

OTHER KINDS OF FORCE FIELDS

WE MAKE A DIGRESSION AS TO WHETHER THE SUCCESS OF THE FIELD CONCEPT MAY BE EXTENDED TO OTHER KINDS OF FORCE LAWS BESIDES THAT OF ELECTRICITY AND MAGNETISM.

CAN GRAVITY BE DESCRIBED BY A VECTOR FIELD?

THE FIELD CONCEPT REPLACES ACTION-AT-A-DISTANCE BY ACTION ON THE IMMEDIATE SURROUNDINGS WHICH IS THEN PROPAGATED THROUGHOUT SPACE, AS DESCRIBED BY A SET OF DIFFERENTIAL EQUATIONS. THE RATE OF PROPAGATION IS FINITE - THE SPEED OF LIGHT. THE CONTINUOUS FIELD CONCEPT SEEMS A PHILOSOPHICAL AS WELL AS A TECHNICAL ADVANCE OVER THE IDEA OF ACTION AT A DISTANCE.

THE GRAVITATIONAL FORCE LAW IS VERY SIMILAR TO THAT OF ELECTROSTATICS. CAN WE APPLY THE FIELD CONCEPT TO GRAVITY AS WELL? THIS QUESTION WAS CONSIDERED BY MAXWELL IN HIS GREAT PAPER OF 1864 IN WHICH HIS EQUATIONS WERE GIVEN. THE SECTION ON GRAVITY IS REPRODUCED BELOW. WE PARAPHRASE HIS ARGUMENT AS FOLLOWS:

THE MAIN DIFFERENCE IN THE FORCE LAW OF GRAVITATION AND ELECTRICITY IS THAT THE GRAVITATIONAL FORCE BETWEEN LIKE OBJECTS IS ATTRACTIVE, WHILE THE ELECTROSTATIC FORCE IS REPULSIVE:

$$\vec{F}_{\text{GRAVITY ON } 1} = -G \frac{M_1 M_2}{r_{12}^2} \hat{r}_{12} \quad (\vec{r}_{12} = \vec{r}_1 - \vec{r}_2)$$

SUPPOSE WE DEFINE THE GRAVITATIONAL FIELD AS \vec{g}

$$\text{ACCORDING TO } \vec{g}_{\text{ON } 1} = -\frac{GM_2}{r_{12}^2} \hat{r}_{12}, \quad \text{SO THAT } \vec{F}_{\text{ON } 1} = M_1 \vec{g}$$

THE 'MAXWELL EQUATIONS' FOR THIS STATIC FIELD ARE CLEARLY

$$\vec{\nabla} \times \vec{g} = 0 \quad \text{AND} \quad \vec{\nabla} \cdot \vec{g} = -4\pi G \rho \quad \text{WHERE } \rho = \text{MASS DENSITY}$$

HENCE WE CAN DERIVE \vec{g} FROM A POTENTIAL: $\vec{g} = -\vec{\nabla} \phi$ WITH $\phi = -G \int \frac{\rho}{r} d\text{vol}$

THE TROUBLE COMES WHEN WE TRY TO ASCRIBE AN INDEPENDENT PHYSICAL REALITY TO THE FIELD \vec{g} SUCH AS AN ASSOCIATED ENERGY DENSITY. IN ASSEMBLING A CONFIGURATION OF MASSES WE STORED UP 'FIELD ENERGY'

$$\begin{aligned} U &= \frac{1}{2} \int \rho \phi d\text{vol} = -\frac{1}{8\pi G} \int (\vec{\nabla} \cdot \vec{g}) \phi d\text{vol} = \frac{1}{8\pi G} \int \vec{g} \cdot \vec{\nabla} \phi d\text{vol} + \text{SURFACE INTEGRAL} \\ &= \frac{-1}{8\pi G} \int g^2 d\text{vol} < 0 \quad \text{ALWAYS} \end{aligned}$$

can shew that electrified bodies in a dielectric medium will act on one another with forces obeying the same laws as are established by experiment.

The energy, by the expenditure of which electrical attractions and repulsions are produced, we suppose to be stored up in the dielectric medium which surrounds the electrified bodies, and not on the surface of those bodies themselves, which on our theory are merely the bounding surfaces of the air or other dielectric in which the true springs of action are to be sought.

Note on the Attraction of Gravitation.

(82) After tracing to the action of the surrounding medium both the magnetic and the electric attractions and repulsions, and finding them to depend on the inverse square of the distance, we are naturally led to inquire whether the attraction of gravitation, which follows the same law of the distance, is not also traceable to the action of a surrounding medium.

Gravitation differs from magnetism and electricity in this; that the bodies concerned are all of the same kind, instead of being of opposite signs, like magnetic poles and electrified bodies, and that the force between these bodies is an attraction and not a repulsion, as is the case between like electric and magnetic bodies.

The lines of gravitating force near two dense bodies are exactly of the same form as the lines of magnetic force near two poles of the same name; but whereas the poles are repelled, the bodies are attracted. Let E be the intrinsic energy of the field surrounding two gravitating bodies M_1, M_2 , and let E' be the intrinsic energy of the field surrounding two magnetic poles, m_1, m_2 , equal in numerical value to M_1, M_2 , and let X be the gravitating force acting during the displacement δx , and X' the magnetic force,

$$X\delta x = \delta E, \quad X'\delta x = \delta E';$$

now X and X' are equal in numerical value, but of opposite signs; so that

$$\delta E = -\delta E',$$

$$E = C - E'$$

or

$$= C - \int \frac{1}{8\pi} (\alpha^2 + \beta^2 + \gamma^2) dV$$

where α, β, γ are the components of magnetic intensity. If R be the resultant

gravitating force, and R' the resultant magnetic force at a corresponding part of the field,

$$R = -R', \text{ and } \alpha^2 + \beta^2 + \gamma^2 = R^2 = R'^2.$$

Hence

$$E = C - \int \frac{1}{8\pi} R^2 dV \dots\dots\dots (47).$$

The intrinsic energy of the field of gravitation must therefore be less wherever there is a resultant gravitating force.

As energy is essentially positive, it is impossible for any part of space to have negative intrinsic energy. Hence those parts of space in which there is no resultant force, such as the points of equilibrium in the space between the different bodies of a system, and within the substance of each body, must have an intrinsic energy per unit of volume greater than

$$\frac{1}{8\pi} R^2,$$

where R is the greatest possible value of the intensity of gravitating force in any part of the universe.

The assumption, therefore, that gravitation arises from the action of the surrounding medium in the way pointed out, leads to the conclusion that every part of this medium possesses, when undisturbed, an enormous intrinsic energy, and that the presence of dense bodies influences the medium so as to diminish this energy wherever there is a resultant attraction.

As I am unable to understand in what way a medium can possess such properties, I cannot go any further in this direction in searching for the cause of gravitation.

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XXV. *A Dynamical Theory of the Electromagnetic Field.*

BY J. C. MAXWELL

(Received October 27,—Read December 8, 1864.)

BUT IT DOESN'T MAKE MUCH PHYSICAL SENSE THAT THE FIELD ENERGY IS ALWAYS NEGATIVE. FOR EXAMPLE, WE WOULD THEN CONCLUDE THAT GRAVITY WAVES CARRY NEGATIVE ENERGY! MAXWELL SKETCHES A POSSIBLE INTERPRETATION WHICH IS A PRECURSOR OF THE 'WAVE THEORY' OF ANTIMATTER, BUT HE REMAINED DIS-SATISFIED. PERHAPS GRAVITY JUST CANNOT BE DESCRIBED BY VECTOR FIELDS...

IN 1915 EINSTEIN SHOWED HOW GRAVITY IS WELL-DESCRIBED BY A THEORY IN WHICH THE POTENTIAL OF THE FIELD IS A TENSOR. THIS IS SOMEWHAT COMPLICATED. IT TURNS OUT THAT THERE HAS BEEN ANOTHER SUCCESS IN FIELD THEORY INVOLVING A SCALAR POTENTIAL...

