Ph 206 LECTURE 17

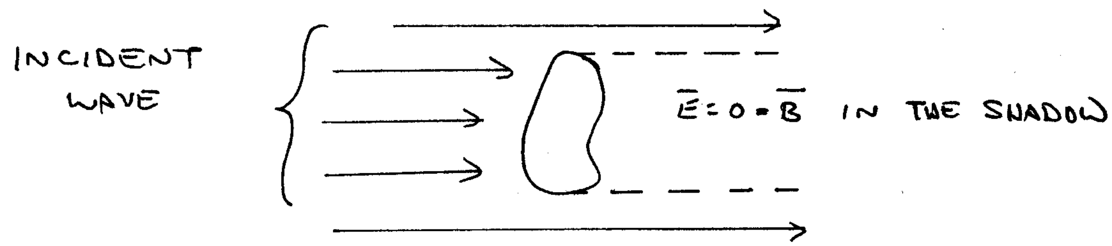
OPTICS AND DIFFRACTION

THIS MATERIAL IS NOT DISCUSSED IN BECKER. YOU MIGHT FIND REFUGE IN MARION OR SCHWARZ. USEFUL BOOKS WITH EMPHASIS ON OPTICS RATHER THAN E & M ARE JENKINS AND WHITE, OR HECHT AND ZAJAC. THE MASTER WORK IS BORN AND WOLF.

IN LECTURE 16 WE BRIEFLY CONSIDERED THE SCATTERING OF E-M WAVES IN THE LONG WAVELENGTH LIMIT. WE NOW TURN TO THE OTHER EXTREME - IN WHICH THE WAVELENGTH IS SHORT COMPARED TO THE SIZE OF THE SCATTERER.

VISIBLE LIGHT HAS WAVELENGTH $\lambda \sim 3-6 \times 10^{-5}$ CM, WHICH IS SHORT BY ORDINARY STANDARDS. THUS OUR TOPIC WILL BE THE RELATION BETWEEN THE LAWS OF OPTICS AND THE THEORY OF E-M WAVES.

GEOMETRICAL OPTICS IS A DESCRIPTION IN WHICH WE TALK OF 'LIGHT RAYS', AND IN WHICH OBJECTS CAST SHARP SHADOWS:



THIS DESCRIPTION HOLDS IN THE LIMIT OF INFINITELY SHORT WAVES.

DIFFRACTION IS THE DESCRIPTION OF THE CASE WHEN THE WAVES ARE SHORT BUT OF FINITE LENGTH. THE CENTRAL QUESTION OF OUR STUDY WILL BE: WHAT HAPPENS WHEN SHORT WAVES STRIKE AN OBSTACLE?

AS A 'SIDELIGHT', WE MENTION A FORMALISM WHICH ATTEMPTS TO RELATE THE RAY CONCEPT TO THE WAVE DESCRIPTION: THIS IS THE EIKONAL METHOD.

SUPPOSE A WAVE OF FREQUENCY ω MOVES IN A MEDIUM WHERE THE INDEX OF REFRACTION VARIES WITH POSITION; $n = n(\bar{x})$.

LET $k \equiv \omega/c$ = WAVE NUMBER WHEN $n=1$.

SUPPOSE ψ IS SOME RELEVANT COMPONENT OF $\bar{E}$ OR $\bar{B}$ SUCH THAT THE WAVE INTENSITY $\sim \psi^2$

THE WAVE ψ OBEYS THE WAVE EQUATION $\nabla^2\psi - \frac{n^2(\bar{x})}{c^2}\ddot{\psi} = 0$

AS A GENERALIZATION OF THE PLANE WAVE $\psi = A e^{i(k\hat{n}\cdot\bar{x} - \omega t)}$

WE SUPPOSE $\psi = A e^{i(kS(\bar{x}) - \omega t)}$

THE FUNCTION $S(\bar{x})$ IS THE EIKONAL $\left(e^{ik\ldots}\right)$

CLEARLY FOR A PLANE WAVE $S = \hat{n}\cdot\bar{x}$, AND THE RAYS ASSOCIATED WITH THE PLANE WAVE POINT ALONG DIRECTION

$$\hat{n} = \bar{\nabla} S$$

WE NOW LOOK FOR THE RAYS IN THE MORE GENERAL CASE.

SUBSTITUTE THE EIKONAL EXPRESSION FOR ψ INTO THE WAVE EQ.:

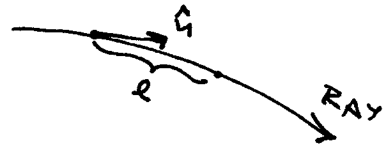
$$\Rightarrow \quad \nabla^2 A - k^2 A\left[(\bar{\nabla}S)^2 - n^2\right] = 0 \qquad \text{(REAL PART)}$$

$$2\bar{\nabla}A\cdot\bar{\nabla}S + A\nabla^2 S = 0 \qquad \text{(IMAGINARY PART)}$$

IF A VARIES SLOWLY $\nabla^2 A$ IS SMALL, AND WE MUST HAVE

$$(\bar{\nabla}S)^2 = n^2 \Rightarrow \bar{\nabla}S = n(\bar{x})\hat{n} \Rightarrow \underline{\hat{n} = \frac{\bar{\nabla}S}{n(\bar{x})} = \text{DIRECTION OF RAY}}$$

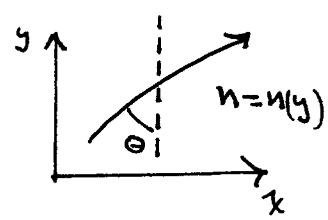
A RAY EQUATION MAY BE OBTAINED: LET ℓ LABEL DISTANCE ALONG THE RAY.



$$\frac{d}{d\ell}(n\hat{n}) = \frac{d}{d\ell}(\bar{\nabla}S) = (\hat{n}\cdot\bar{\nabla})(\bar{\nabla}S) = \left(\frac{\bar{\nabla}S}{n}\cdot\bar{\nabla}\right)\bar{\nabla}S$$

$$= \frac{1}{2n}\bar{\nabla}(\bar{\nabla}S)^2 = \frac{1}{2n}\bar{\nabla}(n^2) = \bar{\nabla}n(\bar{x})$$

EXAMPLE CONSIDER LIGHT RAYS IN THE X-Y PLANE WHEN INDEX $n = n(y)$.



$$\hat{n} = \sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

SO $\frac{d}{d\ell}(n\sin\theta) = \frac{\partial n}{\partial x} = 0$; $\frac{d}{d\ell}(n\cos\theta) = \frac{\partial n}{\partial y}$

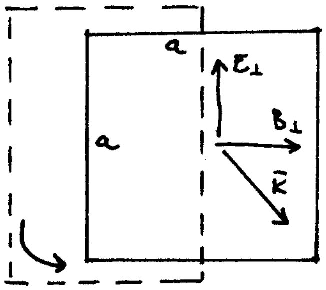
FROM THE FIRST EQUATION, $n\sin\theta = \text{CONSTANT} \Rightarrow$ SNELL'S LAW

DIFFRACTION AND FARADAY'S LAW

BEFORE CONSIDERING A DETAILED APPROACH TO DIFFRACTION, WE PRESENT A BRIEF ARGUMENT AS TO HOW IT IS A CONSEQUENCE OF FARADAY'S LAW.

CONSIDER A PLANE WAVE OF FREQUENCY ω THAT PASSES THROUGH A SQUARE APERTURE OF EDGE a.

CONSIDER ALSO AN IMAGINARY LOOP THAT HAS ONE EDGE ALONG THE MIDDLE OF THE APERTURE $\parallel$ TO $\bar{E}$, AND OPPOSITE EDGE OUTSIDE THE APERTURE.



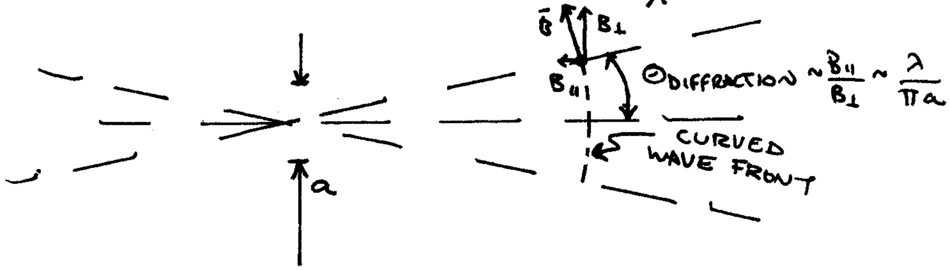
THE $\oint_{\text{LOOP}} \bar{E}\cdot d\bar{l} \sim E_\perp a$

THEN ACCORDING TO FARADAY, THERE MUST BE A TIME-VARYING FLUX OF MAGNETIC FIELD THROUGH THE LOOP, WHICH REQUIRES SOME MAGNETIC FIELD $\parallel$ TO $\bar{k}$.

$$E_\perp a = B_\perp a = -\frac{1}{c}\frac{d}{dt}\int \bar{B}_\parallel \cdot d\bar{S} \approx -\frac{1}{c}\frac{dB_\parallel}{dt}\cdot\frac{a^2}{2} = +\frac{i\omega}{c}B_\parallel\frac{a^2}{2} = +i\frac{\pi a^2}{\lambda}B_\parallel$$

FOR A WAVE $B_\parallel e^{i(kz-\omega t)}$

$$\Rightarrow \quad \frac{|B_\parallel|}{|B_\perp|} \approx \frac{\lambda}{\pi a}$$



FARADAY DOES NOT PERMIT A PURELY PLANE WAVE TO PASS THRU A SMALL APERTURE! SOME LONGITUDINAL FIELD COMPONENT MUST EXIST, AND THE OUTGOING WAVE MUST DIVERGE.

THE CHARACTERISTIC ANGLE OF DIVERGENCE IS $\sim \frac{\lambda}{a}$ \

THE <u>DIFFRACTION ANGLE</u>

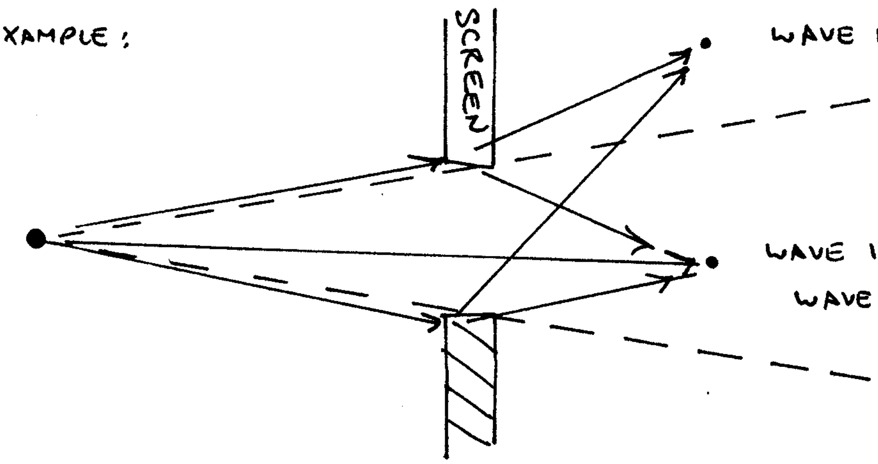
NOTE THAT WE ACTUALLY FOUND $\frac{B_\parallel}{B_\perp} = \frac{\lambda}{i\pi a}$ NEAR THE APERTURE.

THIS IS A PREVIEW OF THE FACT THAT THERE IS A 90° PHASE SHIFT BETWEEN THE "NEAR" AND "FAR" FIELDS OF A DIFFRACTED BEAM.
SEE PP 240 ff.

WE RETURN TO THE MAIN ISSUE: WHAT HAPPENS WHEN SHORT WAVES ENCOUNTER OBSTACLES?

WE KNOW THAT WHEN A WAVE HITS AN OBJECT, THE OBJECT IS EXCITED AND EMITS RADIATION — WHICH THEN INTERFERES WITH THE INCIDENT WAVE TO PRODUCE THE OBSERVED RESULT.

FOR EXAMPLE:

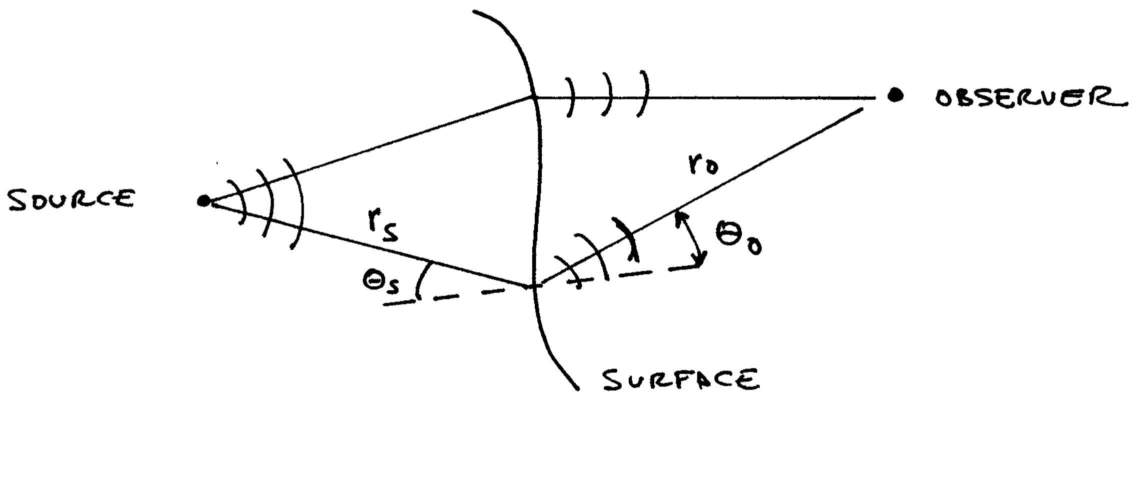


FOR OPAQUE SCREENS WE MAY EXPECT THAT THE OBSERVED WAVE IS THE RESULT OF INTERFERENCE BETWEEN THE DIRECT WAVE AND THE WAVES EMANATING FROM THE BOUNDARY OF THE APERTURE IN THE SCREEN.

THIS APPROACH WAS SUGGESTED BY YOUNG IN 1802, BUT IT IS A BIT DIFFICULT TO CARRY OUT MATHEMATICALLY.

AS EASIER APPROACH IS THAT SUGGESTED BY HUYGENS (~1700) AND PERFECTED BY FRESNEL (1818). ITS JUSTIFICATION IN TERMS OF POTENTIAL THEORY IS DUE TO KIRCHOFF AND HELMHOLTZ.

THE WELL-KNOWN IDEA IS THAT ON ANY SURFACE SURROUNDING THE SOURCE, EACH POINT IS CONSIDERED AS THE SOURCE OF SECONDARY SPHERICAL WAVES - OF AMPLITUDE PROPORTIONAL TO THE STRENGTH OF THE SOURCE WAVE AT THAT POINT. BEYOND THE SURFACE WE IGNORE THE ORIGINAL WAVE, BUT ADD UP THE INTERFERING SPHERICAL WAVES TO FIND THE OBSERVED WAVE.



WE MAKE THIS PICTURE QUANTITATIVE:

THE STRENGTH (AMPLITUDE, NOT INTENSITY) OF THE SOURCE WAVE AT THE SURFACE IS

$$A \frac{e^{i(kr_s - \omega t')}}{r_s} \quad \text{WHERE } t' = \text{TIME OF ARRIVAL AT SURFACE}$$

WE MIGHT EXPECT THE SECONDARY 'WAVELET' SEEN BY THE OBSERVER TO BE

$$A \frac{e^{i(kr_s - \omega t')}}{r} \cdot f(\theta_s, \theta_0) \frac{e^{i[kr_0 - \omega(t - t')]}}{r_0} \cdot d\,\text{AREA}$$

t = TIME OF OBSERVATION.

THE FACTOR $f(\theta_s, \theta_0)$ ALLOWS THE POSSIBILITY THAT THE STRENGTH OF THE SECONDARY WAVE DEPENDS ON THE ANGLES INVOLVED – WE EXPECT NO SECONDARY WAVE HEADING BACK TO THE SOURCE!

ALTOGETHER

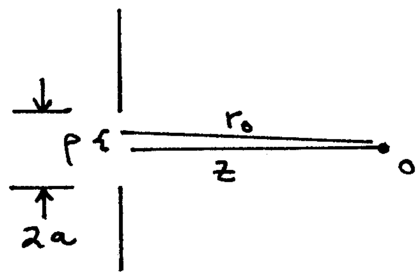
$$\psi_{\text{OBSERVER}} = A \int_{\text{SURFACE}} f(\theta_s, \theta_0) \frac{e^{ik(r_s + r_0)}}{r_s r_0} e^{-i\omega t} \, d\,\text{AREA}$$

WE WILL DROP THE FACTOR $e^{-i\omega t}$ IN MOST OF THE FOLLOWING.

<u>EXAMPLE</u> A PLANE WAVE IS INCIDENT ON AN OPAQUE SCREEN WITH A CIRCULAR APERTURE OF RADIUS a.

PLANE WAVE IN $\Leftrightarrow$ $r_s \to \infty$ (BUT $\frac{A}{r_s} \to$ CONSTANT) AND $\theta_s \to 0$.

$$\text{THEN } \psi_0 \sim \int_{\text{APERTURE}} f(\theta_0) \frac{e^{ikr_0}}{r_0} \, d\,\text{AREA}$$



FOR AN OBSERVER ON THE AXIS, WITH $r_0 \gg a$

WE HAVE $\theta_0 \approx 0$. SO WE JUST WRITE $f_0 = f(0)$ = CONSTANT

ALSO $r_0 = \sqrt{z^2 + \rho^2} \approx z + \frac{\rho^2}{2z}$

$$\text{SO } \psi_0 \sim f_0 \frac{e^{ikz}}{z} 2\pi \int_0^a \rho \, d\rho \, e^{\frac{ik\rho^2}{2z}} = \frac{2\pi i}{k} f_0 e^{ikz} \left(1 - e^{\frac{ika^2}{2z}}\right)$$

AS a GROWS LARGE, THE TERM $e^{\frac{ika^2}{2z}}$ OSCILLATES WILDLY AND SO MUST CONTRIBUTE NOTHING TO THE PHYSICS IN THE LIMIT $a \to \infty$

I.E. $\psi_0 \to \frac{2\pi i f_0}{k} e^{ikz}$ WHEN $a \to \infty$

BUT $a \to \infty \Rightarrow$ NO SCREEN!

$\therefore \psi_0 = e^{ikz} \left(e^{-i\omega t}\right)$ SINCE THIS IS JUST THE PROPAGATION OF A PLANE WAVE!

HENCE WE LEARN THAT $\underline{\underline{f_0 = \frac{k}{2\pi i}}}$

AND THAT THE METHOD IS NOT COMPLETELY CRAZY!

IF THE SOURCE WAS NOT INFINITELY FAR AWAY, WE HAVE

$$\underline{\psi_0 = \frac{Ak}{2\pi i} \int \frac{e^{ik(r_s + r_0)}}{r_s r_0} \, d\,\text{Area}}$$

[WE DISCUSS THE SIGNIFICANCE OF THE PHASE FACTOR, $\frac{1}{i}$, AT THE END OF THIS LECTURE.]

