SOME OUTSIDE DISTURBANCE TO SET UP THE OSCILLATION — IT IS SELF SUSTAINING (<u>IF</u> WE IGNORE COLLISIONS).

More on Relaxation of Charges & Fields in a Conductor

On p.77 we argued that a free charge distribution inside a conductor vanishes with time according to

$$\rho_{FREE} = \rho_0 \, e^{-\frac{4\pi\sigma}{\epsilon}t}$$

However, the characteristic time $\frac{\epsilon}{4\pi\sigma}$ is of order 10^{-18} sec.

We re-examine this phenomenon in view of our understanding of high-frequency behavior in conductors.

For the revised analysis we ignore possible atomic resonances (at frequencies ω_i) but take into account the plasma frequency $\omega_p^2 = \frac{4\pi N e^2}{m}$ and the characteristic collision time τ that leads to the resistance of the conductor.

The equation of motion of a charge in the conductor is then

$$\dot{\bar{v}} + \frac{\bar{v}}{\tau} = \frac{e\bar{E}}{m} \qquad \text{(compare p. 135)}$$

We relate this to the current density via $\bar{J} = Ne\bar{v}$ to find

$$\frac{\partial \bar{J}}{\partial t} + \frac{\bar{J}}{\tau} = \frac{Ne^2}{m}\bar{E}$$

We can apply the continuity equation: $\bar{\nabla}\cdot\bar{J} = -\frac{\partial\rho}{\partial t}$ and Maxwell: $\bar{\nabla}\cdot\bar{E} = 4\pi\rho$

to find $$\frac{\partial^2\rho}{\partial t^2} + \frac{1}{\tau}\frac{\partial\rho}{\partial t} + \omega_p^2\rho = 0.$$

As before we consider solutions $\rho(t) = \rho_0 \, e^{-t/t_0}$

Plugging in, we find $$\frac{1}{t_0} = \frac{1}{2\tau} \pm i\sqrt{\omega_p^2 - \left(\frac{1}{2\tau}\right)^2}$$

The charge distribution dies out as a damped oscillation in characteristic time 2τ; the oscillation frequency is $\omega = \sqrt{\omega_p^2 - \left(\frac{1}{2\tau}\right)^2}$

Recall that conductivity σ is related to τ by $\sigma = \frac{Ne^2\tau}{m}$

For copper, $\sigma \sim 5\times10^{17}$ cgs, $N \sim 9\times10^{22}/\text{cm}^3$ $\Rightarrow \begin{cases} \tau \sim 2\times10^{-14}\,\text{s} \\ \omega_p \sim 10^{16}\,\text{Hz} \end{cases}$

Note: if $\omega_p < \frac{1}{2\tau}$ there is no oscillation. If $\omega_p \ll \frac{1}{2\tau}$, $\frac{1}{t_0} = \frac{1}{\tau}$, $\tau\omega_p^2 = 4\pi\sigma$ as in the discussion of p.77 but this is the case of a poor conductor.

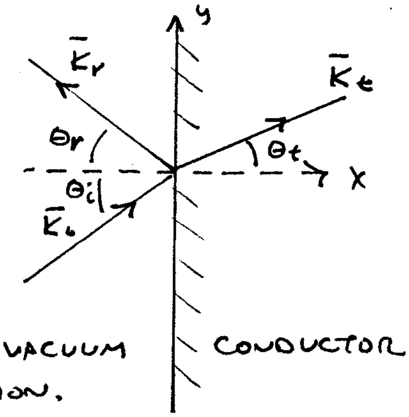
Ph 206 LECTURE 13

PLANE WAVES INCIDENT ON A CONDUCTING SURFACE

WHAT HAPPENS WHEN WE SHINE A WAVE ONTO A CONDUCTOR FROM OUTSIDE THE CONDUCTOR? CERTAINLY THE WAVE CANNOT PENETRATE VERY FAR INTO THE CONDUCTOR, DUE TO THE 'SKIN DEPTH' ATTENUATION FACTOR. BUT THE WAVE DOESN'T JUST VANISH — IT IS LARGLEY REFLECTED - WITH SOME LOSS DUE TO JOULE HEATING.