It has often been noted that magnetism can largely be understood as the consequence of electrostatics plus special relativity (see, for example, chapter 3 of *Principles of Electrodynamics* by M. Schwartz). Similarly, static gravity plus special relativity implies phenomenon often called magnetic gravity (see, for example, D. Bedford and P. Krumm, *Am. J. Phys.* 53, 889, (1985); 55, 362 (1987); H. Kolbenstvedt, *Am. J. Phys.* 56, 523 (1988); E. G. Harris, *Am. J. Phys.* 59, 421 (1991)).

The first recorded discussion of magnetic gravity appears to be by O. Heaviside. See pp. 455-465 of Vol. 1 of his book, *Electromagnetic Theory* ("The Electrician" Printing and Publishing Co., London, 1894).

The static gravitational force on a mass m due to a mass density ρ can be written in close analogy to electrostatics as

$$\mathbf{F} = m\mathbf{g},$$

where the static gravitational field \mathbf{g} obeys

$$\nabla \cdot \mathbf{g} = -4\pi G\rho, \quad \nabla \times \mathbf{g} = 0,$$

where G is Newton's gravitational constant.

Heaviside argued that in analogy to the remaining Maxwell equations we should expect a gravitomagnetic field \mathbf{h} that obeys

$$\nabla \cdot \mathbf{h} = 0, \quad \nabla \times \mathbf{h} = -4\pi H\rho\mathbf{v},$$

where \mathbf{v} is the velocity of the mass that causes field \mathbf{h} and H is a constant (that should be called Heaviside's constant) which characterizes the strength of the gravitomagnetic interaction. The force on mass m is now

$$\mathbf{F} = m\mathbf{g} + m\mathbf{v} \times \mathbf{h},$$

where \mathbf{v} is the velocity of mass m in this expression, in analogy to the Lorentz force law (actually first written down by Heaviside in 1889).

If the constant H (and also G) were larger there might have been an experimental measurement of its value. Then it would have been noted that

$$\sqrt{\frac{G}{H}} = c,$$

the speed of light!

In the preceding I have chosen different units for the field \mathbf{h} than those recommended by Heaviside to emphasize how, if observed, \mathbf{h} might have been interpreted initially as quite distinct from the field \mathbf{g} and having nothing to do with the speed of light.

Lacking evidence of gravitomagnetostatic effects, Heaviside proceeded by analogy to the full Maxwell equations and inferred the time-dependent equations of the gravitational field would be

$$\nabla \cdot \mathbf{g} = -4\pi G\rho, \quad \nabla \times \mathbf{g} = -\frac{\partial \mathbf{h}}{\partial t},$$

and

$$\nabla \cdot \mathbf{h} = 0, \quad \nabla \times \mathbf{h} = -4\pi H\rho\mathbf{v} + \frac{H}{G} \frac{\partial \mathbf{g}}{\partial t}.$$

Heaviside then noted that there should be gravitational waves which propagate with velocity

$$v = \sqrt{\frac{G}{H}},$$

As an example Heaviside considered that the propagation velocity might well be the speed of light. From this assumption the constant H has the value 7.3×10^{-28} m/kg.

Heaviside then noted that the gravitational field of the Sun, taken as moving relative to the 'ether' defined by the fixed stars, would be modified by terms in $(v_{\text{Sun}}/c)^2$ exactly as is the case for the field of a rapidly moving electric charge (which result he had been the first to derive correctly to all orders in v/c in 1888). He then calculated the resulting precession of the Earth's orbit around the Sun and concluded this effect was small enough to have gone unnoticed thus far, and therefore offered no contradiction to the hypothesis that gravitational effects propagate at the speed of light.

Heaviside also considered the effect of the dipole gravitomagnetic field of the rotating Sun, finding the dipole moment to be $-H\mathbf{L}/2$ where \mathbf{L} is the angular momentum of the Sun. However, the effect of this moment on the precession of a planet's orbit has the opposite sign to the observed effect, and is too small in magnitude by a factor $L_{\text{Sun}}/L_{\text{orbital, planet}}$. (Surprisingly, Heaviside seems to be unaware of the long history of measurements of the precession of Mercury's orbit.)

It appears that the first confrontation between experiment and new predictions of gravitational field theory occurred some 20 years before Einstein's celebrated work.

From Heaviside's habit of recording the date on which sections of his book first appeared as short articles in *The Electrician* magazine I infer that gravitation occupied his attention for only three weeks in 1893 and that he never returned to the subject.

Heaviside's work could be called a low-velocity, weak-field approximation to general relativity. This topic was revived in an interesting paper by R.L. Forward, *General Relativity for the Experimentalist*, that is perhaps insufficiently well-known due in part to its place of publication: Proceedings of the Institute for Radio Engineers **49**, 892-904 (1961).

Additional discussions can be found in sec. III of *Laboratory Experiments to Test Relativistic Gravity* by V.B. Braginsky, C.M. Caves and K.S. Thorne, Phys. Rev. D **15**, 2047 (1977) and in the article, *Gravitomagnetism, Jets in Quasars and the Stanford Gyroscope Experiment* by K.S. Thorne in *Near Zero: New Frontiers of Physics*, ed. by J.D. Fairbank *et al.* (W.H. Freeman, New York, 1988).

The precession of planetary orbits is not a good test of gravitomagnetism; that precession is due to corrections of order v^2/c^2 to the field \mathbf{g} that are 'post-Maxwellian'. (The term 'post-Newtonian'

typically used in the literature is perhaps not sufficiently precise in this regard.). However, gravitomagnetism is a useful insight for understanding the precession of orbiting gyroscopes that will hopefully be observed in experiments now under construction.

NUCLEAR FORCES

THE ATOMIC NUCLEUS CONSISTS OF PROTONS ($q = +e$) AND NEUTRONS ($q = 0$). ELECTRICALLY THE NUCLEUS IS NOT STABLE AND TRIES TO BLOW ITSELF APART. BUT IN FACT MANY NUCLEI ARE QUITE STABLE. THEREFORE, WE POSTULATE THE EXISTENCE OF A STRONG FORCE WHICH OVERPOWERS THE ELECTROSTATIC REPULSION, AND BINDS THE NUCLEONS (= PROTONS & NEUTRONS) TOGETHER.

IN 1933 WEINBERG REMARKED THAT THE ENERGY NEEDED TO REMOVE ONE NUCLEON FROM THE NUCLEUS IS INDEPENDENT OF THE SIZE OF THE NUCLEUS. HENCE, THE NUCLEAR FORCE MUST BE SHORT RANGE - ONLY THE IMMEDIATE NEIGHBORS OF A NUCLEON FEEL ITS NUCLEAR FORCE. THE POTENTIAL OF SUCH A FORCE MUST DIE AWAY MUCH MORE QUICKLY THAN $1/r$.

EMPIRICALLY THERE IS AN ATTRACTIVE COMPONENT TO THE NUCLEAR FORCE OF CHARACTERISTIC LENGTH 10^{-13} CM. BUT THERE IS ALSO A REPULSIVE COMPONENT OF CHARACTERISTIC LENGTH $\sim 2 \times 10^{-14}$ CM. THE ATTRACTIVE FORCE HOLDS THE NUCLEUS TOGETHER, WHILE THE 'REPULSIVE CORE' KEEPS THE NUCLEUS FROM COLLAPSING (INTO A BLACK HOLE?)

IN 1935 YUKAWA MADE AN ATTEMPT TO RELATE THE NUCLEAR FORCE TO THE FIELD CONCEPT OF E & M.

FOR E & M WE HAVE $\square \phi_\mu = 4\pi j_\mu$ AS THE EQUATION RELATING THE 4-VECTOR POTENTIAL TO THE CHARGE-CURRENT SOURCES.

WE HAVE STUDIED TWO SIMPLE CASES: $\left\{ \begin{array}{l} \text{THE STATIC SOLUTION,} \\ \text{THE PLANE WAVE SOLUTION.} \end{array} \right.$

THE STATIC EQUATION FOR THE ELECTRIC POTENTIAL IS

$$\nabla^2 \phi = 4\pi \rho$$

IF ρ IS A POINT SOURCE, WE EXPECT SPHERICAL SYMMETRY: $\phi = \phi(r)$, AND THAT $\nabla^2 \phi = 0$ OUTSIDE THE SOURCE.

$$\therefore \nabla^2 \phi = \frac{1}{r} \frac{\partial^2}{\partial r^2} (r\phi) = 0 \Rightarrow r\phi = K \text{ OR } \phi = \frac{K}{r}$$

OF COURSE $K = q = \text{CHARGE OF THE SOURCE}$.