A MORE DETAILED JUSTIFICATION OF THIS MAY BE GIVEN USING A FORM OF GREEN'S THEOREM: THE VALUES OF ψ AND $\vec{\nabla}\psi \cdot \hat{n}$ ON A CLOSED SURFACE COMPLETELY DETERMINE ψ EVERYWHERE WITHIN. IF WE SUPPOSE $\psi = 0 = \vec{\nabla}\psi \cdot \hat{n}$ ON THE OPAQUE PARTS OF THE SCREEN, WE RECOVER THE ABOVE EQUATION, WITH THE PRESENCE OF AN OBLIQUITY FACTOR

$$\frac{\cos\theta_s + \cos\theta_0}{2}$$

FOR NORMAL INCIDENCE ON THE SURFACE, THIS IS $\frac{1 + \cos\theta_0}{2}$, WHICH VERIFIES THAT THERE IS NO SECONDARY WAVELET EXACTLY AT $180°$.

HOWEVER THE USE OF THE BOUNDARY CONDITIONS $\psi = 0 = \vec{\nabla}\psi \cdot \hat{n}$ ON THE SCREEN IS NOT STRICTLY CORRECT – SINCE THE EDGES OF THE APERTURE MUST BE RADIATING. AN EXACT THEORY IS VERY DIFFICULT AND HAS BEEN SUCCESSFULLY APPLIED IN ONLY A VERY FEW CASES. C.F. THE BOOK BY BORN & WOLF.

THE APPROXIMATE THEORY SKETCHED ABOVE IS ADEQUATE FOR MOST PRACTICAL CONSIDERATIONS IN OPTICS.

Scalar Diffraction via Green's Theorem (Jackson, sec. 9.8)

Green told us that for any two scalar functions (in 3 dimensions)

$$\int_V \phi\nabla^2\psi - \psi\nabla^2\phi \, d\mathrm{vol} = \int_S (\phi\,\vec{\nabla}'\psi - \psi\,\vec{\nabla}'\phi)\cdot d\vec{S}' \qquad (\text{p. } 37)$$

We are dealing with waves of frequency ω, i.e., time dependence $e^{-i\omega t}$.
The spatial part of the waves then obey the (Helmholtz) wave equation

$$\nabla^2\psi + k^2\psi = 0 \qquad \left(k = \frac{\omega}{c}\right)$$ when we are not on the bounding surface S.

We choose wave function ϕ to emanate from a point source on the surface S, labelled by $\vec{x}'$. That is, choose $\nabla^2\phi + k^2\phi = -\delta^3(\vec{x}-\vec{x}')$, $\vec{x}$ in V.

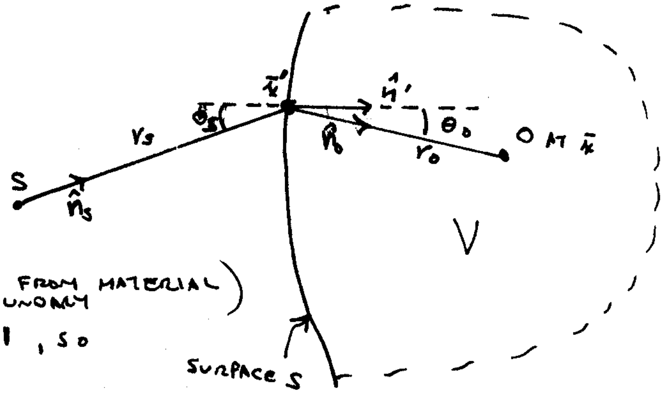
We saw (p. 176) that ϕ is a spherical wave: $\phi = +\dfrac{e^{ikr_0}}{4\pi r_0}$ where $\vec{r}_0 = \vec{x}-\vec{x}'$

Note that $\vec{\nabla}' r_0 = -\hat{n}_0 = -\dfrac{\vec{r}_0}{r_0}$, so $\vec{\nabla}'\phi = -ik\hat{n}_0\left(1+\dfrac{1}{ikr_0}\right)\phi$

Inserting all this in Green's identity we have

$$\psi(\vec{x}) = -\frac{1}{4\pi}\int_S \frac{e^{ikr_0}}{r_0}\,\hat{n}'\cdot\left[\vec{\nabla}'\psi + ik\left(1+\frac{i}{kr_0}\right)\hat{n}_0\psi\right] d\mathrm{Area}'$$

The minus sign arises because we define $\hat{n}'$ to be the <u>inward</u> normal.



If ψ arises from a point source at S outside volume V, then $\psi(\vec{x}') \approx A\dfrac{e^{ikr_s}}{r_s}$

$$\nabla'\psi(\vec{x}') = +ik\hat{n}_s\left(1+\frac{1}{ikr_s}\right)\psi$$

(This is an approximation; it ignores possible radiation from material on the boundary)

In the short wavelength limit, kr_s & kr_0 are $\gg 1$, so

$$\psi \approx \frac{Ak}{4\pi i}\int \frac{e^{ik(r_s+r_0)}}{r_s r_0}\,\hat{n}'\cdot(\hat{n}_s+\hat{n}_0)\, d\mathrm{Area}'$$

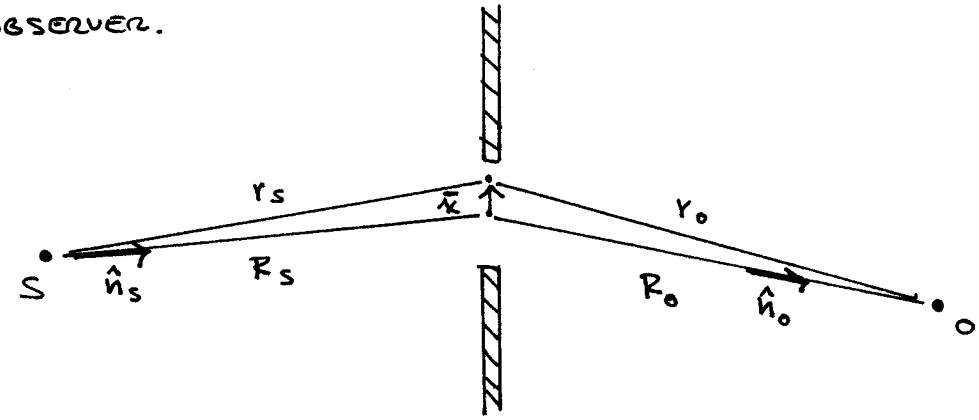
$$= \frac{Ak}{2\pi i}\int \frac{e^{ik(r_s+r_0)}}{r_s r_0}\,\underbrace{\frac{\cos\theta_s + \cos\theta_0}{2}}_{\text{obliquity factor}}\, d\mathrm{Area}'$$

The final approximation is to evaluate the integral only over that part of the boundary where there is no material.

Jackson discusses the merits of the two approximations we have made, and also discusses the more complicated case of vector diffraction.

FRAUNHOFER DIFFRACTION

MANY INTERESTING CASES OF DIFFRACTION INVOLVE APERTURES WHICH ARE SMALL COMPARED TO THE DISTANCES TO THE SOURCE AND OBSERVER.

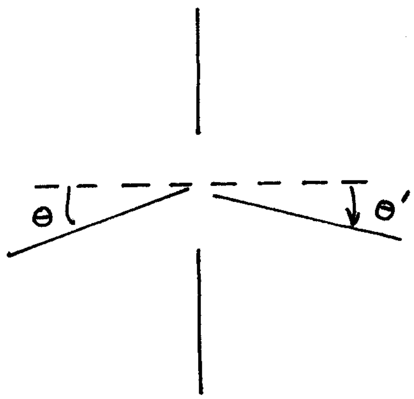


THEN $r_s \sim R_s + \bar{x}\cdot\hat{n}_s$ AND $r_o \sim R_o - \bar{x}\cdot\hat{n}_o$

AND SO $\psi_o \rightarrow \dfrac{AK}{2\pi i}\dfrac{e^{iK(R_s+R_o)}}{R_sR_o}\displaystyle\int_{\text{APERTURE}} e^{iK\bar{x}\cdot(\hat{n}_s-\hat{n}_o)}\, d\,\text{AREA}$

THIS IS THE FRAUNHOFER DIFFRACTION APPROXIMATION

EXAMPLE PLANE WAVES INCIDENT ON A FLAT SCREEN WITH AN INFINITE SLIT OF WIDTH d.



$$\psi_o \sim A'\int_{-d/2}^{d/2} dx\, e^{iKx(\sin\theta+\sin\theta')}$$

$$= A''\frac{\sin u}{u}$$

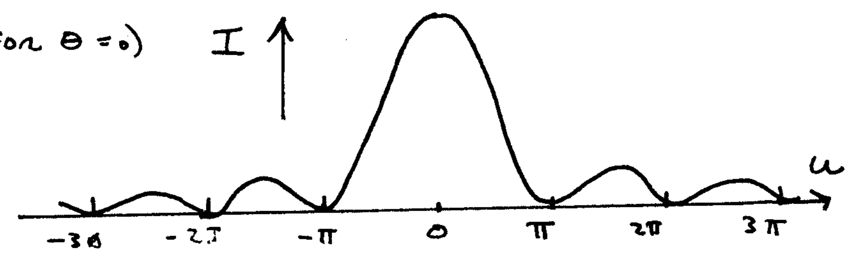
WHERE $u = \dfrac{Kd}{2}(\sin\theta+\sin\theta')$

$= \dfrac{\pi d}{\lambda}(\sin\theta+\sin\theta')$

THE OBSERVED INTENSITY IS $I \sim \psi_o^2 \Rightarrow I = I_o\left(\dfrac{\sin u}{u}\right)^2$

WHERE I_0 = INTENSITY WHEN $\theta' = -\theta \Leftrightarrow$ ALONG THE DIRECT VIEW

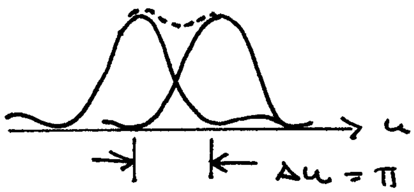
THIS HAS THE FAMILIAR SHAPE (FOR $\theta = 0$)



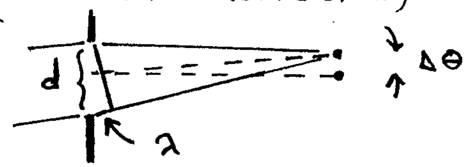
AN IMPORTANT QUESTION IS: IF YOU VIEW TWO OBJECTS THRU A SLIT, WHAT IS THE SMALLEST ANGULAR SEPARATION, $\Delta\theta$, BETWEEN THEM SUCH THAT YOU DISTINGUISH TWO OBJECTS?

AT YOUR EYE EVEN A POINT SOURCE IS 'BLURRED' INTO A DIFFRACTION PATTERN OF FINITE ANGULAR WIDTH.

WE SUPPOSE THAT TWO PATTERNS AS FOUND ABOVE CAN BARELY BE RESOLVED IF THE MAXIMUM OF ONE COINCIDES WITH THE FIRST MINIMUM OF THE OTHER

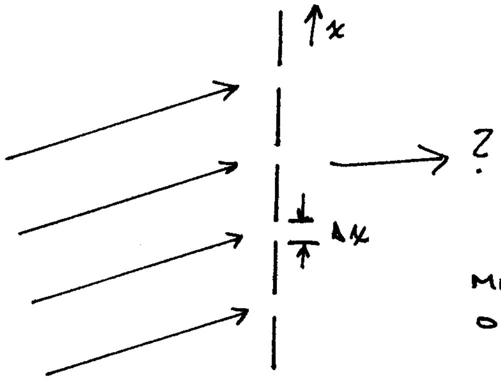


(RAYLEIGH CRITERION)



BUT $\Delta u = \dfrac{\pi d}{\lambda}\Delta\theta$ SO $\underline{\underline{\Delta\theta_{MIN} = \dfrac{\lambda}{d}}}$ SMALL, BUT NON-ZERO

<u>EXAMPLE</u> MULTIPLE SLITS (DIFFRACTION GRATING)



WE LEAVE A QUANTITATIVE DISCUSSION TO YOU, ON THE PROBLEM SET...

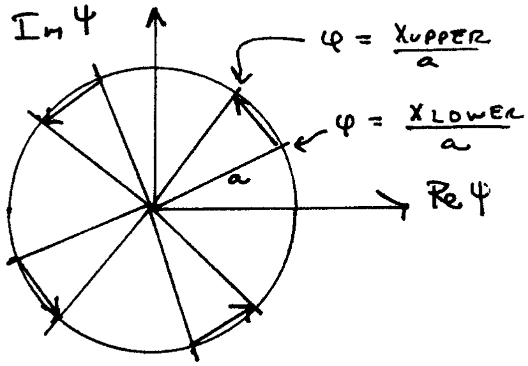
HERE WE PRESENT A GRAPHICAL METHOD WHICH GIVES THE MAIN SENSE OF THE PHYSICS.

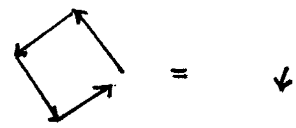
$$\psi_0 \sim \sum_{\text{SLITS}} \int_{x_{LOWER}}^{x_{UPPER}} e^{ikx(\sin\theta + \sin\theta')}\,dx$$

$$\sim \sum_{\text{SLITS}} a\left(e^{i\frac{x_{UPPER}}{a}} - e^{i\frac{x_{LOWER}}{a}}\right)$$

WHERE $a = \dfrac{1}{k(\sin\theta + \sin\theta')} = \dfrac{\lambda}{2\pi(\sin\theta + \sin\theta')}$ = A LENGTH

ψ_0 CAN BE REPRESENTED AS A SUM OF VECTORS WHICH ARE CHORDS OF A CIRCLE OF RADIUS a:



SO ψ_0 = 

FOR 4 SLITS AS SHOWN

IN THE DIAGRAM ↑ THE ARC LENGTH IS $a \cdot \left(\frac{\Delta X}{a}\right) = \Delta X$.

THUS THE CIRCUMFERENCE OF THE CIRCLE IS A 1:1 MAP OF THE SLIT PATTERN – JUST WRAP THE PATTERN AROUND INTO A CIRCLE OF RADIUS a.

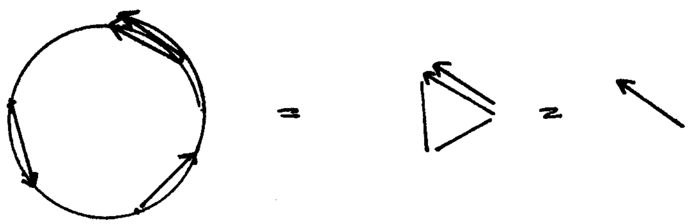
SUPPOSE $\Theta = 0 = \Theta'$. THEN $a = \dfrac{\lambda}{2\pi(\Delta \sin\Theta + \Delta \sin\Theta')} \rightarrow \infty$

AN THE VECTOR PICTURE IS

$\Rightarrow$ $\psi =$  A MAXIMUM

THE FIRST MINIMUM IN THE DIFFRACTION PATTERN VERY NEARLY CORRESPONDS TO THE SITUATION SKETCHED AT THE TOP OF THE PAGE.

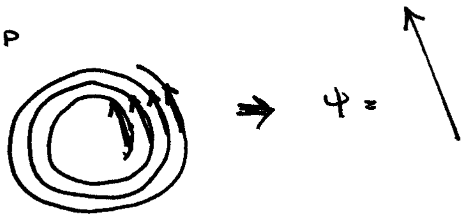
THE SECOND MAXIMUM:

$\psi =$ 

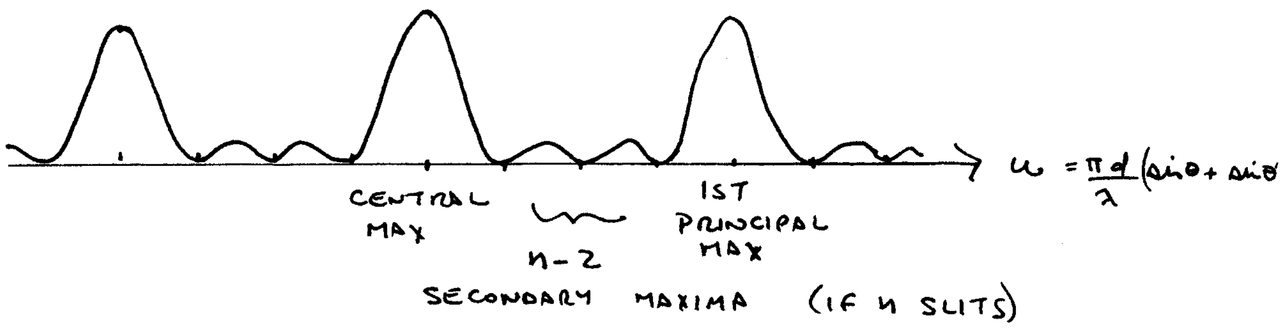
FOR THIS MAX ψ IS ONLY $1/n$ OF THE $\Theta = 0$ MAXIMUM

SO $I \sim \frac{1}{n^2} I_0$

WE GET A SIGNIFICANT SECONDARY MAXIMUM ONLY WHEN ALL n 'SLIT VECTORS' LINE UP



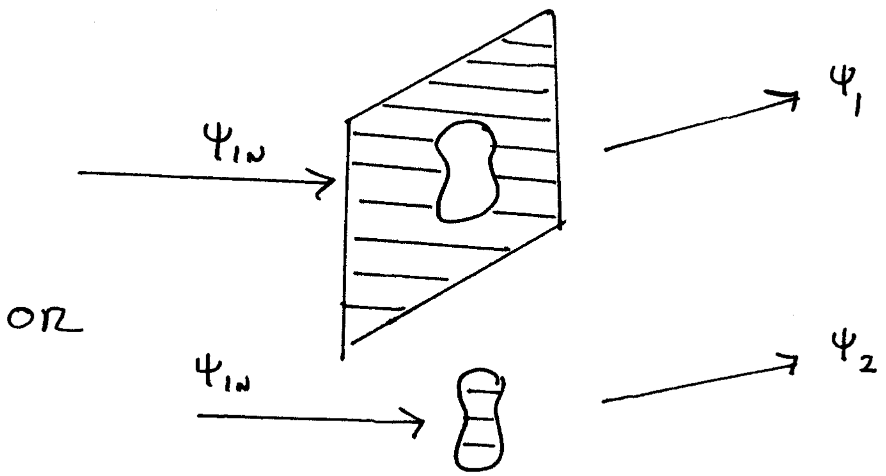
So we form a qualitative picture of the diffraction pattern



We note that at certain angles the single slit diffraction pattern vanishes (p 203). In our vector picture, this occurs when the 'vibration curve' has a radius such that one slit occupies an integer number of turns

and the slit vector vanishes. If the slit separation is exactly m × slit width, with m an integer, then the m'th principal max of the multislit pattern coincides with the 1st single slit minimum — and so will be absent!

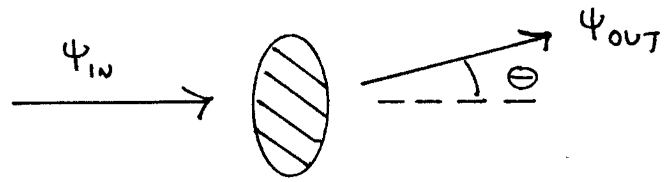
<u>Babinet's Principle</u>. Consider the diffraction by two 'complementary' screens



The sum of ψ_1 and ψ_2 is clearly just what we would have if there were no screen at all!

$$\therefore \quad \psi_{IN} = \psi_1 + \psi_2$$ — <u>Babinet's Principle</u>

EXAMPLE DIFFRACTION AROUND AN OPAQUE DISK



ON THE PROBLEM SET YOU WILL SOLVE THE COMPLEMENTARY PROBLEM: DIFFRACTION THRU A SCREEN WITH A CIRCULAR APERTURE.

THE RESULT IS $\psi_1 = A \dfrac{ka^2}{i} \dfrac{J_1(ka\sin\theta)}{ka\sin\theta}\left[\dfrac{e^{i(kr-\omega t)}}{r}\right]$

SUPPOSING $\psi_{IN} = A e^{i(kz-\omega t)}$ = PLANE WAVE NORMALLY INCIDENT

a = RADIUS OF CIRCLE. J_1 IS A BESSEL FUNCTION.