EXCEPT FOR THE ENERGY LOSS, THIS IS SIMILAR TO THE CASE OF TOTAL INTERNAL REFLECTION INSIDE A DIELECTRIC MEDIUM

SINCE, FOR EXAMPLE, $\bar{E}_{\parallel}$ IS CONTINUOUS AT THE BOUNDARY, IF THE WAVES HAVE THE FORM $e^{i(\bar{K}\cdot\bar{r}-\omega t)}$



WE MUST HAVE $\bar{K}_i\cdot\bar{r} = \bar{K}_r\cdot\bar{r} = \bar{K}_t\cdot\bar{r}$ ON THE BOUNDARY :

$$\Rightarrow \quad K_i \sin\theta_i\, y = K_r \sin\theta_r\, y = K_t \sin\theta_t\, y$$

OF COURSE $K_i = K_r \Rightarrow \theta_i = \theta_r \Rightarrow$ MIRROR REFLECTION.

EVEN FOR THE TRANSMITTED WAVE WE GET A KIND OF SNELL'S LAW.

FORMALLY WE CAN WRITE $\quad \cos\theta_t = \sqrt{1-\left(\frac{K_i}{K_t}\right)^2 \sin^2\theta_i}$

BUT $K_t = \dfrac{1+i}{d}$ WHILE $K_i = \dfrac{\omega}{c}$ (FOR VACUUM) $\Rightarrow K_t \gg K_i$

AS THE WAVE PENETRATES THE CONDUCTOR IT VARIES LIKE

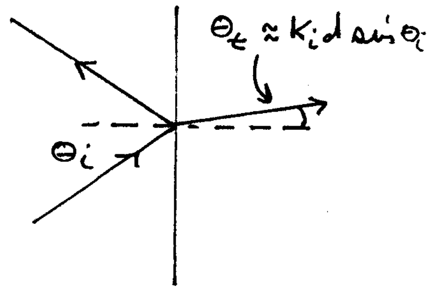
$$e^{i(\bar{K}_t\cdot r-\omega t)} \sim e^{iK_t y\sin\theta_t}\, e^{iK_t x\cos\theta_t}$$

$$\sim e^{iK_i\sin\theta_i y}\, e^{ix\sqrt{K_t^2-K_i^2\sin^2\theta_i}} \approx e^{iK_i\sin\theta_i y}\, e^{iK_t x}$$

or $e^{i\bar{k}_t\cdot\bar{r}} \sim e^{-\frac{x}{d}}\, e^{i\left(k_i \sin\theta_i y + \frac{x}{d}\right)}$

AS EXPECTED THE WAVE DIES OUT QUICKLY, BUT PROPAGATES ALMOST PERPENDICULAR TO THE BOUNDARY

WITH $\theta_t \approx k_i d \sin\theta_i \ll 1$



WHAT ABOUT THE REFLECTED WAVE?

WE HAVE ALREADY SEEN THAT $\theta_r = \theta_i$.

SINCE $\bar{E}$ ~ 0 INSIDE THE CONDUCTOR, AND $\bar{E}_{\parallel}$ IS CONTINUOUS, WE MUST HAVE $\bar{E}_{0r} \sim -\bar{E}_{0i}$ AT <u>NORMAL</u> INCIDENCE

$\Rightarrow$ 180° PHASE CHANGE.

A FINAL QUANTITY OF INTEREST IS THE REFLECTED INTENSITY.

WE CAN PROCEED AS FOR A DIELECTRIC BOUNDARY, MATCHING $\bar{E}$ AND $\bar{H}$ ACROSS THE SURFACE. WE WILL CONSIDER ONLY THE CASE OF NORMAL INCIDENCE.

WE OBTAIN THIS AS THE LIMIT AS $\theta_i \to 0$, AND USE THE FRESNEL FORMULA FOR $\bar{E}$ POLARIZED $\perp$ TO THE SCATTERING PLANE

$$\frac{E_{0r}}{E_{0i}} = -\frac{\sin(\theta_i-\theta_t)}{\sin(\theta_i+\theta_t)} = \frac{\cos\theta_i - \frac{\sin\theta_t}{\sin\theta_i}\cos\theta_t}{\cos\theta_i + \frac{\sin\theta_t}{\sin\theta_i}\cos\theta_t} \xrightarrow[\text{NORMAL INCIDENCE}]{} \frac{1-\frac{k_i}{k_t}}{1+\frac{k_i}{k_t}}$$

$$\approx 1 - \frac{2k_i}{k_t} = 1 - \frac{2k_i d}{1+i} = 1 - k_i d\,(1-i)$$

AND THE REFLECTED INTENSITY $\left|\frac{E_{0r}}{E_{0i}}\right|^2 \sim 1 - 2k_i d \qquad$ IF $k_i d \ll 1$

THIS APPROXIMATION WORKS WELL AT 'LOW' FREQUENCIES, SUCH AS FOR RADIO WAVES. BUT FOR OPTICAL FREQUENCIES ωd IS NO LONGER NEGLIBLE. A REFLECTED INTENSITY OF 90% IS EXTREMELY GOOD. MOST METALS APPEAR DULL TO THE EYE....