THE PLANE WAVE SOLUTIONS TO THE SOURCE-FREE EQUATION $\square \phi = 0$ ARE $\phi = \phi_0 e^{i(kz - \omega t)}$ ETC.

PROVIDED $k^2 - \frac{\omega^2}{c^2} = 0$ OR $k = \omega/c$

YUKAWA'S SUGGESTION WAS TO MODIFY THE WAVE EQUATION TO BE $(\square - \mu^2) \phi = 0$ (IN A SOURCE-FREE REGION)

WHERE μ IS A CONSTANT (LORENTZ INVARIANT) WHOSE SIGNIFICANCE WE MUST INVESTIGATE.

YUKAWA ALSO ALLOWED THE POSSIBILITY THAT THE POTENTIAL MIGHT BE A PURE SCALAR POTENTIAL ϕ I.E. $(\square - \mu^2) \phi = 0$.

IF WE ALSO CONSIDER A POSSIBLE TENSOR POTENTIAL $\phi_{\mu\nu}$, WE MAY NOTE A FEW FEATURES OF GENERAL RELATIVITY AS WELL!

LET'S LOOK FOR SOME SOLUTIONS TO $(\square - \mu^2) \phi = 0$

STATIC CASE $(\nabla^2 - \mu^2) \phi = 0$.

FOR A POINT SOURCE WE AGAIN EXPECT SPHERICAL SYMMETRY: $\phi = \phi(r)$

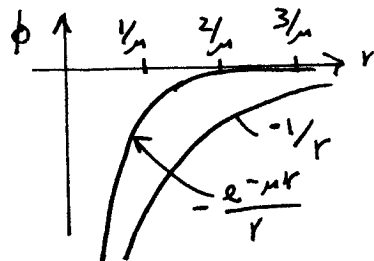
$$\text{THEN } \frac{1}{r} \frac{\partial^2}{\partial r^2} (r\phi) = \mu^2 \phi \text{ OR } \frac{\partial^2}{\partial r^2} (r\phi) = \mu^2 (r\phi)$$

$$\text{SO } r\phi = K e^{\pm \mu r}$$

ONLY MINUS SOLUTION MAKES SENSE FOR A FORCE WHICH VANISHES AS $r \rightarrow \infty$.

$$\text{THEN } \phi = \frac{K e^{-\mu r}}{r} \text{ WHICH IS A SHORT RANGE}$$

POTENTIAL WITH CHARACTERISTIC LENGTH $\frac{1}{\mu}$.



FOR THE ATTRACTIVE NUCLEAR FORCE, $K < 0$ AND $\mu \sim 10^{13} \text{ cm}^{-1}$

FOR THE REPULSIVE FORCE $K > 0$ AND $\mu \sim 5 \times 10^{13} \text{ cm}^{-1}$

PLANE WAVES:

AGAIN WE TRY $\phi = \phi_0 e^{i(kz - \omega t)}$

THIS SATISFIES $(\square - \mu^2) \phi = 0$ IF $-k^2 + \frac{\omega^2}{c^2} - \mu^2 = 0$

$$\text{OR } \omega = c \sqrt{k^2 + \mu^2}$$

$$v_p = \frac{\omega}{k} > c \quad \text{WHILE} \quad v_g = \frac{d\omega}{dk} = \frac{c k}{\sqrt{k^2 + \mu^2}} < c$$

WHILE IS NOT LIKE LIGHT WAVES IN VACUUM.

TO SEE FURTHER WE MUST USE DE BROGLIE'S IDEA OF 'MATTER WAVES', INVOLVING THE RELATIONS

$$E = \hbar \omega \quad \text{AND} \quad p = \hbar k \quad \left(\hbar = \frac{h}{2\pi} = \text{PLANCK'S CONSTANT} \right)$$

WE HAVE ALREADY SUGGESTED THAT THESE RELATIONS ARE TRUE WHEN DISCUSSING 'PHOTONS' = 'PARTICLES' OF LIGHT.

SUBSTITUTING THESE INTO THE ABOVE 'DISPERSION RELATION' WE GET

$$-p^2 + \frac{E^2}{c^2} - \hbar^2 \mu^2 = 0 \quad \text{OR} \quad (\hbar \mu c)^2 = E^2 - p^2 c^2$$

BUT FROM RELATIVITY WE KNOW THAT $E^2 - p^2 c^2 = (m_0 c^2)^2$ WHERE m_0 IS THE MASS ASSOCIATED WITH THE ENERGY-MOMENTUM 4-VECTOR $(E, \vec{p}c)$

THUS $m_0 = \frac{\hbar \mu}{c}$ IS A MASS ASSOCIATED WITH OUR PLANE WAVE SOLUTIONS TO THE NUCLEAR POTENTIAL WAVE EQUATION.

$$\text{NUMERICALLY } \mu \sim 10^{13} \text{ cm}^{-1} \Rightarrow m \sim 170 \text{ MeV}/c^2$$

$$\mu \sim 5 \times 10^{13} \text{ cm}^{-1} \Rightarrow m \sim 850 \text{ MeV}/c^2$$

(WHERE $1 \text{ MeV} = 1 \text{ MILLION ELECTRON VOLTS}$ - AN MKSA UNIT)

$$\text{FOR COMPARISON } m_{\text{ELECTRON}} \sim \frac{1}{2} \text{ MeV}/c^2$$

$$m_{\text{PROTON}} \sim 938 \text{ MeV}/c^2$$

THUS WE ARE LED TO THE IDEA THAT THERE ARE MASSIVE PARTICLES ASSOCIATED WITH THE SHORT RANGE NUCLEAR FORCE FIELD. MASSLESS PARTICLES (PHOTONS) ARE ASSOCIATED WITH THE LONG RANGE ELECTRO MAGNETIC FIELD.

THE GRAVITATIONAL FIELD IS LONG RANGE ($\mu = 0$) AND SO WE WOULD ASSOCIATE MASSLESS PARTICLES (GRAVITONS) WITH THIS FIELD ALSO. THEY HAVE NEVER BEEN OBSERVED...

THE PARTICLES OF THE NUCLEAR FIELD ARE CALLED MESONS. THE π -MESON WAS OBSERVED IN 1947, WITH A MASS $140 \text{ MEV}/c^2$, AND IS INDEED DESCRIBED BY A SCALAR POTENTIAL. IN THE 1960'S THE SO-CALLED ρ , ω AND ϕ MESONS WERE DISCOVERED, WITH MASSES 760 , 780 AND $1020 \text{ MEV}/c^2$. THEY TURN OUT TO BE VECTOR MESONS - I.E. THEY ARE ASSOCIATED WITH A VECTOR POTENTIAL ϕ_V .

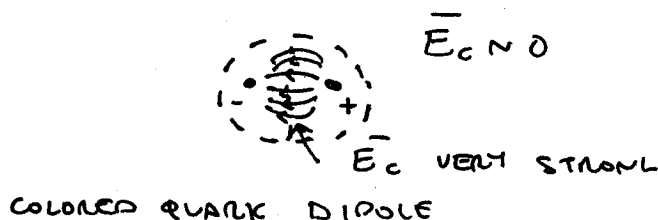
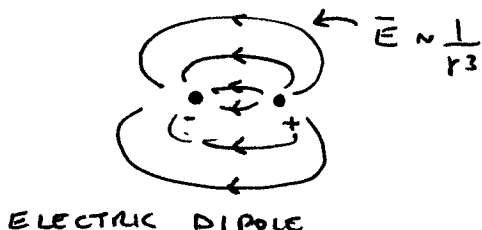
THE π -MESON CAN BE SAID TO CAUSE THE ATTRACTIVE NUCLEAR FORCE, WHILE THE ρ , ω AND ϕ MESONS CAUSE THE REPULSIVE CORE.