THEN BY BABINET'S PRINCIPLE, $\psi_{OUT} = \psi_{IN} - \psi_1$ IS OUR SOLUTION. BUT OUR CASE IS A KIND OF SCATTERING PROBLEM, SO IT MAKES SENSE TO WRITE $\psi_{OUT} = \psi_{IN} + \psi_{SCAT}$

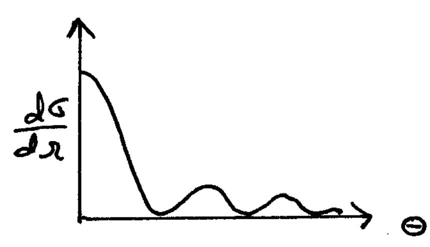
HENCE $\psi_{SCAT} = -\psi_1 = A\, i k a^2 \dfrac{J_1(ka\sin\theta)}{ka\sin\theta}$

IT IS COMMON TO DEFINE THE RELATIVE SCATTERING AMPLITUDE

$$f(\theta) = \frac{\psi_{SCAT}}{A} = i k a^2 \frac{J_1(ka\sin\theta)}{ka\sin\theta}$$

THEN, AS IN LECTURE 16, WE DEFINE THE SCATTERING CROSS SECTION AS THE RELATIVE SCATTERED INTENSITY:

$$\frac{d\sigma_{SCAT}}{d\Omega} = |f(\theta)|^2 = k^2 a^4 \left[\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right]^2 \sim k^2 a^4 \cdot \text{ANGULAR FACTOR}$$



[COMPARE THIS RESULT WITH THAT OF THE LONG WAVELENGTH LIMIT (LECTURE 16) $\dfrac{d\sigma}{d\Omega} \sim k^4 a^6$]

THE TOTAL CROSS SECTION IS

$$\sigma_{SCAT} = \int \frac{d\sigma}{d\Omega} = 2\pi a^2 \int_0^{\pi} \frac{J_1^2(ka\sin\theta)}{\sin^2\theta}\, d\theta = \pi a^2$$

BY A MIRACLE OF BESSELRY

= AREA OF DISK!

OF COURSE, THE OPAQUE DISK ABSORBED ALL THE RADIATION THAT HIT IT. SO WE CAN ALSO DEFINE THE ABSORPTION CROSS SECTION

$$\sigma_{ABS} = \pi a^2 \quad \left[= \text{RELATIVE POWER ABSORBED} = \frac{\text{AREA OF DISK}}{\text{UNIT AREA OF WAVE}} \right]$$

THUS THE TOTAL CROSS SECTION IS REALLY

$$\sigma_{TOT} = \sigma_{SCAT} + \sigma_{ABS} = 2\pi a^2$$

WE MAY NOW ILLUSTRATE A RESULT OF GREAT GENERALITY:

AT 0° SCATTERING ANGLE, THE SCATTERING AMPLITUDE GIVEN ON P207 BECOMES

$$f(0) = i\,\frac{ka^2}{2} \qquad \text{(HOMEWORK PROBLEM)}$$

SO $\underline{\sigma_{TOT} = \frac{4\pi}{k}\, \mathrm{Im}[f(0)]}$ OPTICAL THEOREM

THIS RESULT IS RATHER AMAZING IN THAT IT RELATES ALL SCATTERING EFFECTS, INCLUDING ABSORPTION (OR 'INELASTIC' SCATTERING) TO THE ELASTIC SCATTERING AMPLITUDE AT 0°.

A DIFFRACTION MODEL IS OFTEN INVOKED TO DISCUSS THE SCATTERING OF NUCLEAR PARTICLES: PROTONS, NEUTRONS, α PARTICLES...

σ_{ABS} IS VERY LARGE AND ESSENTIALLY IMPOSSIBLE TO CALCULATE. CRUDELY SPEAKING, ELASTIC SCATTERING CAN ONLY OCCUR IF THE PARTICLES JUST GRAZE ONE ANOTHER. A HEAD-ON COLLISION IS ALMOST ALWAYS INELASTIC. THIS SITUATION IS VERY SIMILAR TO LIGHT SCATTERING AROUND AN OPAQUE DISK — AND INDEED THE SHAPE OF THE DIFFERENTIAL CROSS SECTIONS ARE SIMILAR.

THE FIRST DIFFRACTION MINIMUM FOR PROTON-PROTON SCATTERING WAS ONLY OBSERVED IN 1972. THIS IS BECAUSE VERY HIGH ENERGIES ARE REQUIRED IF THE PROTON IS TO HAVE A SHORT WAVELENGTH, SO THAT DIFFRACTION WORKS. ACCORDING TO DE BROGLIE $\lambda = \frac{h}{\text{MOMENTUM}}$!

FRESNEL DIFFRACTION

THIS IS THE SITUATION WHEN THE SOURCE AND OBJECT DISTANCES ARE SMALL ENOUGH THAT WE CANNOT PULL THEM OUTSIDE THE DIFFRACTION INTEGRAL:

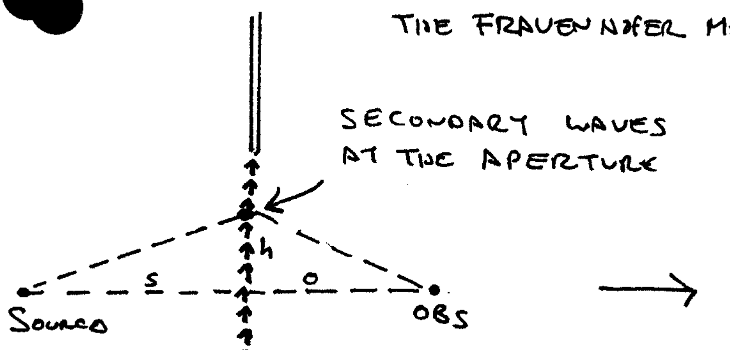
$$\psi_{OBS} = \frac{Ak}{2\pi i}\int \frac{e^{ik(r_s + r_o)}}{r_s r_o}\, d\,\text{AREA}$$

THIS IS HARD TO CALCULATE!

BUT WE CAN EXTEND OUR GRAPHICAL TECHNIQUE TO GET A SENSE OF THE SOLUTION.

FOR FRAUNHOFER DIFFRACTION WE MAPPED THE APERTURE ONTO A CIRCLE – EQUAL LENGTHS ON THE APERTURE $\Rightarrow$ EQUAL ARCS ON THE CIRCLE.

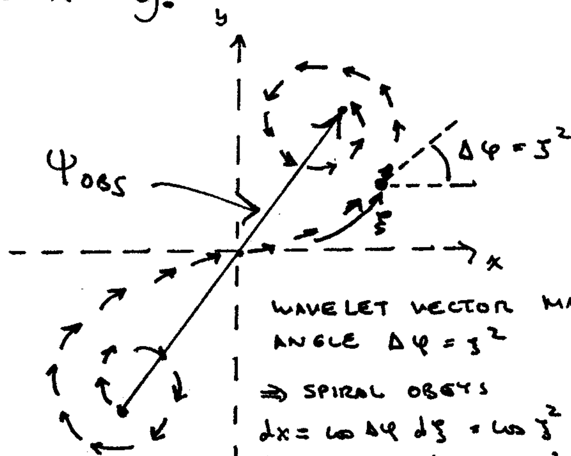
FOR FRESNEL DIFFRACTION WE MUST MODIFY THIS TECHNIQUE TO INCLUDE THE $1/r$ FACTORS. ELEMENTS OF THE APERTURE FARTHER AWAY DON'T COUNT AS MUCH. CONSIDER $\theta = 0 = \theta'$ (FOR WHICH THE FRAUNHOFER MAP WOULD BE A LINE):



PHASE DIFF. FOR WAVELET AT h IS

$$|\Delta\varphi| = \frac{2\pi}{\lambda}\left(\sqrt{s^2+h^2} + \sqrt{o^2+h^2} - s - o\right)$$

$$\approx \frac{\pi h^2}{\lambda}\,\frac{s+o}{so} \equiv \xi^2$$



WAVELET VECTOR MAKES ANGLE $\Delta\varphi = \xi^2$

$\Rightarrow$ SPIRAL OBEYS

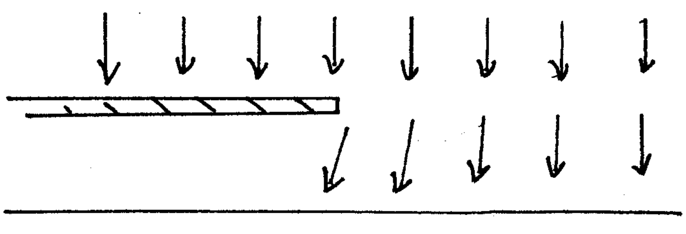
$dx = \cos\Delta\varphi\, d\xi = \cos\xi^2\, d\xi$

$dy = \sin\Delta\varphi\, d\xi = \sin\xi^2\, d\xi$

$$\Rightarrow x = \int_0^{\xi}\cos\xi^2\, d\xi\,,\quad y = \int_0^{\xi}\sin\xi^2\, d\xi$$

INSTEAD OF A CIRCULAR VIBRATION CURVE, WE GET A SPIRAL (OR RATHER 2 SPIRALS) CALLED THE <u>CORNU SPIRAL</u>

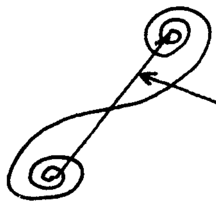
<u>EXAMPLE</u> THE SHADOW CAST BY A STRAIGHT EDGE



WHAT IS THE INTENSITY VS POSITION AT THE LOWER PLANE?

Ph 206 LECTURE 17

IF THERE IS NO STRAIGHT EDGE, THE OBSERVER SEES THE SUM OF THE WHOLE SPIRAL

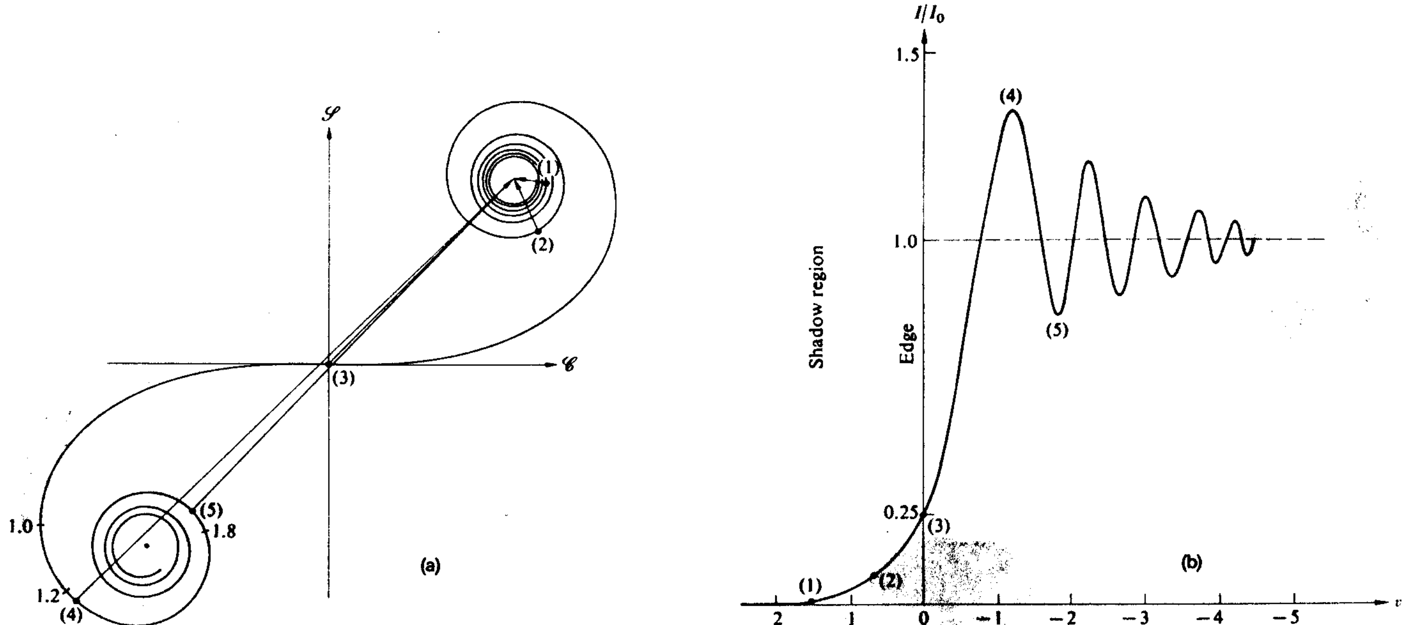


IF THE OBSERVER IS EXACTLY IN LINE WITH THE SHADOW OF GEOMETRICAL OPTICS (S)HE SEES HALF THE CORNU SPIRAL

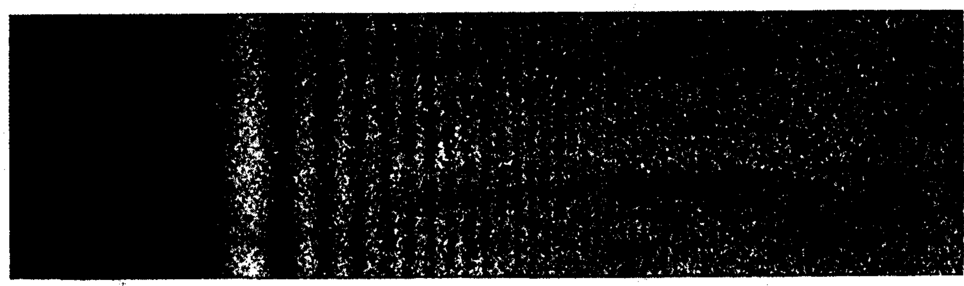
 $\psi_{\text{AT EDGE}}$. SINCE $I \sim \psi^2$, $I_{\text{EDGE}} = \frac{1}{4} I_0$

THE 'SHADOW' PATTERN OSCILLATES AS SHOWN BELOW.

[THIS IS THE SAME PATTERN ENCOUNTERED ON p 146d WHICH DESCRIBED HOW A 'SHARP' WAVE FRONT PROPAGATES INTO A DIELECTRIC MEDIUM.]



(a) The Cornu spiral for a semi-infinite screen. (b) The corresponding irradiance distribution.



The fringe pattern for a half-screen.

APPENDIX: MAXWELL AND THE FLASHLIGHT

WHENEVER YOU TURN ON A FLASHLIGHT YOU GENERATE A BEAM OF LIGHT WHICH MUST BE A SOLUTION TO MAXWELL'S EQUATIONS (IF YOU BELIEVE THAT OPTICS IS INDEED JUST A BRANCH OF ELECTROMAGNETISM). IN THIS APPENDIX WE DISCUSS AN APPROXIMATE SOLUTION TO MAXWELL'S EQUATIONS WHICH HAS FEATURES ASSOCIATED WITH A FLASHLIGHT BEAM, OR MORE RELEVANTLY, A LASER BEAM.

WE LOOK FOR A WAVE PROPAGATING IN THE z DIRECTION, BUT WHICH HAS SIGNIFICANT INTENSITY ONLY NEAR $x = y = 0$. OF COURSE, THE FREE SPACE EQUATIONS

$$\bar{\nabla} \times \bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t} \qquad \bar{\nabla} \times \bar{B} = \frac{1}{c}\frac{\partial \bar{E}}{\partial t}$$

DO NOT ALLOW THE FIELDS TO HAVE A SHARP TRANSITION IN x OR y - SO OUR SOLUTION CANNOT BE EXACTLY LIKE A TUBE OF LIGHT. RATHER WE SHOULD EXPECT A FIELD WHICH IS STRONG AT $x = y = 0$, AND WHICH DIES AWAY SMOOTHLY AS x & y VARY AWAY FROM THE ORIGIN. WE MAKE AN INSPIRED "GUESS" THAT THE x, y DEPENDENCE OF THE BEAM CAN BE DESCRIBED BY A GAUSSIAN. THAT IS, WE LOOK FOR

$$\bar{E}(x,y,z,t) \quad \bar{E}_0 \, e^{-\frac{\rho^2}{w^2}} \, e^{i(kz-\omega t)}$$

$$\rho = \sqrt{x^2+y^2}$$

THE PARAMETER w IS CALLED THE GAUSSIAN WIDTH OF THE BEAM.

BEFORE PLUGGING THIS GUESS INTO MAXWELL'S EQUATIONS WE CAN ALREADY ANTICIPATE THAT IT WON'T WORK! AN IMPORTANT ASPECT OF OPTICAL WAVES IS <u>DIFFRACTION</u>. THIS HAS THE CONSEQUENCE FOR A FLASHLIGHT OR LASER THAT THE BEAM IS NEVER PERFECTLY COLLIMATED, BUT MUST HAVE SOME SPREADING. ROUGHLY, IF THE LIGHT IS EMITTED THRU AN APERTURE OF DIAMETER d, THEN THE BEAM WILL SPREAD SO AS TO OCCUPY A CONE OF ANGLE

$$\theta_0 \sim \lambda/d \, , \quad \lambda = \text{WAVE LENGTH.} \qquad (p. 204)$$

AS WE LIKE THE IDEA OF DESCRIBING THE BEAM BY A GAUSSIAN PROFILE, WE ACCOMODATE THE POSSIBILITY OF DIFFRACTION BY SUPPOSING THAT THE BEAM SIZE PARAMETER w IS NOT A CONSTANT, BUT VARIES "SLOWLY" WITH z. WHEN WE PLUG OUR TRIAL SOLUTION INTO THE WAVE EQUATION WE WILL FIND OUT WHAT 'SLOWLY' MEANS.

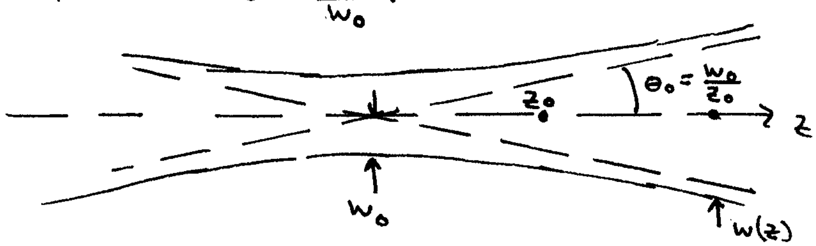
We now go into things in detail.

The width, w, will be a function of the position z along the beam, but there will be a minimum width, $\underline{w_0 \equiv \text{waist}}$ at the focal plane.

The beam will have a diffraction angle $\theta_0 \sim \frac{\lambda}{w_0}$.

The beam will have a 'depth of focus', z_0 where $\theta_0 \sim \frac{w_0}{z_0}$,

and so $z_0 \approx \frac{w_0^2}{\lambda} \sim \frac{\lambda}{\theta_0^2}$ etc.



The depth of focus, z_0, is also called the Rayleigh range in the laser community.

We expect that the desired solution to Maxwell's equation will contain all of these qualitative features.

We can anticipate a technical difficulty at once. Suppose the wave is linearly polarized in the $\hat{x}$ direction. Then if we write

$$\bar{E} = \hat{x} E_0 \, f(x,y,z) \, e^{i(kz-\omega t)}$$

we see that this cannot satisfy the Maxwell equation $\bar{\nabla}\cdot\bar{E} = 0$ unless $\partial f/\partial x = 0$. We certainly want $\partial f/\partial x \neq 0$, so the beam can have a limited extent in the x (and y) direction. It turns out that this implies we must have some E_z as well!