PH 206 LECTURE 13

RAYLEIGH RESISTANCE OF A WIRE (BECKER SEC 60)

THUS FAR WE HAVE NOTED THE DIFFICULTY OF SENDING A WAVE INTO A CONDUCTOR FROM THE OUTSIDE. ON THE OTHER HAND IT IS A WELL KNOWN FEAT OF ELECTRICAL ENGINEERING TO SEND WAVES DOWN WIRES: TELEPHONE, TELEGRAPH, CABLE TV.... HAVE WE JUST PROVED THAT THIS IS IMPOSSIBLE ??

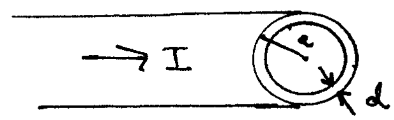
IF WE SEND WAVES DOWN WIRES THERE MUST BE SOME OSCILLATING CURRENT, AND HENCE AN OSCILLATING FIELD INSIDE THE WIRE TO DRIVE THE CURRENT VIA $\bar{J} = \sigma \bar{E}$. THE OSCILLATING $\bar{E}$ LEADS TO AN OSCILLATING $\bar{H}$ FIELD. FINALLY THE BOUNDARY CONDITIONS AT THE SURFACE OF THE WIRE REQUIRE FIELDS $\bar{E}$ AND $\bar{H}$ OUTSIDE THE WIRE, WHICH IS WHERE MOST OF THE FIELD IN FACT RESIDES.

WE HAVE SEEN THAT OSCILLATING FIELDS CANNOT EXIST INSIDE A CONDUCTOR FOR MORE THAN A FEW SKIN DEPTHS BELOW THE SURFACE. HENCE THE CURRENTS OF INTEREST MUST BE VERY NEAR THE SURFACE — IN STRONG DISTINCTION TO THE D.C. CASE OF UNIFORM CURRENT DISTRIBUTION. SINCE THE CURRENT FLOWS THRU A SMALLER VOLUME OF THE CONDUCTOR, THE RESISTANCE OF THE WIRE WILL APPEAR TO RISE.

FOR A WIRE OF RADIUS a, THE D.C. RESISTANCE IS

$$R_0 = \frac{1}{\pi a^2 \sigma} \quad \text{PER UNIT LENGTH ALONG THE WIRE.}$$

BUT IF THE CURRENT IS CONFINED TO A LAYER OF THICKNESS $d = \dfrac{c}{\sqrt{2\pi\sigma\omega\mu}}$



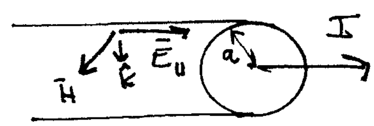
THE RESISTANCE WOULD BE

$$R = \frac{1}{2\pi a d \sigma} = \frac{a}{2d} R_0 \sim \sqrt{\omega}$$

THIS SIMPLE ARGUMENT CAN BE VERIFIED IN A NUMBER OF WAYS.

A VERSION WITH AN ELECTRICAL ENGINEERING FLAVOR IS TO DEFINE THE SURFACE IMPEDANCE AS THE RATIO BETWEEN THE VOLTAGE DROP PER UNIT LENGTH AT THE SURFACE OF THE WIRE TO THE CURRENT:

$$Z = \frac{dV/dl}{I} = \frac{E_\parallel}{I}$$



WE GET AT $E_\parallel \big|_{\text{INSIDE WIRE}}$ IN A ROUNDABOUT WAY. JUST OUTSIDE THE SURFACE OF THE WIRE, AMPERE'S LAW TELLS US $2\pi a H = \frac{4\pi I}{c}$ OR $H = \frac{2I}{ac}$

NOW $H_{\parallel}$ IS CONTINUOUS ACROSS THE SURFACE.

JUST INSIDE THE CONDUCTION $\bar{E}$ AND $\bar{H}$ ARE RELATED AS FOUND ON P.147

$$\bar{H} = \frac{c}{\omega d}(1+i)\,\hat{k}\times\bar{E} \Rightarrow E_{\parallel} = \frac{\omega d}{c}\frac{H}{1+i} = \frac{\omega d}{c}\frac{1-i}{2}\cdot\frac{2I}{ac}, \quad [\mu=1],$$

SO $E_{\parallel} = \dfrac{\omega d}{ac^2}(1-i)\,I \equiv Z\,I.$

[NOTE THAT $\hat{k} \sim \hat{E}\times\hat{H}$ POINTS INWARDS - THE WAVE TRIES TO PENETRATE THE CONDUCTOR BUT DIES OUT...]