IT IS NOT HARD TO IMAGINE HOW A FIELD OF PARTICLES CAN LEAD TO A REPULSIVE FORCE. TWO Ph 206 STUDENTS PLAYING FRISBEE ON A FRICTIONLESS SURFACE WILL BE DRIVEN APART BY THE RECOIL OF CATCHING AND TOSSING THE FRISBEE. HOW THE ATTRACTIVE FORCE ARISES IS A BIT HARDER TO IMAGINE.

IN RECENT TIMES THE THEORY OF NUCLEAR FORCES WAS UNDERGONE A MAJOR CHANGE. BOTH THE NUCLEONS AND THE MESONS ARE THOUGHT TO CONSIST OF MASSIVE 'QUARKS' BOUND TOGETHER BY MASSLESS VECTOR - 'GLUONS'. THE LATTER CREATE A FIELD SOMEWHAT LIKE THE ELECTRIC FIELD, CALLED THE 'COLOR ELECTRIC FIELD'. THE 'COLOR FORCE' IS SHORT RANGE IN THAT 'FREE SPACE' HAS A COLOR DIELECTRIC CONSTANT $\sim \infty$, COMPARED TO ORDINARY DIELECTRIC CONSTANT = 1. THIS HAS THE EFFECT OF 'CONFINING' ALL COLOR FIELD LINES VERY CLOSE TO THEIR SOURCE.

$$\vec{\nabla} \cdot \vec{D}_c = 4\pi \rho_c \quad \vec{E}_c = \frac{\vec{D}_c}{\epsilon_c} \approx 0 \quad \text{OUTSIDE THE SOURCE.}$$

THIS IS INCONSISTENT WITH THE EXISTENCE OF SINGLE POINT SOURCES OF THE COLOR FIELD (= FREE QUARKS). BUT QUARK DIPLES ARE ALLOWED:



THE QUARK DIPLES ARE JUST THE MESONS OF YUKAWA. $\frac{1}{\lambda}$ IS

THE CHARACTERISTIC SIZE OF THE REGION OF NON ZERO COLOR ELECTRIC FIELD (THE 'BAG' SIZE)

THE NUCLEONS ARE QUARK TRIPLETS, A STATE OF SPECIAL STABILITY HAVING TO DO WITH THE FACT THAT THERE ARE 3 'COLORS'....

THE SIGNIFICANCE OF THE LORENTZ GAUGE CONDITION

WE NOW HAVE SEVERAL SETS OF POTENTIALS TO CONTEMPLATE:

SCALAR ϕ , $m \neq 0$, π -MESON

VECTOR ϕ_μ , $m \neq 0$, $E \neq M$; COLOR $E \neq M$

VECTOR ϕ_μ , $m \neq 0$ VECTOR MESONS ρ, ω, ϕ

TENSOR $\phi_{\mu\nu}$, $m = 0$, GRAVITY (GENERAL RELATIVITY)

THE NON-SCALAR POTENTIALS HAVE MORE THAN ONE INDEPENDENT COMPONENT. FOR PLANE WAVE SOLUTIONS THIS POSSIBILITY MANIFESTS ITSELF AS POLARIZATION.

FOR $E \neq M$ WAVES WE KNOW THAT THERE ARE ONLY 2 INDEPENDENT POLARIZATIONS - ALTHOUGH THE POTENTIAL ϕ_μ HAS 4 COMPONENTS.

WHAT HAPPENED TO THE OTHER TWO COMPONENTS?

THE LORENTZ GAUGE CONDITION $\partial^\mu \phi_\mu = 0$ SERVES TO ELIMINATE ONE OF THE COMPONENTS. IN A SENSE, THIS IS THE MAIN SIGNIFICANCE OF THE LORENTZ CONDITION.

FOR THE SCALAR FIELD THE LORENTZ CONDITION DOES NOT EXIST, AND THE SCALAR FIELD RETAINS ITS ONE COMPONENT.

FOR THE GRAVITY TENSOR $\phi_{\mu\nu}$, THE LORENTZ CONDITION IS $\partial^\mu \phi_{\mu\nu} = 0 \Rightarrow 4$ CONSTRAINTS.

IN ADDITION, WE CLAIM THAT THE WAVE SOLUTIONS $\phi_{\mu\nu}$ ARE SYMMETRIC AND TRACELESS. ALTOGETHER THERE ARE 11 CONSTRAINTS ON THE 16 COMPONENTS $\Rightarrow 5$ INDEPENDENT COMPONENTS ONLY.

(QUANTUM EXPERTS WILL NOTE THAT A TENSOR WITH ONLY 5 COMPONENTS CORRESPONDS TO A SPIN-2 OBJECT. IN TURN THIS MEANS THE WAVES DO NOT COUPLE TO DIPOLE SOURCES, BUT ONLY TO QUADRUPOLE AND HIGHER MULTIPOLES, AS WE SAW EARLIER FOR GRAVITY IN A DIFFERENT WAY.)

HOWEVER, WE STILL HAVE NOT EXPLAINED WHY E-M WAVES HAVE ONLY 2 POLARIZATIONS AND NOT 3, AS ALLOWED BY THE LORENTZ CONDITION ALONE.

THIS HAS TO DO WITH THE FACT THAT THE PHOTON IS MASSLESS!

WE CAN SKETCH AN ARGUMENT:

CONSIDER A VECTOR POTENTIAL PLANE WAVE MOVING IN THE +z DIRECTION

$$\phi_\mu = \epsilon_\mu e^{i(kz - \omega t)}$$

$\epsilon_\mu =$ CONSTANT 4-VECTOR - THE POLARIZATION VECTOR.

THE WAVE 4-VECTOR IS $K_\mu = (\frac{\omega}{c}, 0, 0, k)$.

FOR EEM WAVES $k = \omega/c$, FOR MATTER WAVES $k < \omega/c$.

THE LORENTZ GAUGE CONDITION TELLS US THAT

$$\partial^\mu \phi_\mu = 0 \Rightarrow K^\mu \epsilon_\mu = 0.$$

WE CAN EASILY WRITE DOWN 3 INDEPENDENT CASES FOR ϵ_μ :

$$\epsilon_\mu(1) = (0, 1, 0, 0); \epsilon_\mu(2) = (0, 0, 1, 0); \epsilon_\mu(3) = (k, 0, 0, \frac{\omega}{c})$$

$\epsilon_\mu(1)$ AND $\epsilon_\mu(2)$ ARE TRANSVERSE POLARIZATIONS. WE NEED TO SHOW THAT THE LONGITUDINAL POLARIZATION $\epsilon_\mu(3)$ VANISHES FOR EEM WAVES.

WE GET RID OF IT BY A GAUGE TRANSFORMATION.

RECALL THAT \vec{E} AND \vec{B} ARE NOT CHANGED IF

$$\left. \begin{aligned} \phi &\rightarrow \phi - \frac{1}{c} \dot{\Omega} \\ \vec{A} &\rightarrow \vec{A} + \vec{\nabla} \Omega \end{aligned} \right\} \text{THE GAUGE TRANSFORMATION (P174).}$$

THE LORENTZ CONDITION IS STILL SATISFIED SO LONG AS $\nabla^2 \Omega - \frac{1}{c^2} \frac{\partial^2 \Omega}{\partial t^2} = 0$.

IN 4-VECTOR NOTATION $\phi_\mu \rightarrow \phi_\mu - \partial_\mu \Omega$, WHERE $\square \Omega = 0$

IS A TRANSFORMATION WHICH DOES NOT AFFECT THE PHYSICS.

FOR THE PRESENT CASE, CONSIDER $\Omega = e^{i(kz - \omega t)}$,

WHICH OBEYS $\square \Omega = 0$ IF $k = \omega/c$ AS IT DOES FOR E-M WAVES.

THEN $\partial_\mu \Omega = K_\mu e^{i(kz - \omega t)}$

THUS OUR SOLUTIONS $\epsilon_\mu e^{i(kz - \omega t)}$ HAVE THE SAME PHYSICAL SIGNIFICANCE AS THE TRANSFORMED SOLUTIONS $(\epsilon_\mu - K_\mu) e^{i(kz - \omega t)}$.