It is simpler to turn our attention to the vector potential, $\bar{A}$, which obeys $\bar{\nabla}\cdot\bar{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0$, so we don't require $\bar{\nabla}\cdot\bar{A} = 0$, and we can consider $\bar{A} = A_x \hat{x}$ only.

We seek a solution to Maxwell's equations: $\bar{A}(\bar{r},t) = \hat{x}\,\psi(\bar{r})\,g(\varphi)\,e^{i\varphi}$

where $\varphi = kz - \omega t$ is the plane-wave phase, and ψ and g are 'slowly' varying.

The factor $g(\varphi)$ represents a temporal pulse shape. Real beams do not last forever!

The wave equation for $\bar{A}$ in free space is $\nabla^2\bar{A} = \frac{1}{c^2}\frac{\partial^2\bar{A}}{\partial t^2}$.

Inserting our trial solution into this, we find that

$$\nabla^2\psi + 2ik\frac{\partial\psi}{\partial z}\left(1 - i\frac{g'}{g}\right) = 0 \qquad \text{where} \quad g' \equiv \frac{dg}{d\varphi}.$$

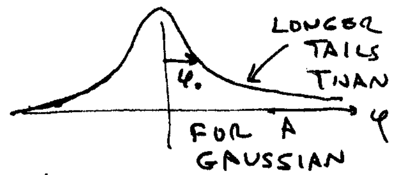
SINCE $\psi = \psi(\bar{r})$ WHILE g IS A FUNCTION OF THE PHASE $\varphi = kz - \omega t$, THIS EQUATION CANNOT BE SATISFIED IN GENERAL.

TO PROCEED FOR A BEAM OF FINITE PULSE LENGTH, WE MUST HAVE

$$|g'/g| << 1.$$

SURPRISINGLY, A GAUSSIAN PULSE SHAPE IN φ DOES NOT SATISFY THIS:

SUPPOSE $g = e^{-\varphi^2/\varphi_0^2}$ THEN $g' = -\frac{2\varphi}{\varphi_0^2} g$ AND $|g'/g| = \frac{|\varphi|}{\varphi_0^2}$ WHICH IS BIG FOR LARGE $|\varphi|$ (WHERE THE PULSE IS SMALL.)



A BETTER FORM FOR THE PULSE SHAPE IS $g = \operatorname{sech}\left(\frac{\varphi}{\varphi_0}\right)$

$$\left|\frac{g'}{g}\right| = \frac{1}{\varphi_0}\left|\tanh\frac{\varphi}{\varphi_0}\right|$$ WHICH IS $<<1$ EVERYWHERE SO LONG AS $\varphi_0 >> 1$.

NOTE: $\int_{-\infty}^{\infty} \operatorname{sech}\frac{\omega t}{\varphi_0}\, dt = \pi \frac{\varphi_0}{\omega}$ SO IF WE WRITE $\varphi_0 = \omega\tau$

THEN $\pi\tau$ IS THE EFFECTIVE FULL WIDTH OF THE PULSE.

FROM NOW ON, WE ASSUME THE TEMPORAL PROFILE OBEYS $|g'/g| << 1$

THEN $$\nabla^2\psi + 2ik\frac{\partial\psi}{\partial z} \approx 0$$

IT IS INSTRUCTIVE TO CHANGE TO SCALED VARIABLES AT THIS POINT:

LET $\xi = \frac{x}{w_0}$ $\quad v = \frac{y}{w_0}$ BUT $\zeta = \frac{z}{z_0}$, ANTICIPATING THAT w_0 AND z_0 ARE THE RELEVANT LENGTH SCALES FOR TRANSVERSE & LONGITUDINAL VARIATION IN ψ. THEN

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial v^2} + \frac{2ikw_0^2}{z_0}\frac{\partial\psi}{\partial\zeta} + \frac{w_0^2}{z_0^2}\frac{\partial^2\psi}{\partial\zeta^2} = 0$$

RECALL THAT $\frac{w_0}{z_0} \approx \Theta_0$ = DIFFRACTION ANGLE, AND $z_0 \approx \frac{w_0^2}{\lambda} \approx k w_0^2$

WE NOW DEFINE $\frac{w_0}{z_0} \equiv \Theta_0$ AND $z_0 \equiv \frac{kw_0^2}{2} = \pi\frac{w_0^2}{\lambda} = \frac{\lambda}{\pi\Theta_0^2}$

SINCE $\Theta_0 = \frac{\lambda}{\pi w_0} = \frac{2}{kw_0}$ ALSO

THEN $$\nabla_\perp^2\psi + 4i\frac{\partial\psi}{\partial\zeta} + \Theta_0^2\frac{\partial^2\psi}{\partial\zeta^2} = 0$$ WITH $\nabla_\perp \equiv \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial v^2}$

SINCE Θ_0^2 IS SMALL, THIS SUGGESTS A SERIES EXPANSION:

$$\psi = \psi_0 + \Theta_0^2\psi_2 + \Theta_0^4\psi_4 + \cdots$$

Plugging the series expansion into the differential equation, and collecting terms of order 1 and order θ_0^2, we find

$$\nabla_\perp^2 \psi_0 + 4i \frac{\partial \psi_0}{\partial \zeta} = 0 \qquad \text{('paraxial wave equation')}$$

and $$\nabla_\perp^2 \psi_2 + 4i \frac{\partial \psi_2}{\partial \zeta} = - \frac{\partial^2 \psi_0}{\partial \zeta^2}$$

An 'educated guess' for ψ_0 is $\psi_0 = h(\zeta)\, e^{-f(\zeta)\rho^2}$ where $\rho^2 = \xi^2 + v^2$

with $f(0) = 1 = h(0)$, measuring ζ from the waist of the beam

That is, we desire ψ_0 to be Gaussian in the transverse coordinate ρ, but the width of the Gaussian grows as $|\zeta|$ increases: $w(z) \approx \theta_0 z = w_0 \zeta \Rightarrow$ $Re(f) \sim \frac{1}{\zeta^2}$ since $f = \left[\frac{w_0}{w(\zeta)}\right]^2$. Also, fields fall off as $\frac{1}{\zeta}$ at large $\zeta \Rightarrow |h| \sim \frac{1}{\zeta}$.

Plugging in, we find $\quad -hf + ih' + \rho^2 h (f^2 - if') = 0$

For this to be true at all ρ, we must have $\begin{cases} f' = -if^2 \\ h' = -ihf \end{cases}$

We see that $h = f$ is a solution! [Paradox: implies $Re(f) \sim 1/\zeta^2$ but $|f| \sim 1/\zeta$]

From the first eq.: $\frac{f'}{f^2} = -i \Rightarrow \frac{1}{f} = a + i\zeta$, and $a = 1$ so $f(0) = 1$

That is, $$f = \frac{1}{1+i\zeta} = \frac{-i}{\zeta - i} = \frac{1-i\zeta}{1+\zeta^2} = \frac{e^{-i\tan^{-1}\zeta}}{\sqrt{1+\zeta^2}}$$

Indeed: $Re\, f = \frac{1}{1+\zeta^2}$, $|f| = \frac{1}{\sqrt{1+\zeta^2}}$, so can have $h = f$!

$$\underline{\underline{\psi_0 = f e^{-f\rho^2}}} = \frac{e^{-i\tan^{-1}\zeta}}{\sqrt{1+\zeta^2}}\, e^{\frac{i\zeta\rho^2}{1+\zeta^2}}\, e^{-\frac{\rho^2}{1+\zeta^2}}$$

Gaussian transverse profile.

Complicated phase

Falloff $\sim \frac{1}{z}$ in far zone

We will show later that the 'complicated phase' is required to satisfy what we know from elsewhere about phases.

People have deduced that $\psi_2 = \left(\frac{f}{2} - \frac{f^3\rho^4}{4}\right)\psi_0$

and also that $\psi_4 = \left(\frac{3f^2}{8} - \frac{3f^4\rho^4}{16} - \frac{f^5\rho^6}{8} + \frac{f^6\rho^8}{32}\right)\psi_0$ [Barton et al, J. Appl. Phys. 66, 2800 (1989)]

So far, we have $\bar{A} = \hat{x}\, \psi_0\, g\, e^{i\varphi}$. To calculate the fields $\bar{E}$ & $\bar{B}$ we also need the scalar potential φ, which is related by the Lorentz condition, $\frac{\partial \phi}{\partial t} = -c\bar{\nabla}\cdot\bar{A}$.

Similar to the form of $\bar{A}$, we suppose $\phi(\bar{r}, t) = \Phi(\bar{r})\, g\, e^{i\varphi}$

Then $\frac{\partial \phi}{\partial t} = -\omega \phi \left(1 - \frac{i g'}{g}\right) \approx -\omega \phi$ if $|g'/g| \ll 1$

Then $\phi = -\frac{i}{k} \bar{\nabla}\cdot\bar{A}$ and (to first order) $\bar{A} = \hat{x}\, \psi_0\, g\, e^{i\varphi}$

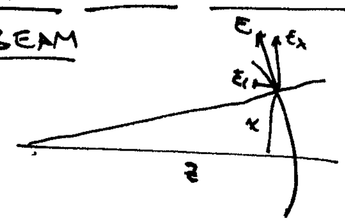
$\bar{E} = -\bar{\nabla}\phi - \frac{1}{c}\frac{\partial \bar{A}}{\partial t} \approx \frac{i}{k}\bar{\nabla}(\bar{\nabla}\cdot\bar{A}) + ik\bar{A}$, again ignoring terms in ig'/g.

$\bar{B} = \bar{\nabla}\times\bar{A}$

Plugging in $A_x = \psi_0\, g\, e^{i\varphi}$ (and dividing out an overall factor of ik) we find

$$E_x = \psi_0\, g\, e^{i\varphi}$$
$$E_y = 0$$
$$E_z = i\frac{\theta_0}{2}\frac{\partial \psi_0}{\partial \xi}\, g\, e^{i\varphi} = -i\theta_0 f \xi E_x$$
$$\left(\to i\frac{x}{z}E_x \text{ at large } z\right)$$
$$B_x = 0$$
$$B_y = E_x$$
$$B_z = -i\theta_0 f \upsilon E_x$$

First-order Gaussian beam



These expressions satisfy $\bar{\nabla}\cdot\bar{E} = 0 = \bar{\nabla}\cdot\bar{B}$ up to terms in θ_0^2.

For the record, adding term ψ_2 to $\bar{A}$ leads to

$$E_x = E_{x0}\left[1 + \theta_0^2\left(f^2\xi^2 - \frac{f^3\rho^4}{4}\right)\right] \quad \text{where } E_{x0} = \psi_0\, g\, e^{i\varphi} = f g\, e^{-f\rho^2} e^{i\varphi}$$
$$E_y = \theta_0^2 f^2 \xi \upsilon E_{x0}$$
$$E_z = -i\theta_0 f \xi E_{x0}\left[1 + \theta_0^2\left(f^2\rho^2 - \frac{f}{2} - \frac{f^3\rho^4}{2}\right)\right]$$
$$B_x = \theta_0^2 f^2 \xi \upsilon E_{x0} = E_y$$
$$B_y = E_{x0}\left[1 + \theta_0^2\left(f^2\upsilon^2 - \frac{f^3\rho^4}{4}\right)\right]$$
$$B_z = -i\theta_0 f \upsilon E_{x0}\left[1 + \theta_0^2\left(f^2\rho^2 - \frac{f}{2} - \frac{f^3\rho^4}{2}\right)\right]$$

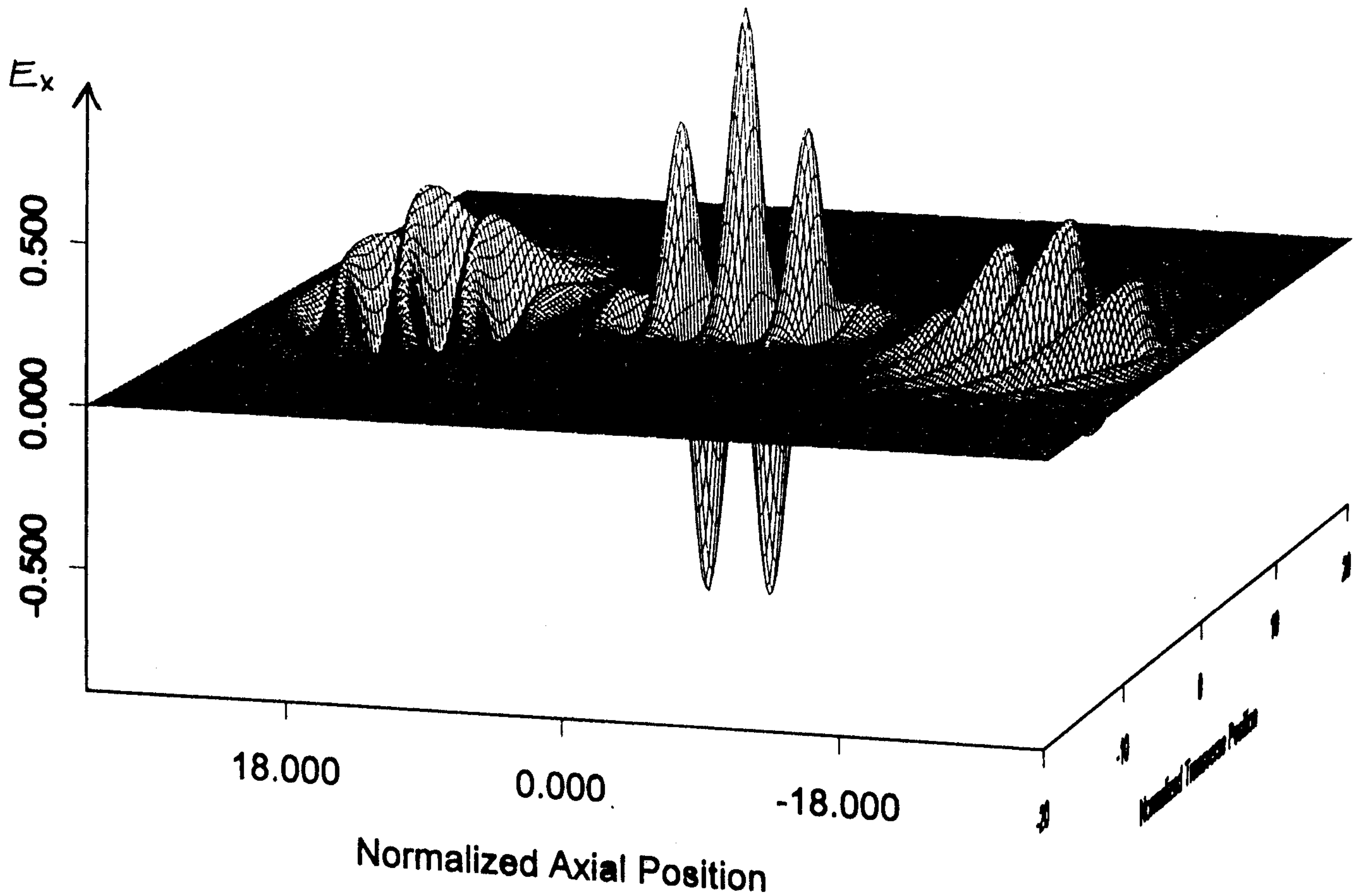
Third-order Gaussian beam

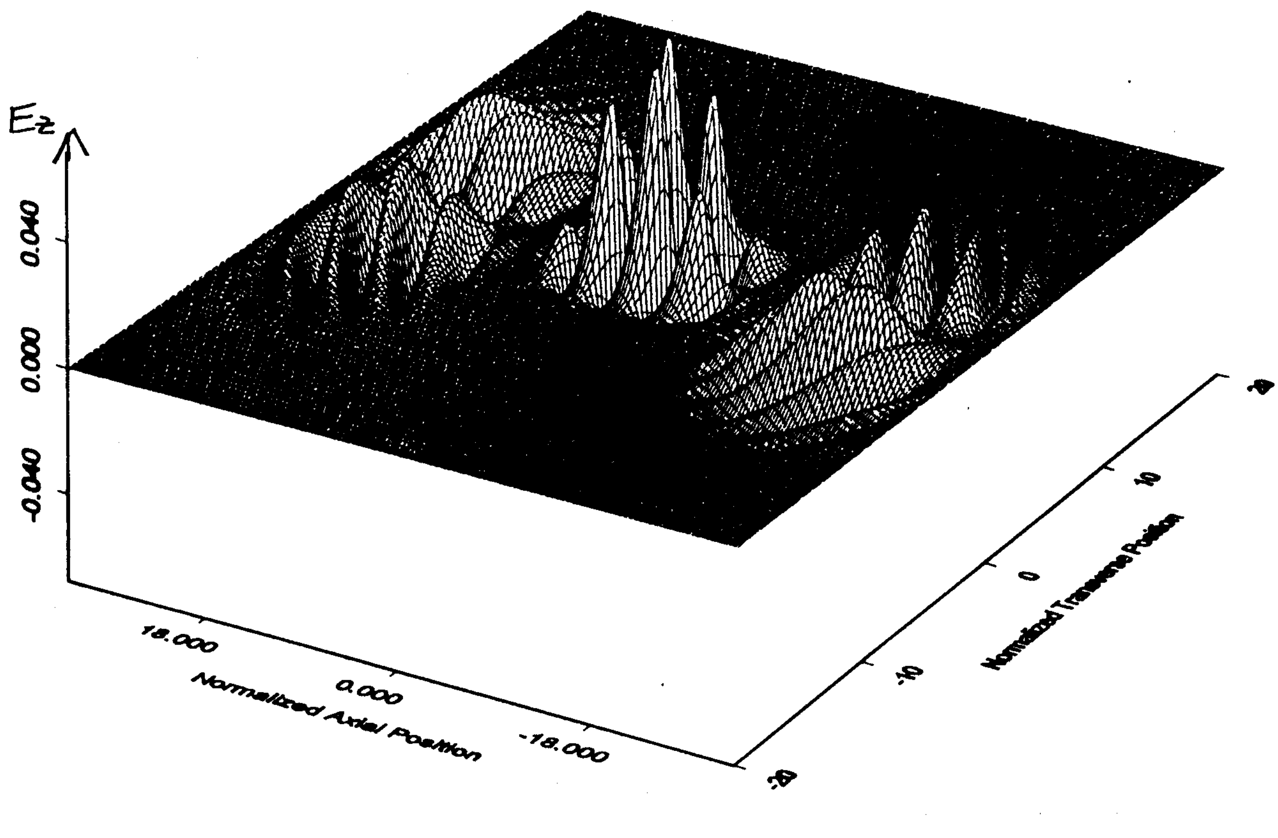
Remarks: The transverse profile is $e^{-\frac{\rho^2}{1+\xi^2}} = e^{-\frac{r_\perp^2}{w_0^2(1+z^2/z_0^2)}}$ where $r_\perp^2 = x^2+y^2$. If we write this as $e^{-r_\perp^2/w(z)^2}$, then $w(z) = w_0\sqrt{1+\frac{z^2}{z_0^2}}$ describes the width of the beam at z. For large z,

$$w(z) \to \frac{w_0 z}{z_0} = \theta_0 z.$$

The beam grows with z, and fills the diffraction cone, as shown on p. 210b.