NOW $\dfrac{\omega d}{ac^2} = \dfrac{\omega}{ac^2}\dfrac{c}{\sqrt{2\pi\sigma\omega}} = \dfrac{1}{2\pi\sigma a}\dfrac{\sqrt{2\pi\sigma\omega}}{c} = \dfrac{a}{2d}\dfrac{1}{\pi a^2\sigma} = \dfrac{a}{2d}R_0$

SO $Z = \dfrac{a}{2d}R_0(1-i).$

THE REAL PART OF THE IMPEDANCE IS THE OHMIC RESISTANCE, $R = \dfrac{a}{2d}R_0$.

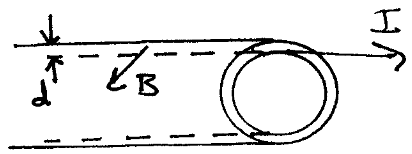
WE ALSO FIND AN IMAGINARY PART TO Z - WHICH WE CAN IDENTIFY AS A SELF-INDUCTANCE TERM:

IN GENERAL $V = IR + L\dot{I} + \dfrac{Q}{C}$

FOR CURRENTS VARYING LIKE $e^{-i\omega t}$; $V = I\left(R - i\omega L + \dfrac{i}{\omega C}\right)$

HENCE WE IDENTIFY $\omega L = \dfrac{a}{2d}R_0$

WE CAN MAKE A <u>NAIVE</u> ESTIMATE OF THE SELF INDUCTANCE [SEE THE PROBLEM SET FOR A MORE EXTENDED DISCUSSION]

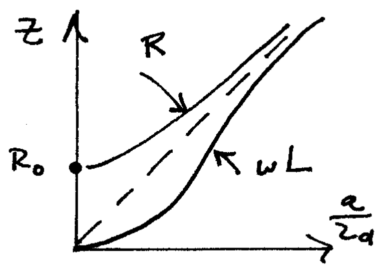


$\Phi_{MAG} = cLI$ (LECTURE 10)

Φ_{MAG} (PER UNIT LENGTH) $\sim \frac{1}{2}Bd = \frac{1}{2}Hd = \dfrac{dI}{ac}$

SINCE THE MAG FIELD IS CONFINED TO A LAYER OF THICKNESS d

SO $L \sim \dfrac{d}{ac^2} = \dfrac{aR_0}{2d\omega}$ AS ABOVE.



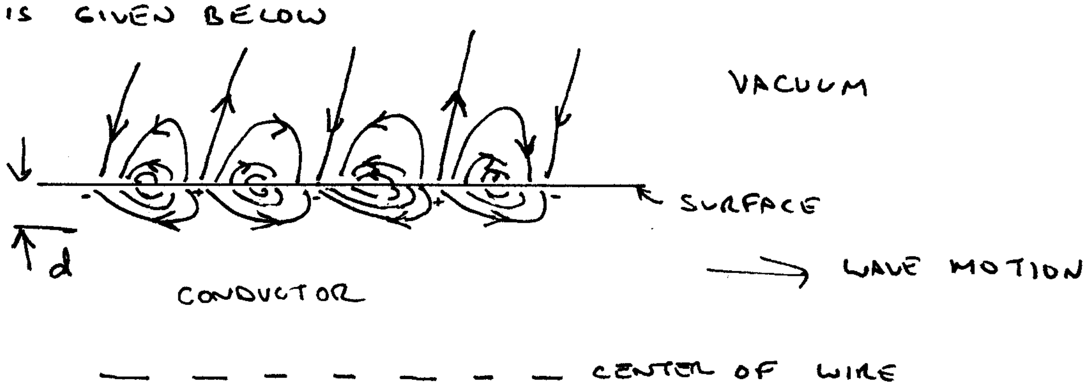
OF COURSE, AS $\omega \to 0$, $R \to R_0$ AND $\omega L \to 0$.

ANOTHER QUESTION CONCERNS THE RELATION OF $\bar{E}$ INSIDE AND OUTSIDE THE WIRE. WE SAW THAT IN FREE SPACE $\bar{E}$ AND $\bar{H}$ ARE $\perp$ TO THE DIRECTION OF THE WAVE. BUT WE CLAIM INSIDE THE WIRE $\bar{E}$ IS PARALLEL TO THE DIRECTION OF THE WAVE SO AS TO DRIVE THE CURRENT.

FURTHERMORE, WE CLAIM THAT $\bar{E}_{\parallel}$ IS CONTINUOUS AT THE SURFACE OF THE CONDUCTOR. CAN WE RECONCILE ALL THESE STATEMENTS?