IN PARTICULAR, THE TRANSFORM OF $E_{\mu}(3)$ IS $(k - \frac{\omega}{c}, 0, 0, \frac{\omega}{c} - k)$

FOR E-M WAVES WITH $k = \omega/c$ THIS VANISHES. ONLY THE TRANSVERSE POLARIZATIONS $E_{\mu}(1)$ AND $E_{\mu}(2)$ HAVE PHYSICAL SIGNIFICANCE!

THIS PROCEDURE CANNOT BE USED TO MAKE $E_{\mu}(3)$ VANISH FOR THE CASE OF MASSIVE VECTOR FIELDS - WHICH ARE INDEED OBSERVED TO POSSESS LONGITUDINAL POLARIZATION (VIA THE RELATION OF POLARIZATION TO ANGULAR MOMENTUM....)

WE CAN TRY OUT THIS PROCEDURE FOR GRAVITY WAVES ALSO:

$$\phi_{\mu\nu} = E_{\mu\nu} e^{i(kz - \omega t)} \quad k_{\mu} = (k, 0, 0, k)$$

$E_{\mu\nu}$ IS THE POLARIZATION TENSOR.

THE LORENTZ CONDITION $\partial^{\mu} E_{\mu\nu} \Rightarrow k^{\mu} E_{\mu\nu} = 0$

FOR OUR CHOICE OF k_{μ} , THIS TELLS US $E_{0\nu} = E_{3\nu}$

THE GAUGE INVARIANCE OF $\phi_{\mu\nu}$ IS THAT A TRANSFORMATION TO

$$\phi_{\mu\nu} + \partial_{\mu} \Omega_{\nu} + \partial_{\nu} \Omega_{\mu} \text{ DOES NOT CHANGE THE PHYSICS, PROVIDED } \square \Omega_{\mu} = 0$$

FOR PLANE WAVES: $\Omega_{\mu} = \Omega_{\mu} e^{i(kz - \omega t)}$ AND THE

TRANSFORMED POTENTIALS ARE $E_{\mu\nu} + k_{\mu} \Omega_{\nu} + k_{\nu} \Omega_{\mu} \equiv E'_{\mu\nu}$

WE CHOOSE THE 4 CONSTANTS Ω_{μ} SO AS TO ELIMINATE THE $E'_{0\nu}$

$$1. E'_{01} = E_{01} + k_0 \Omega_1 + k_1 \Omega_0 = E_{01} + k \Omega_1$$

$$\Rightarrow \text{CHOOSE } \Omega_1 = -E_{01}/k$$

$$2. E'_{02} = E_{02} + k \Omega_2 \Rightarrow \text{CHOOSE } \Omega_2 = -E_{02}/k$$

$$3. E'_{03} = E_{03} + k_0 \Omega_3 + k_3 \Omega_0 = E_{03} + k(\Omega_3 + \Omega_0) \Rightarrow \text{CHOOSE } \Omega_3 + \Omega_0 = -\frac{E_{03}}{k}$$

$$4. E'_{00} = E_{00} + 2k \Omega_0 \Rightarrow \text{CHOOSE } \Omega_0 = -\frac{E_{00}}{2k}$$

SO FAR $E'_{00} = E'_{01} = E'_{02} = E'_{03} = 0$

THE LORENTZ CONDITIONS SAYS $E'_{0\nu} = E'_{3\nu}$

$$\Rightarrow E'_{30} = E'_{31} = E'_{32} = E'_{33} = 0$$

$E'_{\mu\nu}$ IS SYMMETRIC $\Rightarrow E'_{10} = E'_{20} = E'_{13} = E'_{23} = 0$

ALL THAT IS LEFT IS E'_{11} , E'_{22} , E'_{12} AND E'_{21}

BUT $E'_{\mu\nu}$ IS $\left\{ \begin{array}{l} \text{TRACELESS} \Rightarrow E'_{11} = -E'_{22} \\ \text{SYMMETRIC} \Rightarrow E'_{12} = E'_{21} \end{array} \right.$

$$\Rightarrow \epsilon'_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & E'_{11} & E'_{12} & 0 \\ 0 & E'_{12} & -E'_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

\Rightarrow THERE ARE ONLY 2 INDEPENDENT COMPONENTS, AND THEY ARE TRANSVERSE

THUS GRAVITY WAVES, LIKE LIGHT WAVES ARE TRANSVERSE AND HAVE ONLY 2 POLARIZATIONS. (THIS RESULT HOLDS FOR HIGHER TENSIONS AS WELL - IF $\phi_{\mu\nu}$ IS A POTENTIAL WITH $1/r$ DEPENDENCE (\Leftrightarrow MASSLESS PARTICLES) AND IS GAUGE INVARIANT, THEN THE WAVES HAVE ONLY 2 INDEPENDENT, TRANSVERSE POLARIZATIONS). WE ENCOURAGE YOU TO MAKE A PRELIMINARY ENCOUNTER WITH THE PHYSICAL SIGNIFICANCE OF THE POTENTIALS $\phi_{\mu\nu}$ ON THE PROBLEM SET.

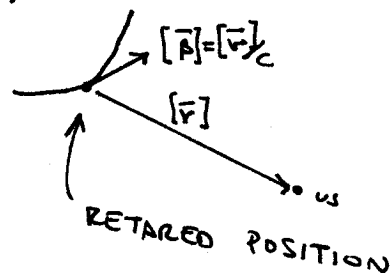
WE NOW RETURN TO OUR STUDY OF E & M

RADIATION BY AN ACCELERATING CHARGE [BECKER SEC. 69]

THE 4-POTENTIAL OF A SINGLE CHARGE WAS FOUND IN LECTURE 18 VIA A LORENTZ TRANSFORMATION:

$$\phi = \left[\frac{e}{r - \vec{r} \cdot \vec{\beta}} \right], \quad \vec{A} = \left[\frac{e\vec{\beta}}{r - \vec{r} \cdot \vec{\beta}} \right],$$

WHERE $[\vec{r}]$ IS THE RETARDED POSITION VECTOR OF THE MOVING CHARGE. $[\vec{\beta}] =$ RETARDED VELOCITY.



IT MAY BE INSTRUCTIVE TO DERIVE THESE RELATIONS DIRECTLY FROM THE RETARDED POTENTIAL SOLUTIONS WE FOUND IN THE CASE OF CONTINUOUS CHARGE AND CURRENT DISTRIBUTIONS (LECTURE 15).

$$\text{WE SAW } \phi = \int \frac{[P]}{r} dvol', \quad \vec{A} = \frac{1}{c} \int \frac{[\vec{j}]}{r} dvol',$$

$[P] = \rho(\vec{x}', t' = t - r/c) =$ RETARDED CHARGE DENSITY.

WE PROCEED BY CONSIDERING A POINT CHARGE AS THE LIMIT OF A SMALL BUT FINITE CHARGE DISTRIBUTION OF TOTAL CHARGE Q . THE ENTIRE CHARGE DISTRIBUTION HAS VELOCITY \vec{v} .

OUR TASK IS TO EVALUATE $[P] dvol'$. THIS IS COMPLICATED BECAUSE THE CHARGE IS MOVING.

SUPPOSE WE CONSIDER A VOLUME ELEMENT

$$dvol = dy dArea$$

AT RETARDED DISTANCE $[r]$.

BY DEFINITION OF THE RETARDED DISTANCE, OUR 'OBSERVATION' OF THE CHARGE IS TO

BE MADE AT TIME $t' = t - [r]/c$.

IN PARTICULAR, THIS TIME IS DIFFERENT BY AMOUNT $dt' = \frac{dr}{c}$

FOR OBSERVATIONS AT $[r]$ AND $[r] + dy$ WHICH BOUND OUR VOLUME ELEMENT.

THAT IS, DURING THE COURSE OF CALCULATING $[p] dvol$ SOME CHARGE LEAVES THE VOLUME. NOTE THAT WE MOVE INWARDS FROM $r + dy$ TO r DURING THE TIME dt' WHILE WE ARE MAKING THE CALCULATION.