Plots of a first-order Gaussian beam:





WHAT ABOUT THE PHASE FACTORS:

$$e^{-i\tan^{-1}s}\, e^{\frac{i s \rho^2}{1+s^2}} = e^{-i\tan^{-1}\frac{z}{z_0}}\, e^{\frac{i z z_0 r_\perp^2}{w_0^2(z^2+z_0^2)}}$$

AT LARGE z THIS BECOMES $e^{-i\frac{\pi}{2}}\, e^{\frac{i k r_\perp^2}{2z}}$ USING $z_0 = \frac{k w_0^2}{2}$

THIS MULTIPLIES THE FACTOR $e^{i\varphi} = e^{i(kz-\omega t)}$

ALTOGETHER: $-i\, e^{i\left[k\left(z + \frac{r_\perp^2}{2z}\right) - \omega t\right]}$

NOW $z + \frac{r_\perp^2}{2z} \approx \sqrt{z^2 + r_\perp^2} = r \Rightarrow$ PHASE FACTOR IS $-i\, e^{i(kr-\omega t)}$

$\Rightarrow$ AT LARGE z, THE PHASE FRONT (= WAVE FRONTS) ARE SPHERICAL

AT SMALL z WE HAVE $e^{-i\frac{z}{z_0}}\, e^{\frac{i z r_\perp^2}{z_0 w_0^2}} = e^{-ikz\frac{\theta_0^2}{2}\left(1-\frac{r_\perp^2}{w_0^2}\right)}$

USING $z_0 = \frac{2}{k\theta_0^2}$. COMBINING THIS WITH $e^{i(kz-\omega t)}$, WE HAVE

$$e^{i\left[kz\left(1-\frac{\theta_0^2}{2}\left(1-\frac{r_\perp^2}{w_0^2}\right)\right)-\omega t\right]}$$

THE PHASE FRONTS ARE PLANES $\Leftrightarrow$ PLANE WAVES. (REALLY ONLY FOR $r_\perp \ll w_0$

THE EFFECTIVE WAVE NUMBER IS $k_{eff} = k\left(1 - \frac{\theta_0^2}{2}\left(1-\frac{r_\perp^2}{w_0^2}\right)\right)$.

THE PHASE VELOCITY IS $v_p = \frac{\omega}{k_{eff}} = \frac{c}{1-\frac{\theta_0^2}{2}\left(1-\frac{r_\perp^2}{w_0^2}\right)} > c$.

THE PHASE STRUCTURE DESCRIBED ABOVE IS PROBABLY NOT ACCURATE FOR $r_\perp > w_0$. INDEED, THE FIRST-ORDER GAUSSIAN APPROXIMATION IS DOUBTFUL AT LARGE $r_\perp$, BECAUSE IT DOES NOT REVEAL THE AIRY DARK RINGS THAT SURROUND A REAL FOCAL SPOT.

IF WE GO TO THE THIRD-ORDER SOLUTION, p 210a, THEN AT $z=0$, THE WAVE INTENSITY IS $I \sim E_x^2$ (TO ORDER θ_0^2) $\sim 1 - \frac{\theta_0^2 r_\perp^2}{2 w_0^2}$,

WHICH HAS A SINGLE DARK RING AT $r_\perp = \sqrt{2}\frac{w_0}{\theta_0} = \sqrt{2}\,\pi\frac{w_0^2}{\lambda}$.

I INFER THAT THE 3RD-ORDER SOLUTION WOULD GIVE A REASONABLE APPROXIMATION TO THE PHASE FOR $r_\perp \lesssim w(z)\frac{w_0}{\lambda}$ ....

Ph 206 Lecture 17

Phase & Group Velocity in more detail:

As on p. 210g, $\varphi_{TOT} = e^{i\left[kz - \tan^{-1}\frac{z}{z_0} + \frac{z z_0 r_\perp^2}{w_0^2(z^2+z_0^2)} - \omega t\right]} \equiv e^{i(h(z)-\omega t)}$

We can expand $h(z)$ about a given value of z, say z_1: $h(z) \approx h(z_1) + \frac{\partial h}{\partial z}(z_1)(z-z_1) + \cdots$

Thus, we can identify $\frac{\partial h}{\partial z}(z_1)$ as $k_{eff}(z_1)$, the effective wave number at z_1.

Then $v_p(z_1) \doteq \frac{\omega}{k_{eff}(z_1)}$ = phase velocity at z_1

It is useful to recall that $z_0 = kw_0^2/2$, so $h = kz - \tan^{-1}\frac{2z}{kw_0^2} + \frac{kzr_\perp^2}{z^2+k^2w_0^4/4}$

$$\frac{dh}{dz} = k - \frac{kw_0^2/2}{z^2+(kw_0^2/2)^2} + \frac{kr_\perp^2}{2(z^2+(kw_0^2/2)^2)} - \frac{kz^2r_\perp^2}{(z^2+(kw_0^2/2)^2)^2}$$

Hence $v_p(z) = c\Big/\left[1 - \frac{w_0^2}{2(z_0^2+z^2)}\left(1 - \frac{r_\perp^2\, z_0^2}{w_0^2(z_0^2+z^2)}\right)\right]$ (agrees with p. 210g for $z \ll z_0$)

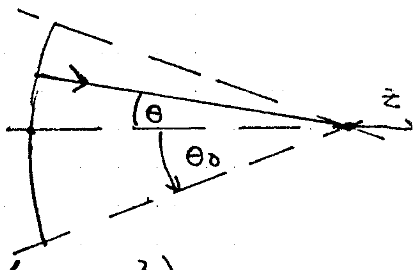
Suppose we now consider v_g as $\frac{d\omega}{dk_{eff}} = \frac{1}{dk_{eff}/d\omega} = \frac{1}{d^2h/dz\,d\omega} = \frac{c}{d^2h/dz\,dk}$

We use $\frac{dh}{dz}(z,k)$ as above – which involves the assumption that the waist, w_0, is not a function of k, $\Rightarrow$ $\theta_0 = \frac{2}{kw_0}$ varies with k.

Then "v_g" $= c\Big/\left[1 + \frac{w_0^2}{2(z_0^2+z^2)}\left(\frac{z_0^2-z^2}{z_0^2+z^2} - \frac{r_\perp^2}{w_0^2}\,\frac{z_0^4-6z_0^2z^2+z^4}{(z_0^2+z^2)^2}\right)\right]$

For $\begin{cases} z \approx 0 \\ r_\perp < w_0 \end{cases}$ this gives "v_g" $\approx c\Big/\left[1 + \frac{\theta_0^2}{2}\left(1 - \frac{r_\perp^2}{w_0^2}\right)\right] < c$, but "$v_g$" $> c$ for $|z| > z_0$!

For $|z| > z_0$, the wave involves both transverse and longitudinal energy flow, and $v_g = \frac{c}{d^2h/dz\,dk}$ is not a good measure of purely longitudinal flow.

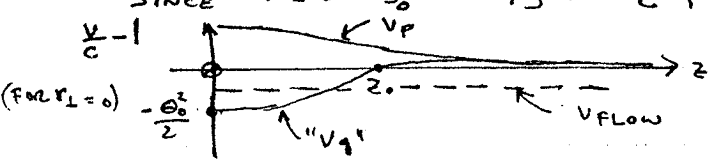


Consider the converging wave as sketched. The wave front has velocity $v_p \approx c$, so the energy at angle θ has $v_z = c\cos\theta < c$. The average z velocity of the energy flow is therefore

$$v_{FLOW} = \frac{c\int_{\cos\theta_0}^{1}\cos\theta\, d\cos\theta}{\int d\cos\theta} = \frac{c}{2}\,\frac{1-\cos^2\theta_0}{1-\cos\theta_0} = \frac{c}{2}(1+\cos\theta_0) = c\left(1 - \frac{\theta_0^2}{4}\right) < c$$

Near the focus, the wave is more like a plane wave, and "v_g" should be meaningful. But v_{FLOW} is the average of "v_g" for $0 < r_\perp \lesssim w_0$.

Since $\langle r_\perp^2\rangle = \int_0^{w_0^2} r_\perp^2\, dr_\perp^2 \Big/ \int dr_\perp^2 = \frac{w_0^2}{2}$, $v_{FLOW} \approx c\left(1 - \frac{\theta_0^2}{4}\right)$ for $z \approx 0$ also.



See also, E. Esarey et al, J. Opt. Soc. Am. B12, 1695 (19

THERE IS YET ANOTHER SUBTLETY TO THE PHASE STRUCTURE.

AT $z=0$, $E_x(0) = e^{-i\omega t}$

AT LARGE z, $E_x(r) \approx \dfrac{-i\, e^{i(kr-\omega t)}}{\sqrt{1+\frac{z^2}{z_0^2}}} \approx -\dfrac{i z_0 e^{i(kr-\omega t)}}{r} = -i\dfrac{z_0}{r} E_x(0)\, e^{ikr}$

[CAN USE ENERGY CONSERVATION TO VERIFY THAT THE MAGNITUDE IS RIGHT:

CONSIDER A PULSE OF LENGTH Δz. $U \sim \dfrac{E^2 \cdot \text{VOL}}{8\pi} = \dfrac{E^2(0)\, w_0^2 \Delta z}{8\pi} = \dfrac{E^2(r)\,(r\theta_0)^2 \Delta z}{8\pi}$

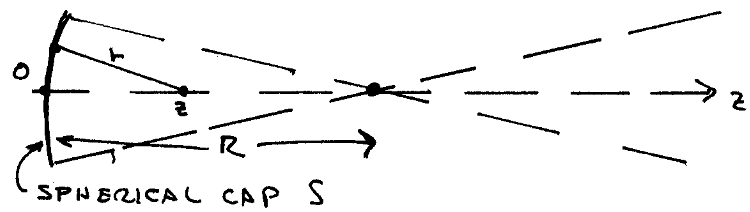
$\Rightarrow E(r) \sim \dfrac{w_0}{\theta_0 r} E(0) = \dfrac{z_0}{r} E(0)$]

BUT THERE IS A 90° PHASE SHIFT BETWEEN $E_x(0)$ AND $E_x(r)$! $r \gg z_0$

THIS IS SOMETIMES CALLED THE GUOY PHASE SHIFT (1892).

WE HAVE ALREADY SEEN THE ESSENCE OF THIS EFFECT IN OUR DISCUSSION OF THE KIRCHKOFF DIFFRACTION INTEGRAL, WHERE THE SECONDARY WAVELET ENTERS WITH A FACTOR $\frac{1}{i}$. (p.202)

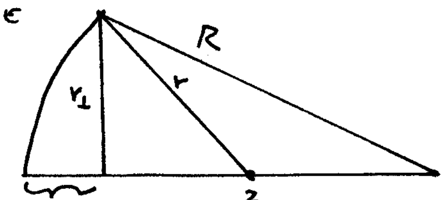
WE APPLY THIS TO A FOCUSSED BEAM (A BIT TRICKY!)



FOR A POINT AT z ALONG THE AXIS (MEASURED FROM A FAR-ZONE SURFACE AT LEFT),

$$\psi(z) \approx \frac{1}{i}\int_S \psi_s \frac{e^{ikr}}{r} \sim \frac{\psi_s}{iz}\int_0^\infty dr_\perp^2\, e^{ik\left(z + \frac{r_\perp^2}{2}\left(\frac{1}{z}-\frac{1}{R}\right)\right)}$$

INTEGRATE OVER SURFACE OF CONSTANT PHASE, SO CAN PULL ψ_s OUT.



$R - \sqrt{R^2 - r_\perp^2} \approx \dfrac{r_\perp^2}{2R}$

$r = \sqrt{\left(z - \frac{r_\perp^2}{2R}\right)^2 + r_\perp^2} \approx \sqrt{z^2 - \frac{z r_\perp^2}{R} + r_\perp^2}$

$\approx z + \dfrac{r_\perp^2}{2z}\left(1 - \dfrac{z}{R}\right) = z + \dfrac{r_\perp^2}{2}\left(\dfrac{1}{z} - \dfrac{1}{R}\right)$

FOR z VERY CLOSE TO R, i.e., AT FOCUS, THE INTEGRAND IS $\approx e^{ikz}$

$\Rightarrow \psi(R) \sim \dfrac{e^{ikz}}{i} = e^{-i\pi/2} e^{ikz}$

OTHERWISE, $\psi(z) \sim \dfrac{\psi_s e^{ikz}}{-kz} \dfrac{\left(e^{\infty} - 1\right)}{\left(\frac{1}{z}-\frac{1}{R}\right)}$ IGNORE RAPID OSC. AT UPPER LIMIT

SO FOR $z<R$, $\psi(z) \sim e^{ikz}$

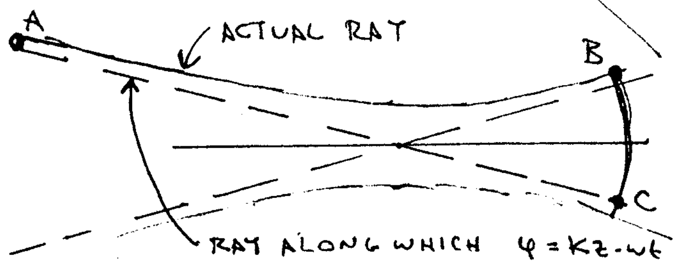
BUT FOR $z>R$ $\psi(z) \sim -e^{ikz} = e^{-i\pi} e^{ikz}$ $\Rightarrow$ 90° PHASE LOSS ON WAY INTO FOCUS, + 90° MORE LOSS ON WAY OUT!

[WAVES RADIATED BY A 'POINT' SOURCE DO NOT SUFFER THIS PHASE SHIFT.]

ANOTHER VERSION USING KIRCHOFF: START FROM FOCAL PLANE, ($z=0$ AGAIN) AND CALCULATE ψ IN FAR FIELD.

$$\psi_z \sim \frac{k}{i}\int_S \psi_s \frac{e^{ikr}}{r} \propto \frac{k}{i}\frac{e^{ikr}}{r}\int_0^\infty e^{-\frac{r_\perp^2}{w_0^2}}\,dr_\perp^2 = \frac{kw_0^2}{i}\frac{e^{ikr}}{r} = -i\frac{z_0}{2r}e^{ikr}$$

SINCE ψ_s IS SIGNIFICANT ONLY FOR $r_\perp \lesssim w_0$, EVEN THE PHASE e^{ikr} DOESN'T VARY MUCH WITH $r_\perp$.

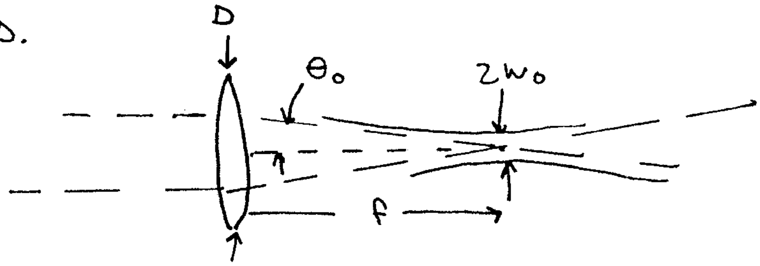


YET ANOTHER VIEW: DIFFRACTION 'BENDS' THE RAYS $\Rightarrow$ OPTICAL PATH ALONG AB $<$ PATH ALONG AC. $\Rightarrow$ PHASE LAG

TO BE QUANTITATIVE, CONSIDER THE RAY ALONG THE WAIST: $r_\perp = w_0 \sqrt{1 + z^2/z_0^2}$ ....
DETAILS MESSY [R.W. BOYD, J. OPT. SOC. AM. 70, 877 (1980)], BUT GUOY PHASE FOLLOWS.

MUCH LORE ABOUT GAUSSIAN BEAMS IS COLLECTED IN THE BOOK <u>LASERS</u> BY SIEGMAN (UNIVERSITY SCIENCE, 1986).

EX: FOCUSING OF A GAUSSIAN BEAM BY A LENS OF FOCAL LENGTH f AND APERTURE D.



WHAT IS THE WAIST, w_0?

WORK BACKWARDS: WAIST IS A DISTANCE f FROM LENS $\Rightarrow$ BEAM WIDTH $\sim 2f\theta_0$ AT LENS. TO CONTAIN THE BEAM, NEED

$$D \sim 4f\theta_0 = \frac{4f\lambda}{\pi w_0}$$

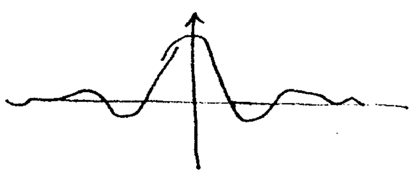
$$\Rightarrow w_0 \sim \frac{4}{\pi}\frac{f}{D}\lambda$$

ETC.

'DIFFRACTION FREE' BEAMS:

THE PARAXIAL WAVE EQUATION, $\nabla_\perp^2 \psi + 2ik\frac{\partial}{\partial z}\psi = 0$, HAS A FORMAL SOLUTION

$$\psi = J_0(k_\perp r_\perp)\, e^{-ik_\perp^2 z/2k} \cdot e^{i(kz-\omega t)} \qquad k_\perp \ll k$$



THIS BEAM DOES NOT DIFFRACT - BUT IT IS NOT REALLY WELL CONTAINED TRANSVERSELY: $J_0(x) \to \frac{1}{\sqrt{x}}$ FOR LARGE x

$\Rightarrow$ ARBITRARILY LARGE FRACTION OF PULSE ENERGY OUTSIDE ANY GIVEN RADIUS....

Radial Polarized Mode

On p. 210e we considered $A_x = \psi_0$ and deduced the fields of a laser beam dominantly polarized along x. If we take $A_y = \psi_0$, clearly we get a beam dominantly polarized in y. $\bar{A} = \psi_0(\hat{x} \pm i\hat{y})$ would lead to circular polarization...

What about $A_z = \psi_0 = f e^{-f\rho^2} g e^{i(kz-\omega t)}$?

$$\bar{\nabla}\cdot\bar{A} = \frac{\partial A_z}{\partial z} \approx ikA_z + g e^{i\varphi}\frac{\partial}{\partial z}\left(f e^{-f\rho^2}\right) = ik\left(A_z - \frac{i g e^{i\varphi}}{k z_0}\frac{\partial}{\partial \varsigma} f e^{-f\rho^2}\right)$$

$$= ik\left(A_z - i\frac{\Theta_0^2}{2} g e^{i\varphi} f'(1-f\rho^2)e^{-f\rho^2}\right) = ikA_z\left(1 - \frac{\Theta_0^2}{2} f(1-f\rho^2)\right) \quad \text{using } f' = -if^2$$

Then $\bar{E} = ik\bar{A} + \frac{i}{k}\bar{\nabla}(\bar{\nabla}\cdot\bar{A})$

$$E_x = \frac{i}{k}\frac{\partial}{\partial x}(\bar{\nabla}\cdot\bar{A}) = \frac{i}{k w_0}\frac{\partial}{\partial \xi}(\bar{\nabla}\cdot\bar{A}) = i\frac{\Theta_0}{2}\left[ik(-2f\xi)A_z + O(\Theta_0^2)\right]$$

$$\approx k\Theta_0 \frac{x}{w_0} f^2 e^{-f\rho^2} e^{i\varphi}$$

$$E_y \approx k\Theta_0 \frac{y}{w_0} f^2 e^{-f\rho^2} e^{i\varphi}$$

$$E_z = ikA_z + \frac{i}{k}\frac{\partial}{\partial z}(\bar{\nabla}\cdot\bar{A}) = ikA_z + \frac{i}{k}(ik)\left[\left(1 - \frac{\Theta_0^2}{2} f(1-f\rho^2)\right)\frac{\partial A_z}{\partial z} + \underbrace{\underbrace{\frac{kA_z}{kz_0}}_{\Theta_0^2/2}\frac{\partial}{\partial \varsigma}\underbrace{\left(1 - \frac{\Theta_0^2 f(1-f\rho^2)}{2}\right)}_{O(\Theta_0^2/2)}}_{O(\Theta_0^4) \text{ so ignore}}\right]$$

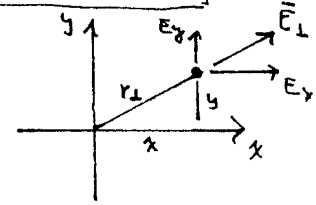
$$\approx -ikA_z\left(1 - \frac{\Theta_0^2}{2} f(1-f\rho^2)\right)^2 \approx ik\Theta_0^2 f^2(1-f\rho^2)e^{-f\rho^2}e^{i\varphi}$$

Divide by $k\Theta_0$:

$$E_x = \frac{x}{w_0} f^2 e^{-f\rho^2} e^{i\varphi}, \quad E_y = \frac{y}{w_0} f^2 e^{-f\rho^2} e^{i\varphi}, \quad E_z = i\Theta_0 f^2(1-f\rho^2)e^{-f\rho^2}e^{i\varphi}$$

$$\bar{E}_\perp = \frac{(x\hat{x} + y\hat{y})}{w_0} f^2 e^{-f\rho^2} e^{i\varphi} = \frac{r_\perp \hat{r}_\perp}{w_0} f^2 e^{-f\rho^2} e^{i\varphi} = \rho\,\hat{r}_\perp f^2 e^{-f\rho^2} e^{i\varphi}$$

<u>Radial polarization</u>



At large ς, the transverse field peaks at $\rho = \frac{\varsigma}{\sqrt{2}}$, or $\theta = \frac{r_\perp}{z} = \frac{\Theta_0}{\sqrt{2}}$

Near this peak $E_\perp \sim \rho f^2 \sim \frac{1}{\varsigma}$ for large ς, as expected.