WE CANNOT EXPECT THE FIELDS OUTSIDE BUT NEAR TO THE WIRE TO EXACTLY RESEMBLE THE FREE-SPACE FIELDS. BUT AS WE MOVE AWAY FROM THE WIRE WE SHOULD APPROACH THE FREE SPACE CASE.

THERE IS NO PROBLEM WITH $\bar{H}$. IT CIRCULATES AROUND THE WIRE WHETHER INSIDE OR OUTSIDE THE SURFACE. A QUALITATIVE PICTURE OF $\bar{E}$ IS GIVEN BELOW



INSIDE THE WIRE, $\bar{E}$ IS PRIMARILY ALONG THE WIRE, BUT THE STRENGTH OF $\bar{E}$ OSCILLATES. THE FIELD LINES BEGIN ON THE SURFACE AND LOOP BACK TO THE SURFACE. OUTSIDE THE WIRE SOME FIELD LINES BEGIN AND END ON THE SURFACE, BUT OTHERS 'ESCAPE' TO FORM THE TRANSVERSE $\bar{E}$ FIELD AT LARGE DISTANCES – THE FREE-SPACE FIELD.

TRANSMISSION LINES (BECKER SEC. 61)

SINCE WE CAN TRANSMIT PLANE WAVES THRU FREE SPACE, WHAT IS THE ADVANTAGE OF 'TIEING' THE WAVES TO A WIRE AND SUFFERING THE JOULE HEAT LOSSES?

PLANE WAVES TRAVEL IN STRAIGHT LINES, AND OCCUPY A LARGE AREA ALONG THE WAVEFRONT. IN THE CASE OF WAVES TIED TO WIRES IT IS EASY TO IMAGINE THAT IF WE BEND THE WIRE, THE WAVE WILL BE GUIDED AROUND THE CORNER. THE PRINCIPAL ROLE OF THE WIRE IS TO GUIDE THE WAVE, NOT TO CARRY IT. ALL OF THE USEFUL POWER FLOW OCCURS OUTSIDE THE SURFACE OF THE WIRE. FURTHERMORE, THE FIELDS ASSOCIATED WITH THE WIRE FALL OFF AS WE GO AWAY FROM THE WIRE, AND SO CAUSE LESS DISTURBANCE TO THE SURROUNDINGS – AND IN TURN ARE LESS PERTURBED BY THE ENVIRONMENT.

AS CONSIDERED ABOVE IT IS REMARKABLE THAT WE COULD TRANSMIT WAVES DOWN A SINGLE WIRE! A RETURN

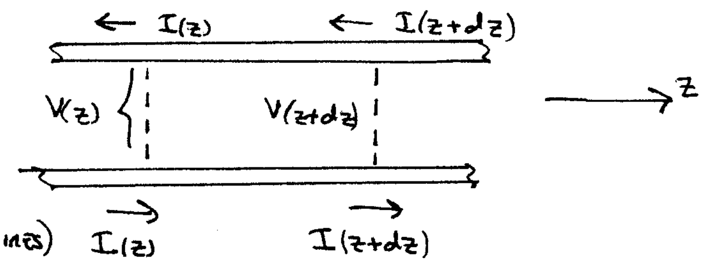
WIRE IS NOT NEEDED, AS IS THE CASE IN D.C. CIRCUITS. HOWEVER, TRANSMISSION OF WAVES ON SINGLE WIRES IS NOT VERY PRACTICAL – THE FIELDS OUTSIDE THE WIRE REMAIN PRONE TO INTERFERENCE WITH THE EXTERNAL ENVIRONMENT OF THE WIRE. [STRICTLY SPEAKING ONLY PULSES COULD BE TRANSMITTED DOWN A SINGLE WIRE. SINE WAVES MUST LAST FOREVER, AND WOULD RESULT IN STANDING WAVE PATTERNS..]

PRACTICAL TRANSMISSION LINES INVOLVE AT LEAST TWO WIRES. THE CLASSIC EXAMPLES ARE THE PARALLEL-WIRE LINE AND THE COAXIAL CABLE:

THE CURRENTS OSCILLATE, AND LOCALLY ALWAYS MOVE IN OPPOSITE DIRECTIONS ON THE 2 CONDUCTORS. THE COAXIAL CABLE ESPECIALLY SOLVES THE PROBLEM OF INTERFERENCE, AS THE FIELDS ARE NON-ZERO (EXCEPT FOR SKIN EFFECTS) ONLY IN THE REGION BETWEEN THE TWO CONDUCTORS.