THE TOTAL CHARGE OBSERVED WHILE EVALUATING $[p] dvol$ IS

$$de = \underbrace{[p] dvol}_{\substack{\text{CHARGE INSIDE} \\ \text{IF } \vec{v} = 0}} - \underbrace{[\hat{r} \cdot \vec{v}] dt' dArea [p]_{t'}}_{\text{CHARGE WHICH LEFT DURING TIME } dt'}$$

$$= [p] dvol \left(1 - \left[\hat{r} \cdot \frac{\vec{v}}{c} \right] \right),$$

$$\text{so } \frac{[p] dvol}{r} = \frac{de}{[r - \vec{r} \cdot \vec{\beta}]}$$

$$\Rightarrow \phi = \int \frac{de}{[r - \vec{r} \cdot \vec{\beta}]} \rightarrow \frac{e}{[r - \vec{r} \cdot \vec{\beta}]} \quad \text{FOR A POINT CHARGE}$$

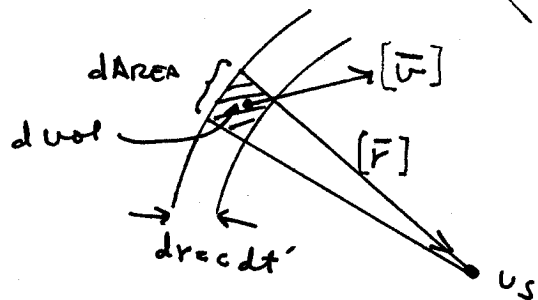
$$\text{LIKEWISE } \vec{A} = \frac{1}{c} \int \frac{[p] \vec{v} dvol}{r} \rightarrow \frac{e \vec{\beta}}{[r - \vec{r} \cdot \vec{\beta}]}$$

THESE RESULTS ARE THE SAME AS THOSE OBTAINED BY THE LORENTZ TRANSFORMATION. THEY ARE KNOWN AS THE LIENARD-WIECHERT POTENTIALS.

ON THE PROBLEM SET YOU HAVE ALSO USED A LORENTZ TRANSFORMATION TO FIND THE \vec{E} AND \vec{B} FIELDS OF A UNIFORMLY MOVING POINT CHARGE.

WE NOW FACE THE GENERAL QUESTION AS TO WHAT ARE \vec{E} AND \vec{B} FOR ARBITRARY MOTION OF A POINT CHARGE?

WE COULD MAKE A LORENTZ TRANSFORMATION OF THE CASE OF A CHARGE WHICH IS INSTANTANEOUSLY AT REST, BUT WHICH HAS NON-ZERO ACCELERATION. OR WE CAN DIFFERENTIATE THE LIENARD-WIECHERT POTENTIALS.



EITHER WAY THE RESULT IS

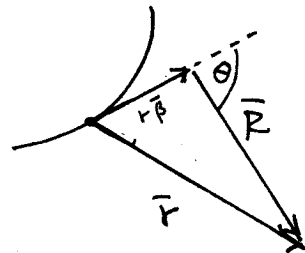
$$\vec{E} = \left[\frac{e}{\gamma^2 (r - \vec{r} \cdot \vec{\beta})^3} \right] + e \left[\frac{\vec{r} \times [(\vec{r} - r\vec{\beta}) \times \dot{\vec{\beta}}]}{c (r - \vec{r} \cdot \vec{\beta})^3} \right]$$

$$\vec{B} = [\hat{r}] \times \vec{E} \quad \text{WITH } [\vec{r}] = \text{RETARDED DISTANCE}$$

THE DETAILED DEMONSTRATION OF THESE RESULTS IS QUITE TEDIOUS - WITH PLENTY OF CHANCE FOR ALGEBRAIC ERROR. BECKER SEC 69 SKETCHES ONE METHOD. AS WE WILL FIND IT MORE USEFUL TO OBTAIN RESULTS FOR PARTICULAR EXAMPLES VIA LORENTZ TRANSFORMATIONS, WE WILL NOT BELABOR THE DERIVATION FOR \vec{E} AND \vec{B} .

FOR THE MOMENT WE MERELY NOTE THAT \vec{E} MAY BE WRITTEN $\vec{E} = \vec{E}_{\text{NEAR}} + \vec{E}_{\text{FAR}}$.

WE NOTE THAT $\vec{R} = \vec{r} - r\vec{\beta}$ WOULD BE THE PRESENT POSITION OF THE PARTICLE IF THE VELOCITY WERE UNIFORM



$$\text{SO } \vec{E}_{\text{NEAR}} = \frac{e \vec{R}}{\gamma^2 S^3} \sim \frac{1}{R^2}$$

$$\text{WITH } S = r - \vec{r} \cdot \vec{\beta} = R \sqrt{1 - \beta^2 \sin^2 \theta} \quad (\text{AS ON P 223}).$$

THIS IS IN FACT JUST THE FIELD OF A UNIFORMITY MOVING CHARGE WHICH YOU DERIVED VIA A LORENTZ TRANSFORMATION.

$$\vec{E}_{\text{FAR}} = \frac{e}{c^2 S^3} \vec{r} \times (\vec{R} \times \vec{a}) \sim \frac{1}{R} \quad (\vec{a} = c \dot{\vec{\beta}} = \text{ACCELERATION})$$

THIS IS OF COURSE THE RADIATION FIELD.

LOW VELOCITY LIMIT

IF $\frac{v}{c} \rightarrow 0$ THEN $\vec{R} \rightarrow \vec{r}$ AND $S \rightarrow r$,

$$\text{SO } \vec{E}_{\text{RADIATION}} \rightarrow \frac{e}{c^2} \left[\frac{\vec{r} \times (\vec{r} \times \vec{a})}{r^3} \right] = \frac{e}{c^2} \left[\frac{\hat{n} \times (\hat{n} \times \vec{a})}{r} \right] \quad (\hat{n} = \hat{r} = \frac{\vec{r}}{r})$$

$$\vec{B}_{\text{RAD}} = \hat{n} \times \vec{E}_{\text{RAD}} = \frac{e}{c^2} \left[\frac{\vec{a} \times \hat{n}}{r} \right] \quad [\vec{a} \text{ AND } \vec{r} \text{ ARE RETARDED QUANTITIES}]$$

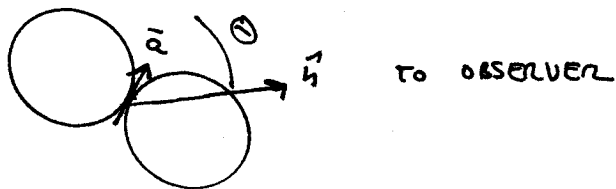
$$\text{AS NOTED EARLIER, } \vec{E}_{\text{RAD}} = -e \left[\frac{\vec{a}_{\perp}}{c^2 r} \right]$$

THE RADIATED POWER IS

$$\frac{dP}{d\Omega} = r^2 \bar{S} \cdot \hat{n} = \frac{c\gamma^2}{4\pi} \hat{n} \cdot \bar{E} \times \bar{B}$$

$$= \frac{e^2 (\hat{n} \times \ddot{a})^2}{4\pi c^3} = \frac{e^2 a^2 \sin^2 \theta}{4\pi c^3}$$

$$P_{TOT} = \frac{2e^2 a^2}{3c^3}$$

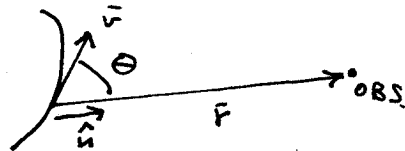


THESE RESULTS ARE THE LARMOR FORMULAE, WHICH WE DERIVED PREVIOUSLY FOR CONTINUOUS CHARGE DISTRIBUTIONS. (LECTURE 16)

IN THE NEXT LECTURE WE ILLUSTRATE NOW INTERESTING RESULTS FOR THE RELATIVISTIC CASE V < c CAN BE OBTAINED FROM THE LARMOR FORMULA.