Also $\frac{E_z}{E_{r_\perp}} \rightarrow -i\Theta_0 f\rho = -\Theta_0\frac{\rho}{\varsigma} = -\frac{r_\perp}{z}$ at large ς, as expected for a spherical wavefront.

The transverse fields show a Guoy phase shift of $e^{-2i\tan^{-1}\varsigma}$ from the factor f^2.

The total phase shift between $z = 0$ and ∞ is $180°$.

$$E_z(\rho=0) \sim i\Theta_0 f^2 e^{i\varphi} \sim 1/\varsigma^2 \text{ for } \varsigma \gg 1$$

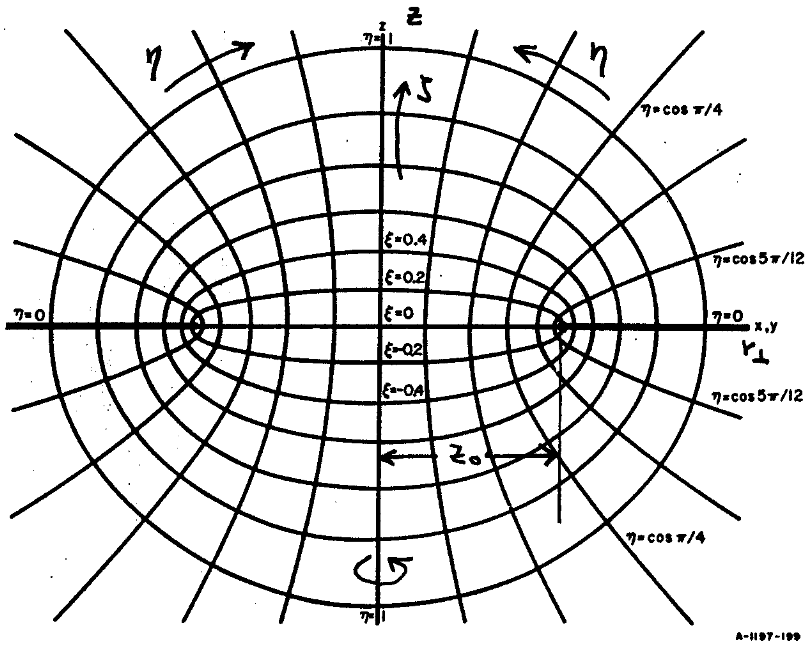
$$B_\perp = B_z = 0, \quad B_\phi = E_\perp$$

Ph 206 Lecture 17

210k

Higher Modes of Laser Beams

Besides the fundamental Gaussian mode deduced above, laser beams can arise in an of a sequence of modes. At least one of these higher modes is of physical interest to the topic of laser acceleration of particles. Further, a systematic analysis of these modes is of some pedagogical interest, being a good example of the advantages of a curvilinear coordinate system.



Indeed, on examining the famous set of 11 coordinate systems in which $\nabla^2\psi = 0$ is separable, we note that oblate spheroidal coordinates seem well matched to the physics of Gaussian laser beams.

This coordinate system is based on the oblate spheroid, and hyperboloids of revolution, derived from foci at $x = \pm z_0$, rotating the curves about the z-axis.

Analyses of the Helmholtz wave equation, $\nabla^2\psi + k^2\psi = 0$, in this coordinate system were developed in the 1930's (Stratton, Morse...), but applied mainly to somewhat esoteric antenna problems. Much of these notes is abstracted from the book 'Spheroidal Wave Functions' by C. Flammer (1957). The key formulae from this book appear in Chap. 21 of Abramowitz & Stegun.

From the figure, we can note an important fact, perhaps not otherwise readily discernable: Gaussian laser beams are based on solutions to the wave equation in situations where there is an iris of radius z_0. The first-order Gaussian beams are further based on the approximation that the wave is large only close to the z-axis, for $z \approx 0$. That is, the characteristic waist, w_0, of the beam is small compared to z_0:

That is $\quad \theta_0 = \dfrac{w_0}{z_0} \ll 1.$

In general, $w_0 > \lambda$, so an equivalent condition is $\underline{kz_0 \gg 1}$.

As has been hinted earlier, the Gaussian beams are not very accurate at transverse coordinates $r_\perp > w_0$ (for $z < z_0$).

Historically, interest in Gaussian laser modes grew out of a physical situation in which the beams were confined in z by (curved) mirrors, but without any aperture at the symmetry plane $z=0$.

That these Gaussian modes are based on the tacit assumption of an iris of radius z_0, the Rayleigh range, has generally escaped notice. For very tight focusing, $z_0 \sim \lambda$, the Gaussian beams become a rather poor approximation.

Notation for oblate spheroidal coordinates is not entirely standard. We follow Flammer – but soon make a change of variable of the more 'confusing' coordinate, η, to one of more obvious significance to laser beams.

The oblate spheroidal coordinates are (ς, η, ϕ).

ς (called ξ in Flammer) goes over to spherical coordinate $\frac{r}{z_0}$ for large ς, and can be called the radial coordinate.

η goes over to $\cos\theta$ of a spherical coord. system for large r, and can be called an angular coordinate.

ϕ is the familiar azimuthal angle about the z axis.

The transformations to rectangular coordinates (x, y, z) are

$$\left.\begin{aligned} x &= z_0\sqrt{1+\varsigma^2}\,\sqrt{1-\eta^2}\,\cos\phi \\ y &= z_0\sqrt{1+\varsigma^2}\,\sqrt{1-\eta^2}\,\sin\phi \end{aligned}\right\} \quad r_\perp = \sqrt{x^2+y^2} = z_0\sqrt{1+\varsigma^2}\,\sqrt{1-\eta^2}$$

$$z = z_0\,\varsigma\,\eta$$

As mentioned above, the defining length z_0 will prove to have the significance of the Rayleigh range. Recall: $z_0 = \frac{k w_0^2}{2} = \frac{2}{k\theta_0^2}$

$\theta_0 = \frac{w_0}{z_0}$, so $k z_0 = \frac{2}{\theta_0^2}$, etc.

We now consider the scalar wave equation $\nabla^2\psi + k^2\psi = 0$.

Following the transformation procedure outlined in the Appendix, we find

$$\frac{\partial}{\partial\varsigma}(1+\varsigma^2)\frac{\partial\psi}{\partial\varsigma} + \frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial\psi}{\partial\eta} + \frac{\varsigma^2+\eta^2}{(1+\varsigma^2)(1-\eta^2)}\frac{\partial^2\psi}{\partial\phi^2} + (kz_0)^2(\varsigma^2+\eta^2)\psi = 0$$

We are interested in $kz_0 \gg 1$, and $\eta \approx 1$ (waves near the z axis)

It is helpful to make a change of variables:

$$\sigma = \frac{kz_0}{2}(1-\eta^2) = \frac{1}{\theta_0^2}(1-\eta^2) = \frac{z_0^2}{w_0^2}(1-\eta^2). \quad \to \frac{\sin^2\theta}{\theta_0^2} \text{ for large } r.$$

Note: $r_\perp^2 = z_0^2(1+\varsigma^2)(1-\eta^2) = w_0^2\,\sigma\,(1+\varsigma^2)$ so $\sigma = \frac{r_\perp^2}{w_0^2(1+\varsigma^2)} = \frac{\rho^2}{1+\varsigma^2}$.

$\eta^2 = 1 - \Theta_0^2 \sigma$ AND $\eta \approx 1 - \frac{\Theta_0^2}{2}\sigma$ FOR η NEAR 1

HENCE $\varsigma = \frac{z}{z_0 \eta} \approx \frac{z}{z_0}\left(1 + \frac{\Theta_0^2}{2}\sigma\right) = \frac{z}{z_0}\left(1 + \frac{r_\perp^2}{2(z^2+z_0^2)}\right)$

SO FOR LARGE z, $\varsigma \to \frac{z}{z_0}\left(1 + \frac{r_\perp^2}{2z^2}\right) \approx \frac{r}{z_0}$.

WAVE FRONTS OF CONSTANT ς, AS WILL BE FOUND, ARE THEREFORE SPHERICAL FOR LARGE r.

TO REPLACE η BY σ IN THE WAVE EQUATION, WE NEED THAT $2\eta\, d\eta = -\Theta_0^2 d\sigma$.

SO FOR $\eta \approx 1$, $d\eta \approx \frac{\Theta_0^2}{2} d\sigma$. THUS,

$$\frac{\partial}{\partial \varsigma}(1+\varsigma^2)\frac{\partial \psi}{\partial \varsigma} + \frac{4}{\Theta_0^2}\frac{\partial}{\partial \sigma}\sigma\frac{\partial \psi}{\partial \sigma} + \frac{1+\varsigma^2 - \Theta_0^2\sigma}{(1+\varsigma^2)\Theta_0^2\sigma}\frac{\partial^2\psi}{\partial\phi^2} + \frac{4}{\Theta_0^4}\left(1+\varsigma^2 - \Theta_0^2\sigma\right)\psi = 0.$$

THE WAVE EQUATION IS SEPARABLE: TRY $\psi = Z(\varsigma)\, S(\sigma)\, \Phi(\phi)$.

WE READILY ANTICIPATE THAT $\Phi(\phi) = e^{im\phi}$. HENCE, DIVIDING BY ψ,

$$\frac{1}{Z}\frac{d}{d\varsigma}(1+\varsigma^2)\frac{dZ}{d\varsigma} + \frac{4}{\Theta_0^2 S}\frac{d}{d\sigma}\sigma\frac{dS}{d\sigma} + \frac{4}{\Theta_0^4}(1+\varsigma^2) + \frac{m^2}{1+\varsigma^2} - \frac{4}{\Theta_0^2}\sigma - \frac{m}{\Theta_0^2\sigma} = 0 = \lambda_m - \lambda_m$$

ON INTRODUCING THE SEPARATION CONSTANT λ. THE SEPARATED EQUATIONS ARE

$$\frac{d}{d\varsigma}(1+\varsigma^2)\frac{dZ}{d\varsigma} = \left(\lambda_m - \frac{4}{\Theta_0^4}(1+\varsigma^2) - \frac{m^2}{1+\varsigma^2}\right)Z$$

$$\frac{d}{d\sigma}\sigma\frac{dS}{d\sigma} = -\frac{\Theta_0^2}{4}\left(\lambda_m - \frac{4\sigma}{\Theta_0^2} - \frac{m^2}{\Theta_0^2\sigma}\right)S$$

WE ATTACK THE 'ANGULAR' FUNCTION S FIRST.

IT IS HELPFUL TO RECALL OUR SOLUTION FOR THE FUNDAMENTAL GAUSSIAN BEAM. FROM THIS, WE EXPECT THAT S CONTAINS A FACTOR $e^{-\sigma}$ SINCE $\sigma = \frac{\rho^2}{1+\varsigma^2}$

ALSO, FROM EXPERIENCE WITH THE SCHRÖDINGER EQUATION, WE ANTICIPATE THAT S CONTAINS A FACTOR $r_\perp^m \sim \sigma^{m/2}$.

THAT IS, WE TRY $S = \sigma^{m/2} e^{-\sigma} L(\sigma)$.

PLUGGING INTO THE DIFFERENTIAL EQUATION FOR S, WE DEDUCE THAT

$$\sigma L'' + (m+1-2\sigma)L' + \nu L = 0 \quad \text{WHERE } \nu = \frac{\Theta_0^2}{4}\lambda_m - m - 1$$

OR. $\lambda_m = \frac{4}{\Theta_0^2}(\nu + m + 1)$

THIS IS THE DIFFERENTIAL EQUATION FOR GENERALIZED LAGUERRE POLYNOMIALS. (SEEN ALSO IN THE RADIAL PART OF THE SCHRÖDINGER EQUATION THESE ARE, HOWEVER, THE 'OFFICIAL' LAGUERRE POLYNOMIALS!)

WE LABEL THE SOLUTIONS FOR L BY INDEX n = ORDER OF HIGHEST POWER. BY DIRECT CALCULATION, ONE QUICKLY FINDS

$$\nu_0 = 0,\ L_0^m = 1 \ ;\ \nu_1 = 2,\ L_1^m = 1 - \frac{2}{m+1}\sigma \ ;\ \nu_2 = 4,\ L_2^m = 1 - \frac{4}{m+1}\sigma - \frac{4}{(m+1)(m+2)}\sigma^2 \ldots$$

IN GENERAL, $\nu_n = 2n$, SO $\lambda_{m,n} = \frac{4}{\theta_0^2}(2n+m+1)$

AND $S_{m,n}(\sigma) = \sigma^{m/2} e^{-\sigma} L_n^m(\sigma)$

WE NOW TURN TO THE Z EQUATION.

$Z(\zeta)$ SHOULD CONTAIN THE PHASE FACTOR e^{ikz} (LARGE z, SMALL θ)

SINCE $z = z_0\zeta\eta \approx z_0\zeta$, $kz \approx kz_0\zeta = \frac{2\zeta}{\theta_0^2}$.

THUS, WE WRITE $Z(\zeta) = F(\zeta)\, e^{\frac{2i\zeta}{\theta_0^2}}$

PLUGGING IN TO THE Z DIFFERENTIAL EQUATION, AND USING $\lambda_{m,n}$ FROM ABOVE, WE FIND

$$(1+\zeta^2)\left(F'' + \frac{4i}{\theta_0^2}F'\right) + 2\zeta\left(F' + \frac{2iF}{\theta_0^2}\right) = \left[\frac{4}{\theta_0^2}(2n+m+1) - \frac{m^2}{1+\zeta^2}\right]F$$

WE ARE CONTENT WITH SOLUTIONS IN THE REGIME $kz_0 \gg 1 \Rightarrow \frac{1}{\theta_0^2}$ BIG

KEEPING ONLY THE TERMS IN $1/\theta_0^2$, WE ARRIVE AT THE FIRST-ORDER DIFFERENTIAL EQUATION:

$$(1+\zeta^2)F' = -(\zeta + i(2n+m+1))F.$$

FOR $n = 0 = m$, WE KNOW THAT $F_0^0 = \frac{1}{1+i\zeta} = \frac{1-i\zeta}{1+\zeta^2} = \frac{e^{-i\tan^{-1}\zeta}}{\sqrt{1+\zeta^2}}$

AT LARGE ζ, $F \sim 1/\zeta$, WHICH WE EXPECT TO HOLD FOR ALL n AND m, SINCE OUR SOLUTIONS REPRESENT RADIATION FIELDS. SO F_n^m SHOULD BE RELATED TO F_0^0 BY AT MOST A PHASE CHANGE.

AFTER SOME PLAYING, WE FIND THAT A SUITABLE FORM IS

$$F_n^m = \frac{e^{-i a_{mn}\tan^{-1}\zeta}}{\sqrt{1+\zeta^2}} = \frac{\left(e^{i\tan^{-1}\zeta}\right)^a}{\sqrt{1+\zeta^2}} = \left(\frac{1-i\zeta}{\sqrt{1+\zeta^2}}\right)^a \frac{1}{\sqrt{1+\zeta^2}} = \frac{(1-i\zeta)^a}{(1+\zeta^2)^{\frac{a+1}{2}}}$$

(WHICH REVERTS TO F_0^0 FOR $a = 1$.)

Inserting this hypothesis for F_n^m into the differential equation, we find that it is satisfied for

$$\boxed{a_{min} = 2n + m + 1}$$

$$\boxed{Z_n^m = \frac{e^{-i(2n+m+1)\tan^{-1}\varsigma}\, e^{ikz_0\varsigma}}{\sqrt{1+\varsigma^2}}}$$

$$\rho = \frac{r_\perp}{w_0}, \quad z_0 = \frac{k w_0^2}{2}, \quad \theta_0 = \frac{w_0}{z_0}$$

$$\sigma = \frac{\rho^2}{1+\varsigma^2}, \quad \varsigma = \frac{z}{z_0}\left(1 + \frac{r_\perp^2}{2(z^2+z_0^2)}\right)$$

$$\boxed{\Psi_n^m = Z_n^m S_n^m e^{im\phi} = \frac{\sigma^{m/2} L_n^m(\sigma)}{\sqrt{1+\varsigma^2}}\, e^{im\phi}\, e^{-i(2n+m+1)\tan^{-1}\varsigma}\, e^{-\sigma}\, e^{ikz_0\varsigma}}$$

Cylindrical Gaussian modes.

By related procedures, one finds in rectangular coords (ξ, ν, ς),

$$\boxed{\Psi_{m,n} = \frac{H_m(\sqrt{2}\,\xi)\, H_n(\sqrt{2}\,\nu)}{\sqrt{1+\varsigma^2}}\, e^{-i(n+m+1)\tan^{-1}\varsigma}\, e^{-\sigma}\, e^{ikz_0\varsigma} \qquad \left(\xi = \frac{x}{w_0},\ \nu = \frac{y}{w_0}\right)}$$

Rectangular Gaussian modes

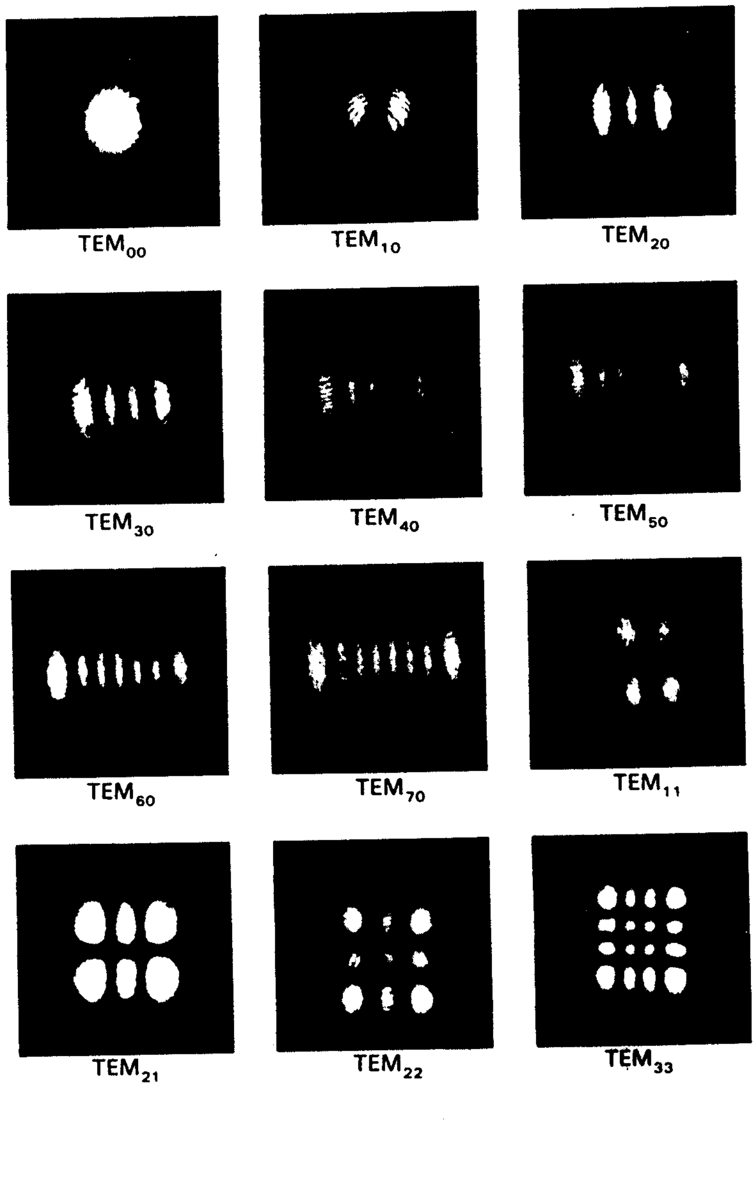
where $H_n(x)$ is a Hermite polynomial:

$$H_0 = 1, \quad H_1 = x, \quad H_2 = 4x^2 - 2 \ \ldots$$

The pictures are from laboratory production of various higher modes (Kogelnik & Rigrod, 1962).