WE FIRST DERIVE A WAVE EQUATION FOR TRANSMISSION LINES BY CONSIDERATIONS OF THE CURRENT AND VOLTAGE DROPS IN THE SPIRIT OF D.C. CIRCUIT ANALYSIS. THEN WE RETURN TO THE FIELDS.

WE MEASURE THE VOLTAGE DROP BETWEEN THE TWO WIRES AT A GIVEN POSITION z.



LET R = RESISTANCE / LENGTH (OF BOTH WIRES)
C = CAPACITANCE / LENGTH (BETWEEN WIRES)
L = INDUCTANCE / LENGTH (OF THE CIRCUIT FORMED BY JOINING THE WIRES AT $z = \pm\infty$)

APPLYING KIRCHOFF'S CIRCUIT RULE FOR THE LOOP SHOWN, WE GET

$$V(z+dz) - I\frac{R}{2}dz - V(z) - I\frac{R}{2}dz - L\,dz\,\dot{I} = 0,$$

OR $$\frac{\partial V}{\partial z} = L\frac{\partial I}{\partial t} + IR.$$

ALSO, THE CHARGE WHICH ACCUMULATES ON LENGTH dz OF THE UPPER WIRE IN TIME dt IS

$$dQ = dt\,(I(z+dz) - I(z))$$

SO $$\frac{\partial Q}{\partial t} = \frac{\partial I}{\partial z}dz.$$

BY THE DEFINITION OF CAPACITANCE, $dQ = C dz \cdot V$,

so $$\frac{\partial V}{\partial t} = \frac{1}{C}\frac{\partial I}{\partial z}.$$

Thus $$\frac{\partial^2 V}{\partial z \partial t} = \frac{1}{C}\frac{\partial^2 I}{\partial z^2} = L\frac{\partial^2 I}{\partial t^2} + R\frac{\partial I}{\partial t},$$

OR $$\underline{\frac{\partial^2 I}{\partial z^2} = LC\frac{\partial^2 I}{\partial t^2} + RC\frac{\partial I}{\partial t}}.$$ (TELEGRAPHER'S EQUATION)

(DERIVED BY KELVIN PRIOR TO THE MAXWELL THEORY BUT WITHOUT INDUCTANCE, 1854)
(HEAVISIDE ADDS INDUCTANCE, 1876 ; ADDS LEAKAGE CURRENTS, 1887 : SEE PROBLEM SET)
THE WAVES PROPAGATE WITH VELOCITY $v = \frac{1}{\sqrt{LC}}$, BUT DIE AWAY DUE TO THE TERM $RC\frac{\partial I}{\partial t}$.

FOR A COAXIAL CABLE, YOU HAVE SHOWN THAT $C = \frac{\epsilon}{2 \ln b/a}$

AND $L = \frac{2}{c^2} \ln b/a$ IF a = INNER RADIUS
b = OUTER RADIUS (PROB 10, SET 5)

NOTE THAT DUE TO THE SKIN EFFECT WE ONLY NEED CALCULATE THE SELF INDUCTANCE IN THE GAP BETWEEN THE CONDUCTORS WHICH LEADS TO THE ABOVE SIMPLE RESULT.

Thus $v = \frac{c}{\sqrt{\epsilon}} = \frac{c}{n}$ WHERE n = INDEX OF REFRACTION OF THE MEDIUM OUTSIDE THE CONDUCTORS

LIKEWISE AFTER SOME EFFORT, ONE CAN SHOW THAT FOR TWO PARALLEL WIRES OF RADII b, DISTANCE $2a$ APART,

$$C = \frac{\epsilon}{4 \ln\left(\frac{a+\sqrt{a^2-b^2}}{b}\right)} \qquad L = \frac{4}{c^2}\ln\left(\frac{a+\sqrt{a^2-b^2}}{b}\right) \qquad (\mu = 1)$$

SO AGAIN $v = \sqrt{LC} = \frac{c}{\sqrt{\epsilon}} = \frac{c}{n}$

THESE 'COINCIDENCES' THAT $v = \frac{c}{n}$ ARE NOT FORTUITOUS, BUT MUST BE A GENERAL CONSEQUENCE OF MAXWELL'S EQUATIONS — AS $v = \frac{c}{n}$ IS THE PROPAGATION VELOCITY OF ELECTROMAGNETIC WAVES OUTSIDE THE CONDUCTORS.

TEM WAVES

WE RETURN TO CONSIDERATION OF THE FIELDS ASSOCIATED WITH WAVES ON WIRES.