FOR LATER REFERENCE WE WRITE THE FIELDS \bar{E} & \bar{B} IN A SLIGHTLY DIFFERENT FORM:

DEFINE θ AS THE ANGLE BETWEEN \vec{v} AND \vec{r} , THE RETARDED POSITION VECTOR



THEN $S = r - \vec{r} \cdot \vec{\beta} = r(1 - \beta \cos \theta) = r(1 - \hat{n} \cdot \vec{\beta})$

WE ALSO DEFINE $\hat{n} = \vec{r} / r$

THEN $\bar{E} = \frac{e}{r^2} \frac{\hat{n} - \vec{\beta}}{(1 - \beta \cos \theta)^3} + \frac{e}{c} \frac{\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}]}{r(1 - \beta \cos \theta)^3}$

$\bar{B} = \hat{n} \times \bar{E}$

ALL QUANTITIES ARE RETARDED.

THE RADIATED POWER $\frac{dP}{d\Omega} = \frac{c\gamma^2}{4\pi} \hat{n} \cdot \bar{E}_{RAD} \times \bar{B}_{RAD} = \frac{c\gamma^2 E_{RAD}^2}{4\pi}$

SO $\frac{dP}{d\Omega} = \frac{e^2}{4\pi c} \frac{|\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}]|^2}{(1 - \beta \cos \theta)^6} = \text{ANGULAR DISTRIBUTION OF RADIATION ACCORDING TO A FIXED OBSERVER.}$

APPENDIX: POTENTIALS & FIELDS OF A MOVING CHARGE OBTAINED VIA DELTA FUNCTIONS. (BECKER SECS. 66, 69, EYEGES, SEC. 14.3)

ANOTHER APPROACH TO THE PROBLEM OF A POINT CHARGE e WITH VELOCITY $\vec{v}(t)$ IS TO USE A DIRAC DELTA FUNCTION:

$$\phi = \int \frac{[\rho]}{r} dvol \rightarrow \int \frac{e}{r} \delta(t' - t + r/c) dt', \quad \vec{A} = \int \frac{[\vec{j}]}{rc} dvol \rightarrow \int \frac{e \vec{v}(t')}{rc} \delta(t' - t + r/c) dt'$$

BECAUSE r IS A FUNCTION OF t AND t' WE CANNOT IMMEDIATELY PERFORM THE INTEGRAL OVER THE δ -FUNCTION. RECALL

$$\int \delta(f(t)) dt = \int \delta(f) \frac{df}{df/dt} = \frac{1}{df/dt|_{f=0}}$$

IN OUR CASE $f = t' - t + r/c$ AND $\frac{df}{dt} = 1 + \frac{1}{c} \frac{dr}{dt} = 1 - \frac{\vec{r} \cdot \vec{v}}{rc}$ USING $r = \sqrt{(\vec{x} - \vec{x}') \cdot (\vec{x} - \vec{x}')}$
AND $\vec{r} = \vec{x} - \vec{x}'$

THUS $r \frac{df}{dt} = r - \vec{r} \cdot \vec{v} = r - \vec{r} \cdot \vec{\beta} = S$ (AS ON P. 234).

HENCE, $\phi = \left[\frac{e}{r - \vec{r} \cdot \vec{\beta}} \right]$, $\vec{A} = \left[\frac{e \vec{\beta}}{r - \vec{r} \cdot \vec{\beta}} \right]$ AS ALSO FOUND ON P. 223 & P. 233.

THE ABOVE CAN BE REGARDED AS A DEMONSTRATION THAT THE δ -FUNCTION METHOD IS CONSISTENT.

TO FIND \vec{E} & \vec{B} , WE DON'T USE THE LIENARD-WIECHERT POTENTIALS, BUT GO BACK TO THE FORMS AT THE TOP OF THE PAGE.

AMONG THE MANY WAYS TO PROCEED, WE CHOOSE A PATH THAT LEAVES PART OF \vec{E} AS $\frac{d}{dt}$ (SOMETHING).

$\vec{E} = -\vec{\nabla} \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$ $\vec{\nabla}$ ACTS ONLY ON r IN OUR EXPRESSION FOR ϕ

SO WE CAN WRITE $\vec{\nabla} = \frac{\partial r}{\partial \vec{r}} \frac{\partial}{\partial r} = \hat{n} \frac{\partial}{\partial r}$

THEN $\vec{E} = -e \int \hat{n} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \right) \right) \delta(t' - t + r/c) - e \int \frac{\hat{n}}{r} \left(\frac{\partial}{\partial r} \delta(t' - t + r/c) \right) - \frac{e}{c} \frac{d}{dt} \int \frac{\vec{\beta} \delta(t' - t + r/c)}{r}$
 $= e \int \frac{\hat{n}}{r^2} \delta(t' - t + r/c) - \frac{e}{c} \int \frac{\hat{n}}{r} \delta'(t' - t + r/c) dt' - \frac{e}{c} \frac{d}{dt} \int \frac{\vec{\beta} \delta(t' - t + r/c)}{r} dt'$

TRICK: $\frac{d}{dt} \delta(t' - t + r/c) = -\delta'(t' - t + r/c)$ WHERE $\delta'(f)$ MEANS $\frac{d}{df} \delta(f)$

THEN SINCE $\frac{\hat{n}}{r}$ IS NOT A FUNCTION OF t BEFORE INTEGRATION OVER t'

WE HAVE $\vec{E} = e \int \frac{\hat{n}}{r^2} \delta(t' - t + r/c) dt' + \frac{e}{c} \frac{d}{dt} \int \frac{\hat{n} \vec{\beta}}{r} \delta(t' - t + r/c) dt'$

INTEGRATING OVER THE δ -FUNCTION AS ON TOP OF P. 235a, WE GET

$$\bar{E} = \left[\frac{e \hat{n}}{rS} \right] + \frac{e}{c} \frac{d}{dt} \left[\frac{\hat{n} - \bar{B}}{S} \right]$$

MORE TRICKS: $t' = t - r/c$ AND $t = t' + r/c$ FOR []

THUS $\frac{dt}{dt'} = 1 - \frac{1}{c} \frac{dr}{dt'} = 1 - \hat{n} \cdot \bar{B} = \frac{S}{r}$

HENCE $\frac{dt'}{dt} = \frac{r}{S} = 1 - \frac{1}{c} \frac{dr}{dt}$ (USING $t' = t - r/c$)

ANOTHER USEFUL IDENTITY IS $\bar{B} = \frac{1}{c} \frac{d\bar{r}'}{dt'} = -\frac{1}{c} \frac{d\bar{r}}{dt'} = -\frac{1}{c} \frac{dt}{dt'} \frac{d\bar{r}}{dt} = -\frac{S}{rc} \frac{d(r\hat{n})}{dt}$

$$\begin{aligned} \bar{E} &= e \left[\frac{\hat{n}}{r^2} \left(1 - \frac{1}{c} \frac{dr}{dt} \right) \right] + \frac{e}{c} \frac{d}{dt} \left[\frac{\hat{n}}{r} \left(1 - \frac{1}{c} \frac{dr}{dt} \right) + \frac{1}{rc} \frac{d(r\hat{n})}{dt} \right] \\ &\quad + \frac{e}{c} \frac{d}{dt} \left[r \frac{\hat{n}}{r^2} - \frac{\hat{n}}{rc} \frac{dr}{dt} + \frac{\hat{n}}{rc} \frac{dr}{dt} + \frac{1}{c} \frac{d\hat{n}}{dt} \right] \\ &= \left[e \frac{\hat{n}}{r^2} - \frac{e \hat{n}}{cr^2} \frac{dr}{dt} + \frac{e}{c} \frac{\hat{n}}{r^2} \frac{dr}{dt} + \frac{er}{c} \frac{d}{dt} \left(\frac{\hat{n}}{r^2} \right) + \frac{e}{c^2} \frac{d^2 \hat{n}}{dt^2} \right] \end{aligned}$$

$$\boxed{\bar{E} = e \left[\frac{\hat{n}}{r^2} \right] + \frac{e}{c} \left[r \frac{d}{dt} \frac{\hat{n}}{r^2} \right] + \frac{e}{c^2} \left[\frac{d^2 \hat{n}}{dt^2} \right]}$$

HEAVISIDE, ELECTRO-MAGNETIC THEORY, VOL III, P 438 (1912)
REDISCOVERED BY PEYMAN (1962)