The name "TEM" is descriptive only for $|z| \gg z_0$.

For $|z| \lesssim z_0$, the mode have both longitudinal $\bar{E}$ and $\bar{B}$ components, in general.



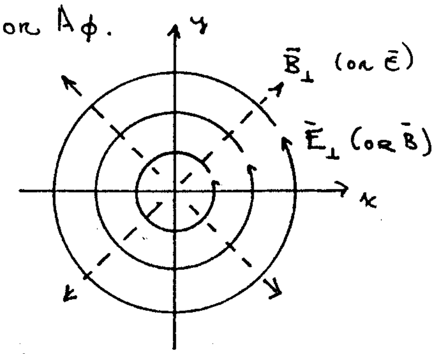
TEM_{00}, TEM_{10}, TEM_{20}, TEM_{30}, TEM_{40}, TEM_{50}, TEM_{60}, TEM_{70}, TEM_{11}, TEM_{21}, TEM_{22}, TEM_{33}

EXAMPLE: CYLINDRICAL MODE ψ_0^1

WE INTERPRET THE WAVE FUNCTION ψ_n^m AS A COMPONENT OF THE VECTOR POTENTIAL $\bar{A}$, AND RELATE THIS TO ELECTRIC & MAGNETIC FIELDS VIA

$$\bar{E} = ik\bar{A} + \frac{i}{k}\bar{\nabla}(\bar{\nabla}\cdot\bar{A}) \quad , \quad \bar{B} = \bar{\nabla}\times\bar{A} \qquad \text{(p. 210e)}$$

IN CYLINDRICAL COORDS, ψ_n^m COULD STAND FOR EITHER $A_{r_\perp}$ OR A_ϕ.



IF A_ϕ, THEN $\bar{E}_\perp$ CIRCULATES ABOUT THE z AXIS, & $\bar{B}_\perp$ IS RADIAL.
IF $A_{r_\perp}$, THEN $\bar{E}_\perp$ IS RADIAL AND $\bar{B}_\perp$ CIRCULATES ABOUT $\hat{z}$.

$\psi_0^0 \Rightarrow E_\perp, B_\perp \neq 0$ AT ORIGIN, WHICH CONTRADICTS $\bar{\nabla}\cdot\bar{E} = 0 = \bar{\nabla}\cdot\bar{B}$.

INDEED, PLUGGING IN A_ϕ OR $A_{r_\perp} = \psi_0^0$ LEADS TO BLOWUP AT ORIGIN.

$\Rightarrow$ SIMPLEST CYLINDRICAL MODE IS BASED ON $\psi_0^1 \sim \sigma^{1/2} \sim \frac{\rho}{\sqrt{1+\varsigma^2}} \sim r_\perp$, FOR WHICH THE TRANSVERSE $\bar{E}$ & $\bar{B}$ FIELDS VANISH ON THE z AXIS.

I. $$A_\phi = \frac{\psi_0^1}{ik} = \frac{\rho}{ik}\left(1 - \frac{\rho^2}{1+\varsigma^2}\right)\frac{e^{-\rho^2/1+\varsigma^2}}{1+\varsigma^2}\, e^{i\phi}\, e^{-2i\tan^{-1}\varsigma}\, e^{ikz_0\varsigma} \equiv \frac{\rho}{ik}E_0^1$$

DIVIDE BY ik TO MAKE E LOOK NICER.

$$\bar{\nabla}\cdot\bar{A} = \frac{1}{r_\perp}\frac{\partial A_\phi}{\partial\phi} = \frac{1}{w_0\rho}\frac{\partial A_\phi}{\partial\phi} = \frac{E_0^1}{kw_0} = \frac{\theta_0}{2}E_0^1$$

$$E_{r_\perp} = \frac{i}{k}\frac{\partial\,\bar{\nabla}\cdot\bar{A}}{\partial r_\perp} = \frac{i\theta_0}{2kw_0}\frac{\partial E_0^1}{\partial\rho} = i\frac{\theta_0^2}{4}\frac{\partial E_0^1}{\partial\rho}$$ (SMALL, BUT NEEDED SO E_z CAN FORM LOOPS)

$$E_\phi = \rho E_0^1 + \frac{i}{kw_0\rho}\frac{\partial}{\partial\phi}\left(\frac{\theta_0}{2}E_0^1\right) = \rho E_0^1 - \frac{\theta_0^2}{4}\frac{E_0^1}{\rho} \approx \underline{\rho E_0^1}$$

$$E_z = \frac{i}{k}\frac{\partial\,\bar{\nabla}\cdot\bar{A}}{\partial z} \approx \underline{-\frac{\theta_0}{2}E_0^1}$$ (SINCE LEADING z DEPENDENCE IS e^{ikz})

$$B_{r_\perp} = -\frac{\partial A_\phi}{\partial z} \approx -\rho E_0^1 = -E_\phi$$

$$B_\phi = 0$$

$$B_z = \frac{1}{r_\perp}\frac{\partial}{\partial r_\perp}r_\perp A_\phi = \frac{1}{ikw_0\rho}\frac{\partial}{\partial\rho}\rho^2 E_0^1 = -i\theta_0\left[1 - \frac{3\rho^2}{1+\varsigma^2} + \frac{\rho^4}{(1+\varsigma^2)^2}\right]\frac{e^{-\rho^2/1+\varsigma^2}}{1+\varsigma^2}\, e^{i\phi}\, e^{-2i\tan^{-1}\varsigma}\, e^{ikz_0\varsigma} \approx -i\theta_0 E_0$$

II. $$A_{r_\perp} = \psi_0^1/ik = \frac{\rho}{ik}E_0^1 \qquad \bar{\nabla}\cdot\bar{A} = \frac{1}{r_\perp}\frac{\partial}{\partial r_\perp}r_\perp A_{r_\perp} \approx -i\theta_0 E_0^1$$ (LIKE B_z JUST ABOVE)

$$E_{r_\perp} \approx \underline{\rho E_0^1}\ , \quad E_\phi \approx \theta_0^2\ , \quad E_z = \frac{i}{k}\frac{\partial\,\bar{\nabla}\cdot\bar{A}}{\partial z} \approx \underline{i\theta_0 E_0^1}$$

$$B_{r_\perp} = 0\ , \quad B_\phi = \frac{\partial A_{r_\perp}}{\partial z} = \rho E_0^1 = E_{r_\perp}\ , \quad B_z = -\frac{1}{r_\perp}\frac{\partial A_{r_\perp}}{\partial\phi} = \frac{-1}{ikw_0}\frac{\partial E_0^1}{\partial\phi} = -\frac{\theta_0}{2}E_0^1$$

REMARKS: THE LEADING TRANSVERSE FIELD VARY AS $\rho E_0^1 \Rightarrow$ VANISH ON AXIS, REACH A MAXIMUM AT $\rho \sim \varsigma/\sqrt{3}$ AND FALL OFF AT LARGE ρ. THE FACTOR $\rho \sim \varsigma/\sqrt{3}$ CANCELS ONE POWER OF ς IN THE DENOMINATOR FOR LARGE ς $\Rightarrow$ FIELDS VARY AS $1/\varsigma$

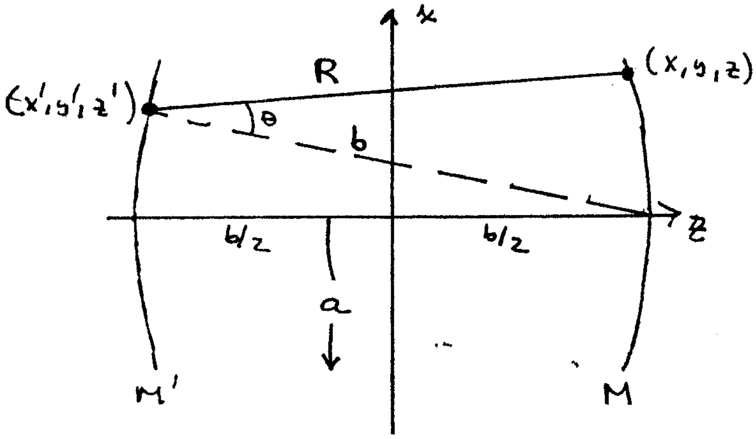
THE <u>GUOY PHASE SHIFT</u> IS $e^{-2i\tan^{-1}\varsigma} \to e^{-i\pi} = -1$ AT LARGE $\varsigma \Rightarrow 180°$ PHASE SHIFT BETWEEN $\varsigma = 0$ & ∞.

FOR A_ϕ, E_ϕ AND E_z ARE $180°$ OUT OF PHASE, BUT FOR $A_{r_\perp}$, $E_{r_\perp}$ AND E_z ARE $90°$ OUT OF PHASE.

Higher Modes via a Huygens-Kirchoff Integral

The higher laser modes became well known in the 1960's following an argument due to Boyd & Gordon, Bell. Syst. Tech. J. **40**, 489 (1961).

Inspired by practical design considerations of a laser oscillator, they considered a confocal resonant cavity. This is defined by



Two spherical mirrors of radii of curvature b, separated by b along the z axis. The mirrors are square of edge length $2a \ll b$.

The basic idea is that the fields at mirror M must have the same form as the fields at mirror M', for a stable radiation pattern bouncing back and forth between them.

Because width a is finite, some radiation 'leaks' out of the cavity sideways, and the amplitude of the fields decreases slightly in going from M' to M.

In rectangular coords, we look for fields at mirror M' of form

$$\bar{E} = E_0 \hat{x}\, f_m(x')\, g_n(y') \qquad (\text{on } M')$$, and then expect that

$$\bar{E} = \lambda_m \lambda_n E_0 \hat{x}\, f_m(x)\, g_n(y) \quad \text{on } M$$, where the eigenvalues λ_m, λ_n

are complex in general (phase change in going from M' to M), and have magnitude close to 1 for a finite.

But $\bar{E}(M)$ can also be related to $\bar{E}(M')$ by a Huygens-Kirchoff integral. It suffices to work in the Fraunhofer approximation for $a \ll b$: (p. 203)

$$\lambda_m \lambda_n f_m(x)\, g_n(y) \approx \frac{k}{2\pi i b} \int e^{ikR} f_m(x')\, g_n(y')\, dx'\, dy'$$, using $R \approx b$ in the denom.

Mirror M obeys $x^2+y^2+(z+b/2)^2 = b^2 \Rightarrow z \approx b/2 - \frac{x^2+y^2}{2b}$; likewise $z' \approx -b/2 + \frac{x'^2+y'^2}{2b}$

so $R = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \approx b - \frac{xx'+yy'}{b} + \cdots$

So $$\lambda_m \lambda_n f_m(x)\, g_n(y) = \frac{k e^{ikb}}{2\pi i b} \int_{-a}^{a} dx'\, f_m(x')\, e^{-\frac{ikxx'}{b}} \int_{-a}^{a} dy'\, g_n(y')\, e^{-\frac{ikyy'}{b}}$$

Clearly, we have $$\lambda_m f_m(x) = \sqrt{\frac{k e^{ikb}}{2\pi i b}} \int_{-a}^{a} dx'\, f_m(x')\, e^{-\frac{ikxx'}{b}}$$, an integral equation.

BOYD & GORDON HAPPENED TO KNOW THAT THE SOLUTION TO THIS INTEGRAL EQUATION IS

$$f_m(x) \sim S_{0,n}(x) \sim H_m(x)$$

WHERE $S_{0,n}$ IS THE FUNCTION INTRODUCED ON p. 210n, AND H_m IS A HERMITE POLYNOMIAL.

THEY ALSO KNEW THE EIGENVALUES, SO AFTER SOME TIDYING UP, THEY ARRIVED AT THE MODE FORM IN RECTANGULAR COORDS GIVEN ON p. 210o.

APPENDIX III

VECTOR RELATIONS IN CURVILINEAR COORDINATES

A list of vector operators expressed in the common orthogonal curvilinear coordinates is often useful in the solution of physical problems. For the derivation of these relations, it is possible to procede quite formally from the definition of the operator ∇ in Cartesian coordinates and the transformation equations to other coordinate systems, but for physical applications it is advantageous to work from the geometrical definitions of gradient, divergence, and curl. One may first specify the coordinate system and derive the required expressions, or make a general derivation valid for any curvilinear coordinates and only then specify the coordinates. We shall follow the latter plan, first outlining a derivation valid for any right-handed system of orthogonal coordinates for which the line element is known, and then writing the particular forms for Cartesian, cylindrical, and spherical polar coordinates.

A line element in three dimensions is an infinitesimal displacement in space. If only one of three orthogonal coordinates q_1, q_2, q_3 is varied, the corresponding line element may be written

$$ds_1 = h_1\,dq_1, \tag{1}$$

together with similar expressions in q_2 and q_3. For any infinitesimal displacement,

$$ds^2 = h_1^2\,dq_1^2 + h_2^2\,dq_2^2 + h_3^2\,dq_3^2. \tag{2}$$

Now the gradient of a scalar function ψ is defined by the requirement that

$$\nabla\psi \cdot d\mathbf{s} = d\psi, \tag{3}$$

giving the change in ψ corresponding to the space displacement $d\mathbf{s}$. Then

$$(\nabla\psi)_1 = \lim_{ds_1 \to 0} \frac{\psi(q_1 + dq_1) - \psi(q_1)}{ds_1} = \frac{1}{h_1}\frac{\partial\psi}{\partial q_1} \tag{4}$$

is the general form of a gradient component.

To find the divergence, we shall consider an infinitesimal volume $dv = ds_1\,ds_2\,ds_3$ bounded by the surfaces $q_1 =$ constant, $q_1 + dq_1 =$ constant, etc., as indicated in Fig. III-1. Let us apply Gauss's divergence theorem, $\int \nabla \cdot \mathbf{A}\,dv = \int \mathbf{A} \cdot d\mathbf{S}$, to a vector $\mathbf{A}(q_1, q_2, q_3)$ with components A_1, A_2, A_3, integrating over this infinitesimal volume. The integral of the outward normal component of $\mathbf{A}$ over the two surfaces

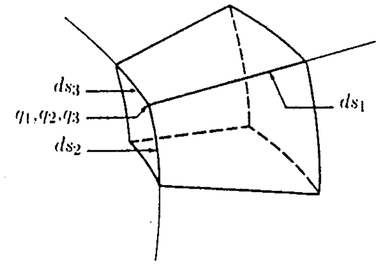


FIG. III-1. Element of volume for computing the divergence.

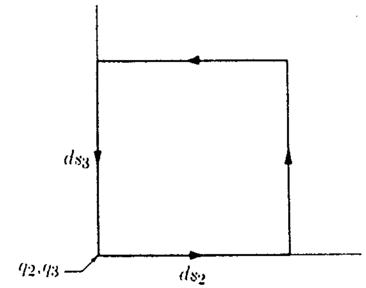


FIG. III-2. Element of area in the q_2, q_3 plane for finding the q_1 component of the curl. Arrows show the direction of the path of integration.

perpendicular to the direction of increasing q_1 is

$$\begin{aligned}(A_1\,ds_2\,ds_3)_{q_1+dq_1} - (A_1\,ds_2\,ds_3)_{q_1} &= \frac{\partial}{\partial q_1}(A_1\,ds_2\,ds_3)\,dq_1 \\ &= \frac{\partial(h_2h_3A_1)}{\partial q_1}\,dq_1\,dq_2\,dq_3 \\ &= \frac{1}{h_1h_2h_3}\frac{\partial(h_2h_3A_1)}{\partial q_1}\,dv,\end{aligned}$$

and analogous expressions hold for the other two sets of surfaces. Since the sum of these three terms is, by Gauss's theorem, equal to $\nabla \cdot \mathbf{A}\,dv$, the divergence is given explicitly by

$$\nabla \cdot \mathbf{A} = \frac{1}{h_1h_2h_3}\left(\frac{\partial(h_2h_3A_1)}{\partial q_1} + \frac{\partial(h_3h_1A_2)}{\partial q_2} + \frac{\partial(h_1h_2A_3)}{\partial q_3}\right). \tag{5}$$

The Laplacian of a scalar function can be written down immediately, since it is just the divergence of the gradient:

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial q_3}\right)\right]. \tag{6}$$

To obtain a particular component of the curl of a vector, we may apply Stokes' theorem to an infinitesimal area at right angles to the direction of the desired component. Consider the area defined by ds_2 and ds_3, as in Fig. III-2. By Stokes' theorem, $\oint \mathbf{A} \cdot d\mathbf{s} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$, which, in this application, becomes

$$(A_2\,ds_2)_{q_3} + (A_3\,ds_3)_{q_2+dq_2} - (A_2\,ds_2)_{q_3+dq_3} - (A_3\,ds_3)_{q_2} = (\nabla \times \mathbf{A})_1\,ds_2\,ds_3.$$