FOR PERFECTLY CONDUCTING WIRES, WE CLAIM THAT THE $\bar{E}$ AND $\bar{H}$ WAVE FIELDS CAN BE PURELY TRANSVERSE, AND THAT $\bar{E}$ AND $\bar{H}$ HAVE THE FORM OF SOLUTIONS TO ELECTRO- AND MAGNETOSTATICS PROBLEMS.

IF $\sigma \to \infty$, THE SKIN DEPTH $d \to 0$ SO THE FIELDS DON'T PENETRATE THE CONDUCTORS: $\bar{E}_{IN} = 0 = \bar{H}_{IN}$. THE CURRENTS FLOW ENTIRELY ON THE SURFACE OF THE CONDUCTORS. THEN $\bar{E}_{\parallel}$ MUST VANISH AT THE SURFACE OF A CONDUCTOR, SO THE BOUNDARY CONDITION ON $\bar{E}$ IS LIKE THAT FOR ELECTROSTATICS.

SUPPOSE THE WIRES RUN ALONG THE z-AXIS. WE LOOK FOR SOLUTIONS

$$\bar{E} = \bar{E}_{\perp}(x,y)\, e^{i(kz-\omega t)}$$

$$\bar{H} = \sqrt{\epsilon}\,\hat{k} \times \bar{E} \qquad (\mu = 1)$$

WITH $k = \sqrt{\epsilon}\,\frac{\omega}{c}$ SO THE WAVE VELOCITY IS $\frac{c}{\sqrt{\epsilon}}$

THESE FIELDS CERTAINLY SATISFY THE WAVE EQUATIONS $\nabla^2 \bar{E} = \frac{\epsilon}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2}$ ETC.

ALSO $\bar{\nabla}\cdot\bar{E} = 0$ SO $\underline{\bar{\nabla}_{\perp}\cdot\bar{E}_{\perp} = 0}$ WHERE $\bar{\nabla}_{\perp} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right)$

AND $\bar{\nabla}\times\bar{E} = -\frac{1}{c}\frac{\partial \bar{H}}{\partial t} = -\frac{\sqrt{\epsilon}}{c}\,\hat{k}\times\bar{E}_{\perp}\frac{\partial}{\partial t}e^{i(kz-\omega t)} = i\underbrace{\frac{\sqrt{\epsilon}\,\omega}{c}}_{k}\,\hat{k}\times\bar{E}_{\perp}\,e^{i(kz-\omega t)}$

$$\shortparallel$$

$$\left(\bar{\nabla}_{\perp}\times\bar{E}_{\perp} + \hat{z}\frac{\partial}{\partial z}\times\bar{E}_{\perp}\right)e^{i(kz-\omega t)}$$

$$= \left(\bar{\nabla}_{\perp}\times\bar{E}_{\perp} + ik\,\hat{z}\times\bar{E}_{\perp}\right)e^{ikz-\omega t}, \qquad (\hat{k} = \hat{z})$$

$$\Rightarrow \quad \underline{\bar{\nabla}_{\perp}\times\bar{E}_{\perp} = 0}$$

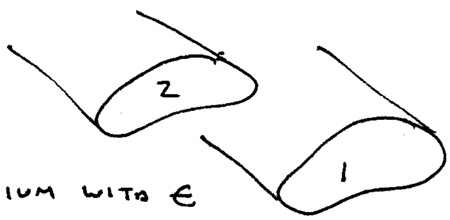
HENCE $\bar{E}_{\perp}(x,y)$ OBEYS THE 2-DIMENSIONAL ELECTROSTATIC MAXWELL EQUATIONS, AS WELL AS THE ELECTROSTATIC BOUNDARY CONDITIONS.

SIMILARLY $\bar{\nabla}_{\perp}\cdot\bar{H}_{\perp} = 0$ AND $\bar{\nabla}_{\perp}\times\bar{H}_{\perp} = 0$ OUTSIDE THE CONDUCTORS ....

HENCE WE CAN HAVE PURELY TRANSVERSE WAVES SO LONG AS AN ELECTROSTATIC SOLUTION EXISTS. NOTE THAT THIS EXCLUDES PURELY TRANSVERSE WAVES INSIDE HOLLOW CONDUCTING PIPES (IF THERE IS NO CENTER CONDUCTOR AS FOR A COAXIAL CABLE).

WITH THE KNOWLEDGE THAT $\bar{E}$ AND $\bar{H}$ BEHAVE LIKE STATIC FIELDS MULTIPLIED BY $e^{ikz-\omega t}$, AND THE RELATION $\bar{H} = \sqrt{\epsilon}\, \hat{k} \times \bar{E}$

WE CAN DEMONSTRATE THAT $\dfrac{1}{\sqrt{LC}} = \dfrac{c}{\sqrt{\epsilon}}$ FOR ARBITRARY CONDUCTORS.