LIKEWISE $\bar{B} = \nabla \times \bar{A} = e \int \nabla \times \frac{\bar{B}}{r} \delta(\epsilon' - t + r/c) dt'$ ($\nabla \times \bar{B} = 0$ WHEN INSIDE INTEGRAL)

$$\begin{aligned} &= -e \int \bar{B} \times \left(\frac{\nabla}{r} \right) \delta(\epsilon' - t + r/c) dt' - e \int \bar{B} \times \frac{1}{r} \nabla \delta(\epsilon' - t + r/c) dt' \\ &= +e \int \bar{B} \times \frac{\hat{n}}{r^2} \delta(\epsilon' - t + r/c) dt' - e \int \bar{B} \times \frac{\hat{n}}{rc} \delta'(\epsilon' - t + r/c) dt' \\ &= e \left[\frac{\bar{B} \times \hat{n}}{rS} \right] + e \frac{d}{dt} \int \frac{\bar{B} \times \hat{n}}{rc} \delta(\epsilon' - t + r/c) dt' \\ &= \frac{e}{c} \left[-\frac{d(r\hat{n})}{dt} \times \frac{\hat{n}}{r^2} \right] + \frac{e}{c} \frac{d}{dt} \left[\frac{\bar{B} \times \hat{n}}{S} \right] \\ &= \left[\hat{n} \times \frac{e}{cr} \frac{d\hat{n}}{dt} \right] + \frac{e}{c} \frac{d}{dt} \left[-\frac{d(r\hat{n})}{dt} \times \frac{\hat{n}}{r} \right] \\ &= \left[\hat{n} \times \frac{er}{c} \frac{d}{dt} \frac{\hat{n}}{r} \right] + \frac{e}{c^2} \left[\hat{n} \times \frac{d^2 \hat{n}}{dt^2} \right] = [\hat{n}] \times \bar{E} \end{aligned}$$

SO $\boxed{\bar{B} = [\hat{n}] \times \bar{E}}$

A SECOND FORM FOR \vec{E} CAN BE FOUND BY RETURNING TO THE TOP OF P. 235b:

$$\vec{E} = \left[\frac{e \hat{n}}{rs} \right] + \frac{e}{c} \frac{d}{dt} \left[\frac{\hat{n} - \vec{\beta}}{s} \right]$$

$$\frac{d}{dt} = \frac{dt'}{dt} \frac{d}{dt'} = \frac{r}{s} \frac{d}{dt'} \quad \text{HENCE}$$

$$\vec{E} = e \left[\frac{\hat{n}}{rs} \right] + \frac{e}{c} \left[r \frac{\dot{\hat{n}} - \dot{\vec{\beta}}}{s^2} - \frac{r\dot{s}}{s^3} (\hat{n} - \vec{\beta}) \right] \quad \text{WHERE } [\dot{f}] \equiv \left[\frac{df}{dt'} \right]$$

Now $\dot{\vec{r}} = -c(\hat{n} \cdot \vec{\beta})$, $\dot{\vec{r}} = -c\vec{\beta}$

AND $\dot{s} = \dot{r} - \dot{\vec{r}} \cdot \vec{\beta}$ so $\dot{s} = \dot{r} - \dot{\vec{r}} \cdot \vec{\beta} - \vec{r} \cdot \dot{\vec{\beta}} = -c(\hat{n} \cdot \vec{\beta}) + c\beta^2 - \vec{r} \cdot \dot{\vec{\beta}}$

ALSO, $\hat{n} = \frac{\vec{r}}{r}$ so $\dot{\hat{n}} = \frac{\dot{\vec{r}}}{r} - \frac{\vec{r}\dot{r}}{r^2} = \frac{c}{r} (-\vec{\beta} + \hat{n}(\hat{n} \cdot \vec{\beta}))$

$$\begin{aligned} \vec{E} &= e \left[\frac{\hat{n}}{rs} + \frac{1}{s^2} (-\vec{\beta} + \hat{n}(\hat{n} \cdot \vec{\beta})) - \frac{r\dot{\vec{\beta}}}{cs^2} - \frac{r(\hat{n} - \vec{\beta})}{cs^3} (-c(\hat{n} \cdot \vec{\beta}) + c\beta^2 - \vec{r} \cdot \dot{\vec{\beta}}) \right] \\ &= e \left[\hat{n} \left(\frac{1}{rs} + (\hat{n} \cdot \vec{\beta}) \left(\frac{1}{s^2} + \frac{r}{s^3} \right) - \frac{r\beta^2}{s^3} \right) - \vec{\beta} \left(\frac{1}{s^2} + \frac{r(\hat{n} \cdot \vec{\beta})}{s^3} - \frac{r\beta^2}{s^2} \right) - \frac{r\dot{\vec{\beta}}}{cs^2} + \frac{r^2}{cs^3} (\hat{n} - \vec{\beta})(\hat{n} \cdot \dot{\vec{\beta}}) \right] \\ &\quad \begin{array}{l} \hat{n} \left(\frac{s^2 + sr(\hat{n} \cdot \vec{\beta}) + r^2(\hat{n} \cdot \vec{\beta})}{rs^3} \right) \quad \frac{s + r(\hat{n} \cdot \vec{\beta})}{s^3} \quad \frac{-rs\dot{\vec{\beta}} + (\vec{r} - r\vec{\beta})(\vec{r} \cdot \dot{\vec{\beta}})}{cs^3} \\ \hat{n} \frac{sr + r^2(\hat{n} \cdot \vec{\beta})}{rs^3} \quad \frac{r}{s^3} \quad \frac{-r^2\dot{\vec{\beta}} + r(\vec{r} \cdot \vec{\beta})\dot{\vec{\beta}} + (\vec{r} - r\vec{\beta})(\vec{r} \cdot \dot{\vec{\beta}})}{cs^3} \\ \frac{r\hat{n}}{s^3} \quad \frac{-r\hat{n}\beta^2}{s^3} \quad -\frac{r\vec{\beta}}{s^3} + \frac{r\beta^2}{s^3} \end{array} \end{aligned}$$

$$\boxed{\vec{E} = e \left[\frac{\vec{r} - r\vec{\beta}}{r^2(r - \vec{r} \cdot \vec{\beta})^3} + \frac{\vec{r} \times ((\vec{r} - r\vec{\beta}) \times \dot{\vec{\beta}})}{c(r - \vec{r} \cdot \vec{\beta})^3} \right]}$$

LIÉNARD - WIECHERT (1898-1900)

NOTING THAT $1 - \beta^2 = \frac{1}{\gamma^2}$

OF COURSE $\vec{B} = [\hat{n}] \times \vec{E}$ STILL

FOOTNOTE: THE EXPRESSION FOR \vec{E} AT THE TOP OF THE PAGE CAN BE DERIVED FROM THE GENERAL FORM FOR \vec{E} ON P. 178 - WITH CARE. IT APPEARS THAT WE MUST MAKE THE TRANSFORMATION $\int [\dot{p}] d\text{vol} \rightarrow - \int e \delta'(t' - t + r/c) dt'$ AND NOT +, PERHAPS AGREEING THAT t AND NOT t' IS WHAT THE \bullet ACTS ON IN THIS CASE. IF SO

$$\begin{aligned} \vec{E} &= \int \left(\frac{[\rho]}{r^2} \hat{n} + \frac{1}{c} \frac{[\dot{\rho}]}{r} \hat{n} - \frac{1}{c^2} \frac{[\dot{j}]}{r} \right) d\text{vol} \rightarrow e \int \frac{\hat{n}}{r^2} \delta(t' - t + r/c) dt' - \frac{e}{c} \int \frac{\hat{n} - \vec{\beta}}{r} \delta'(t' - t + r/c) dt' \\ &= e \left[\frac{\hat{n}}{rs} \right] + \frac{e}{c} \frac{d}{dt} \int \frac{\hat{n} - \vec{\beta}}{r} \delta(t' - t + r/c) dt' = e \left[\frac{\hat{n}}{rs} \right] + \frac{e}{c} \frac{d}{dt} \left[\frac{\hat{n} - \vec{\beta}}{s} \right]. \end{aligned}$$