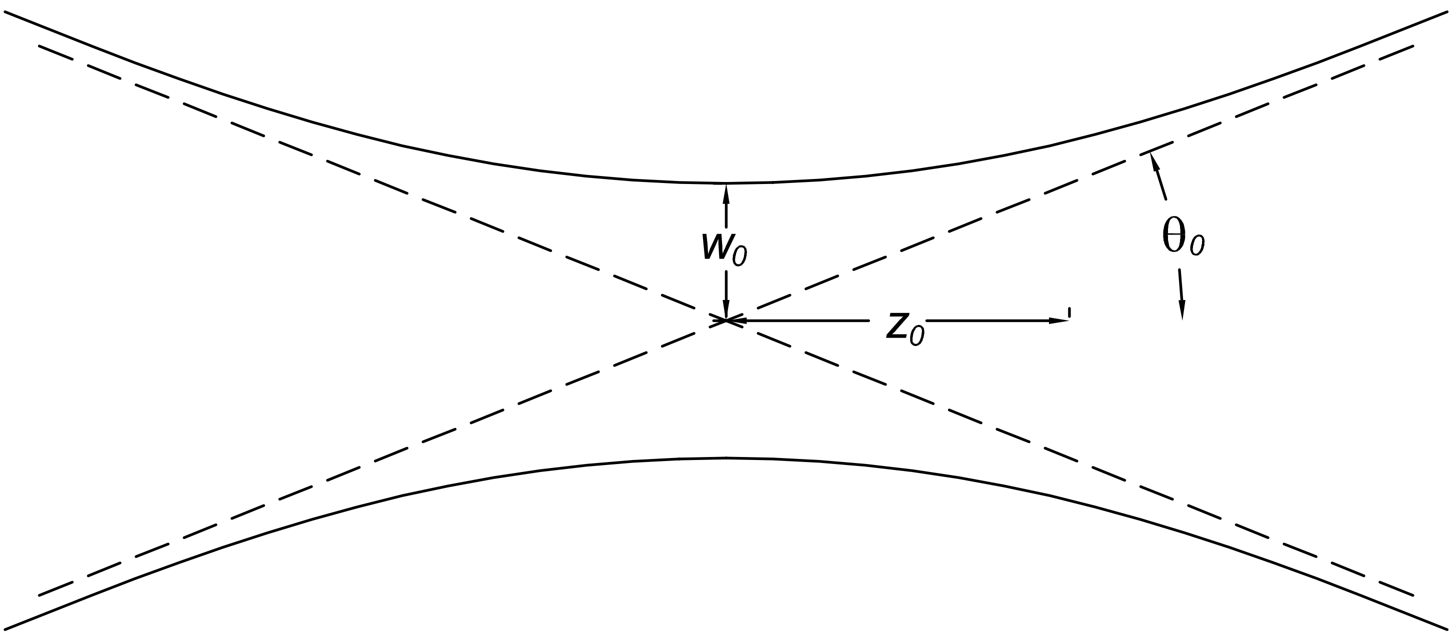
Parameters of a Laser Focus

Diffraction angle θ_0,

Waist w_0,

Depth of focus (Rayleigh range) z_0:

$$\theta_0 = \frac{w_0}{z_0} = \frac{2}{kw_0}, \qquad \text{and} \qquad z_0 = \frac{kw_0^2}{2} = \frac{2}{k\theta_0^2}$$

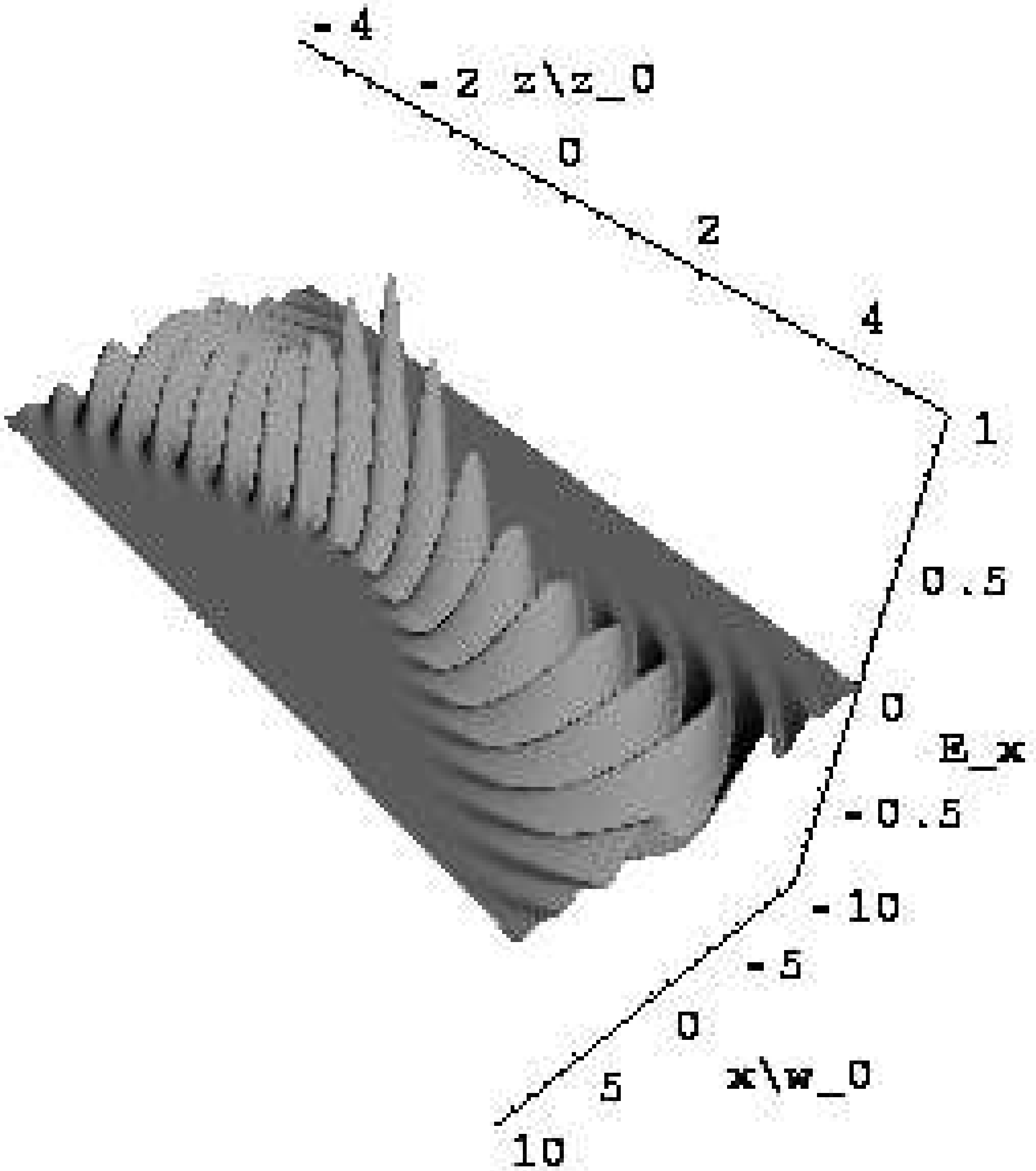


$$\xi = \frac{x}{w_0}, \qquad \upsilon = \frac{y}{w_0}, \qquad \rho = \frac{r_\perp}{w_0}, \qquad \varsigma = \frac{z}{z_0}$$

$$\text{Paraxial wave equation:} \qquad \nabla_\perp^2 \psi + 4i \frac{\partial \psi}{\partial \varsigma} = 0$$

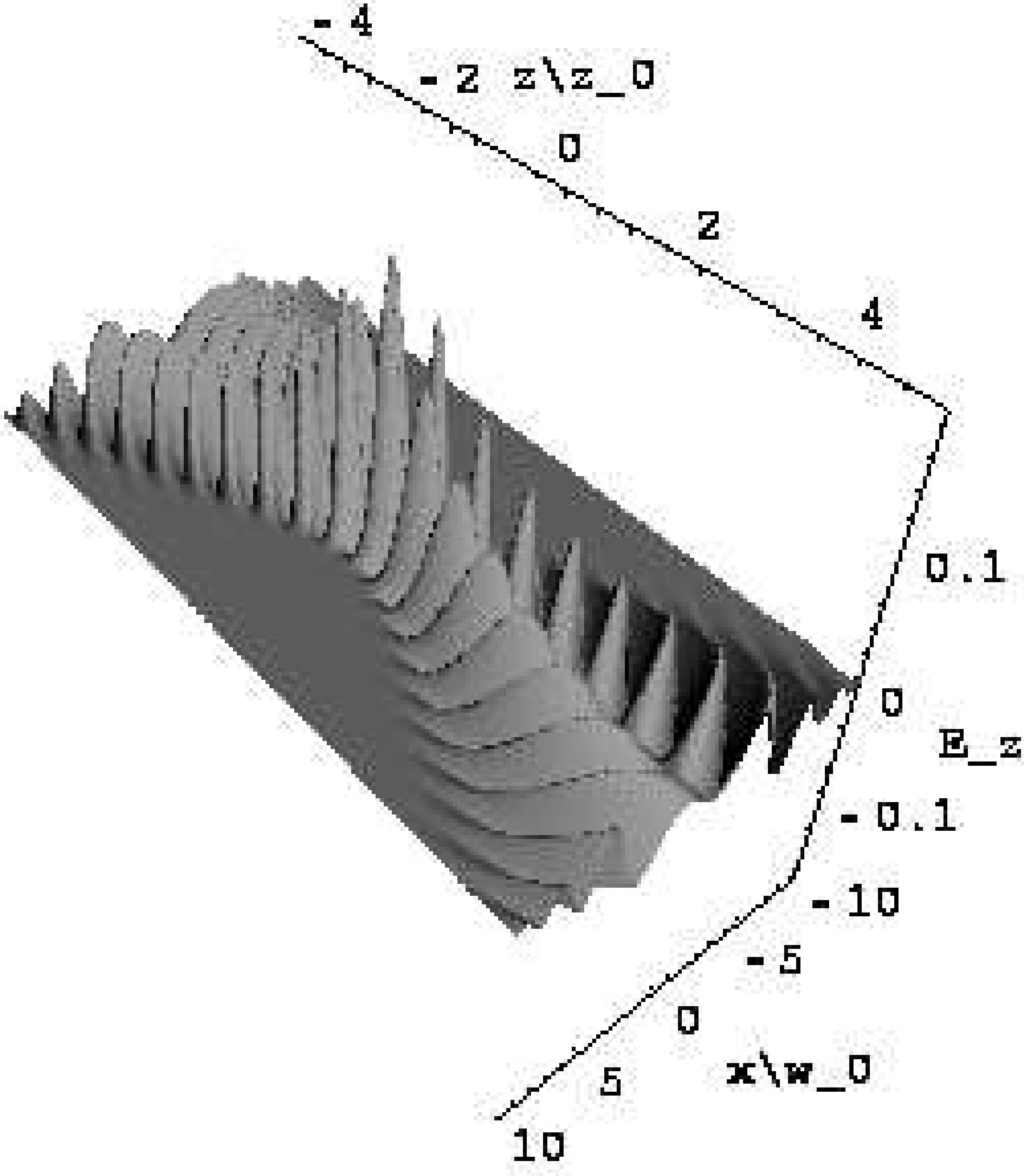
$$\psi = f e^{-f\rho^2}, \qquad f = \frac{1}{1 + i\varsigma} = \frac{1 - i\varsigma}{1 + \varsigma^2} = \frac{e^{-i \tan^{-1} \varsigma}}{\sqrt{1 + \varsigma^2}}$$

Linearly Polarized Gaussian Laser Beam, $\mathbf{A} \propto \psi\hat{\mathbf{x}}$



$$
\begin{aligned}
E_x &= E_0 f e^{-f\rho^2} e^{i(kz-\omega t)} \\
&= \frac{E_0 e^{-r_\perp^2/w_0^2(1+z^2/z_0^2)}}{\sqrt{1+z^2/z_0^2}} e^{i\{kz[1+r_\perp^2/2(z^2+z_0^2)]-\omega t-\tan^{-1}(z/z_0)\}}
\end{aligned}
$$

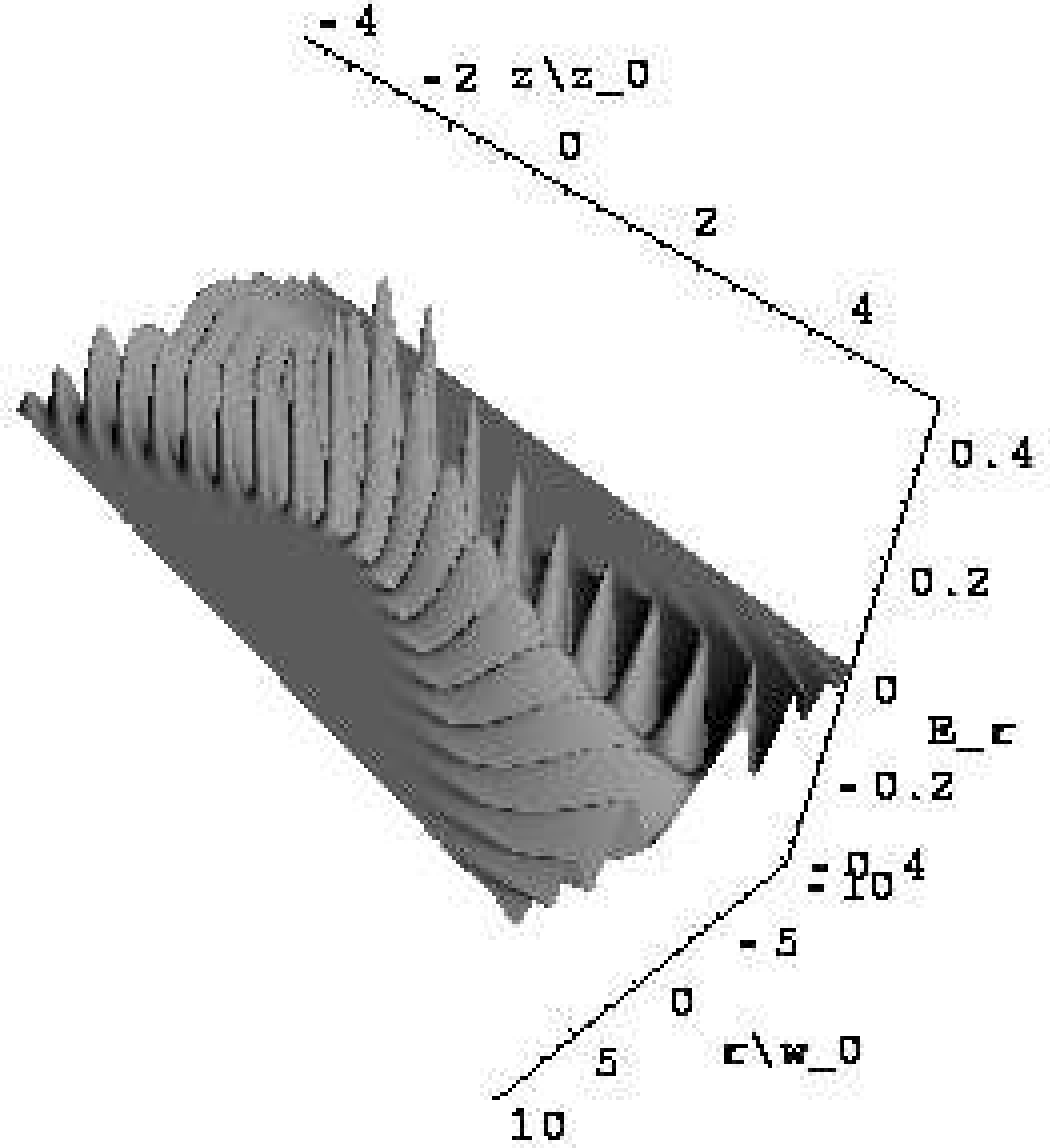
Linearly Polarized Gaussian Laser Beam, $\mathbf{A} \propto \psi \hat{\mathbf{x}}$



$$E_z = -i\theta_0 \xi E_0 f^2 e^{-f\rho^2} e^{i(kz-\omega t)}$$

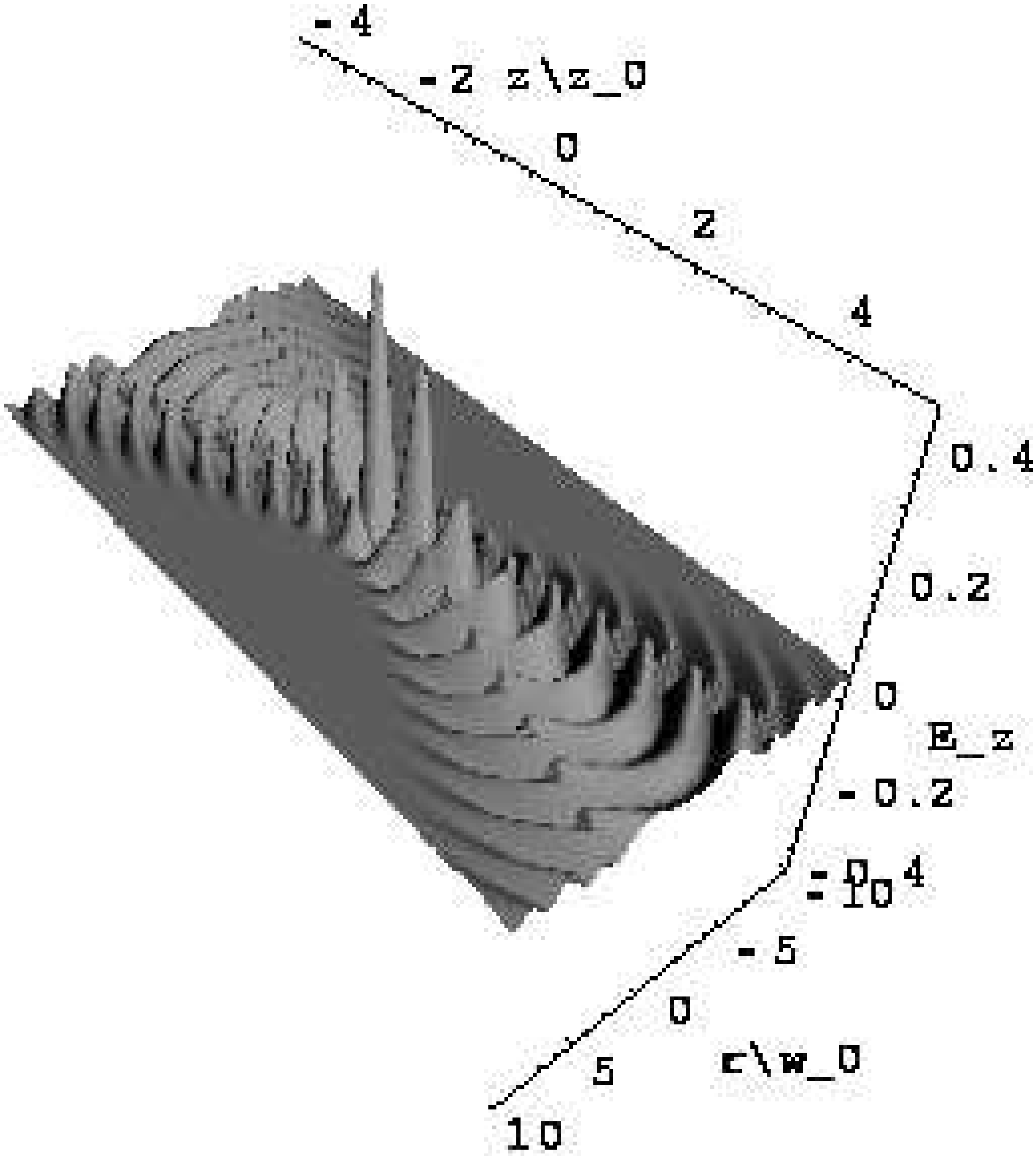
$$= -i \frac{x e^{-i\tan^{-1}(z/z_0)}}{\sqrt{z^2+z_0^2}} E_x$$

Axicon Gaussian Laser Beam, $\mathbf{A} \propto \psi \hat{\mathbf{z}}$



$$
\begin{aligned}
E_{r_\perp} &= \rho E_0 f^2 e^{-f\rho^2} e^{i(kz-\omega t)} \\
&= \frac{r_\perp E_0 e^{-r_\perp^2/w_0^2(1+z^2/z_0^2)}}{1+z^2/z_0^2} e^{i\{kz[1+r_\perp^2/2(z^2+z_0^2)]-\omega t-2\tan^{-1}(z/z_0)\}}
\end{aligned}
$$

Axicon Gaussian Laser Beam, $\mathbf{A} \propto \psi \hat{\mathbf{z}}$



$$E_z = i\theta_0 E_0 f^2(1 - f\rho^2)e^{-f\rho^2}e^{i(kz-\omega t)}$$
$$= i\frac{\lambda}{\pi r_\perp}\left(1 - \frac{r_\perp^2}{z^2+z_0^2}\frac{e^{-i\tan^{-1}(z/z_0)}}{\theta_0^2}\right)E_{r_\perp}$$