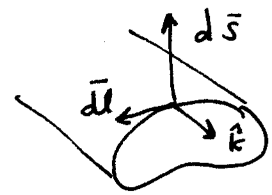
MEDIUM WITH ϵ

THE CAPACITANCE (PER UNIT LENGTH) IS $C = \dfrac{Q}{V} = \dfrac{Q}{-\int_1^2 \bar{E}\cdot d\bar{l}}$

WE CAN FIND Q VIA $\bar{\nabla}\cdot\bar{D} = 4\pi\rho$

SO $$Q = \frac{1}{4\pi}\int_{\text{SURFACE OF CONDUCTOR 1}} \bar{D}\cdot d\bar{S} = \frac{\epsilon}{4\pi}\int_1 \bar{E}\cdot d\bar{S}$$

NOW $\bar{H} = \sqrt{\epsilon}\,\hat{k}\times\bar{E} \Rightarrow \bar{E} = -\dfrac{1}{\sqrt{\epsilon}}\,\hat{k}\times\bar{H}$

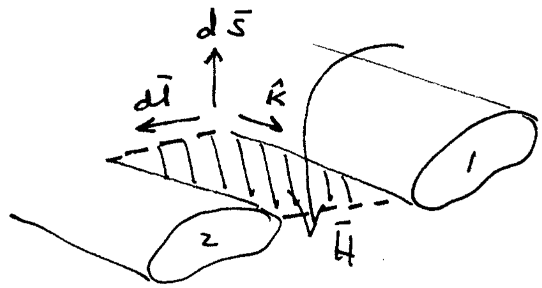


SO $$Q = -\frac{\sqrt{\epsilon}}{4\pi}\int_1 \hat{k}\times\bar{H}\cdot d\bar{S}$$ $$d\bar{S} = d\bar{l}\times\hat{k}\,dz$$

$$= -\frac{\sqrt{\epsilon}}{4\pi}\oint_{\text{LOOP AROUND CONDUCTOR 1}} (\hat{k}\times\bar{H})\cdot(d\bar{l}\times\hat{k})\,dz = -\frac{\sqrt{\epsilon}}{4\pi}dz\oint \underbrace{(\hat{k}\cdot d\bar{l})}_{0}(\bar{H}\cdot\hat{k}) - \hat{k}^2(\bar{H}\cdot d\bar{l})$$

$$= \frac{\sqrt{\epsilon}\,dz}{4\pi}\oint \bar{H}\cdot d\bar{l} = \frac{\sqrt{\epsilon}\,I\,dz}{c}$$ BY AMPERE'S LAW

SO $$C = -\frac{\sqrt{\epsilon}\,I\,dz}{c\int_1^2 \bar{E}\cdot d\bar{l}}$$



FOR PERFECT CONDUCTORS, $\bar{B}$ & $\bar{H} = 0$ INSIDE

WE CALCULATE THE INDUCTANCE VIA

$$\Phi_M = cLI$$

$$\Phi_m = \int_{\text{SURFACE BETWEEN 1 AND 2}} \bar{B}\cdot d\bar{S}$$

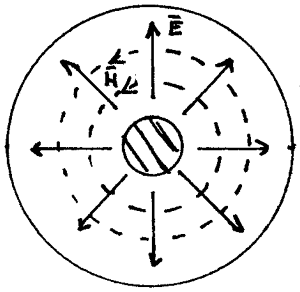
$$= \int \bar{H}\cdot d\bar{S} = \int_1^2 \bar{H}\cdot d\bar{l}\times\hat{k}\,dz = dz\,\sqrt{\epsilon}\int_1^2(\hat{k}\times\bar{E})\cdot(d\bar{l}\times\hat{k}) = -\sqrt{\epsilon}\,dz\int_1^2 \bar{E}\cdot d\bar{l}$$

SO $$L = \frac{\Phi_M}{cI} = -\frac{\sqrt{\epsilon}\,dz}{cI}\int_1^2 \bar{E}\cdot d\bar{l}$$

DIVIDING BY dz TO OBTAIN C AND L PER UNIT LENGTH,

$$LC = \frac{\epsilon}{c^2}$$ AS CLAIMED.

THE FIELD LINES FOR A COAXIAL CABLE ARE SIMPLE



FOR THE 2 WIRE LINE, THE FIELDS ARE RELATIVELY STRAIGHT FORWARD - SINCE THE ELECTROSTATIC EQUIPOTENTIALS ARE ACTUALLY CYLINDERS ....