

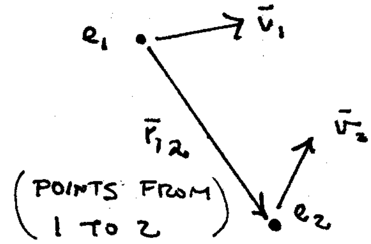
MECHANICS AND ELECTROMAGNETISM

IN THIS LECTURE WE PRESENT ARGUMENTS THAT OUR DISCUSSIONS OF MECHANICS AND OF ELECTROMAGNETISM ARE CONSISTENT WITH ONE ANOTHER.

NEWTON'S THIRD LAW [WE FOLLOW PAGE & ADAMS, AM J. PHYS 13, 141 (1945)]

FROM NEWTON'S 3RD LAW WE DEDUCED THAT THE TOTAL MOMENTUM OF AN ISOLATED SYSTEM MUST BE CONSTANT. AND FURTHER, IF THE FORCES ARE DIRECTED ALONG THE LINES OF CENTERS OF THE PARTICLES THEN THE TOTAL ANGULAR MOMENTUM OF THE SYSTEM IS A CONSTANT.

WE PREVIOUSLY MENTIONED A SIMPLE EXAMPLE WHICH APPEARS TO CONFOUND NEWTON. A CHARGE e_1 IS MOVING WITH VELOCITY $\bar{v}_1$ WHILE CHARGE e_2 MOVES WITH $\bar{v}_2$.



IF $\bar{v}_1$ IS NOT $\parallel$ TO $\bar{v}_2$ WE READILY CONVINCE OURSELVES THAT THE FORCE ON 1 IS NOT EQUAL NOR OPPOSITE TO THE FORCE ON 2!

DOES THIS INVALIDATE MECHANICS, OR ELECTRICITY?

AS HINTED EARLIER, WE RESTORE CONSISTENCY BY TAKING INTO ACCOUNT THE MOMENTUM OF THE FIELDS. WE WISH TO SHOW THAT THE SUM OF THE MECHANICAL MOMENTUM PLUS FIELD MOMENTUM IS CONSTANT. SO ALTHOUGH WE DO HAVE TO GIVE UP NEWTON'S 3RD LAW, WE DO NOT GIVE UP ITS MOST IMPORTANT CONSEQUENCE: CONSERVATION OF MOMENTUM FOR ISOLATED SYSTEMS.

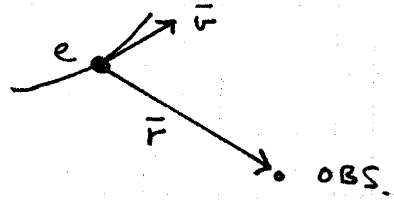
TO MAKE A DEMONSTRATION OF THIS IT IS DESIREABLE TO A HAVE AN EXPRESSION FOR THE FIELDS OF AN ACCELERATING CHARGE IN TERMS OF <u>PRESENT</u> RATHER THAN RETARDED QUANTITIES. THIS IS POSSIBLE IN PRINCIPLE IF THE MOTION IS KNOWN. HOWEVER PROBABLY THE ONLY CASES IN WHICH THE EXACT SOLUTIONS ARE KNOWN ARE FOR UNIFORMLY ACCELERATED MOTION (WITH $\bar{a} = 0$ AS A SPECIAL CASE: PROBLEM 10, SET 9)

FOR THE GENERAL CASE IT IS POSSIBLE TO GIVE POWER SERIES EXPANSIONS FOR THE FIELDS. IN THE APPENDIX TO THIS LECTURE WE INDICATE HOW TO PERFORM ONE SUCH EXPANSION, WHICH IS ORGANIZED BY THE POWER OF $1/c$.

THE RESULTS, ACCURATE TO ORDER $(1/c)^2$ ARE

$$\bar{E} = e\frac{\bar{r}}{r^3}\left(1 + \frac{v^2}{2c^2} - \frac{3}{2}\frac{(\hat{r}\cdot\bar{v})^2}{c^2}\right) - \frac{e}{2c^2 r}\left(\bar{a} + (\bar{a}\cdot\hat{r})\hat{r}\right) + O\left(\frac{1}{c^3}\right)$$

$$\bar{B} = \frac{e\,\bar{v}\times\bar{r}}{c\,r^3} + O\left(\frac{1}{c^3}\right)$$



ALL QUANTITIES EVALUATED AT THE PRESENT TIME OF THE OBSERVER.

THE FORM OF $\bar{E}$ IS NOT IMPLAUSIBLE ON COMPARISON WITH THE EXPRESSION FOR $\bar{E}$ IN TERMS OF RETARDED QUANTITIES (p234)

OUR VERSION OF $\bar{B}$ GIVES NO SIGN OF THE RADIATION FIELDS! THEY APPEAR IN HIGHER ORDER. HOWEVER THIS IS FINE FOR OUR PURPOSES, IN WHICH WE WILL SUPPOSE THE ACCELERATION IS SO SMALL THAT WE MAY IGNORE THE RADIATION ALTOGETHER. OUR ARGUMENTS WILL EMPHASIZE PROPERTIES OF THE FIELDS AT RELATIVELY SMALL r, FOR WHICH THE ABOVE EXPANSION IS SUFFICIENT.

FOR OUR EXAMPLE OF 2 CHARGES WE MAY NOW CALCULATE THE ELECTROMAGNETIC FORCES.

$$\bar{F}_{\text{ON }2} = e_2\left(\bar{E}_{\text{FROM }1} + \frac{\bar{v}_2}{c}\times\bar{B}_{\text{FROM }1}\right) \qquad (r \equiv |\bar{r}_{12}|)$$

$$= \frac{e_1 e_2}{r^3}\left\{\bar{r}_{12}\left(1 + \frac{v_1^2}{2c^2} - \frac{3}{2}\left(\frac{\hat{r}_{12}\cdot\bar{v}_1}{c}\right)^2\right) - \frac{r^2}{2c^2}\left(\bar{a}_1 + (\bar{a}_1\cdot\hat{r}_{12})\hat{r}_{12}\right) + \frac{\bar{v}_2\times(\bar{v}_1\times\bar{r}_{12})}{c^2}\right\}$$

AND

$$\bar{F}_{\text{ON }1} = \frac{e_1 e_2}{r^3}\left\{\bar{r}_{21}\left(1 + \frac{v_2^2}{2c^2} - \frac{3}{2}\left(\frac{\hat{r}_{12}\cdot\bar{v}_2}{c}\right)^2\right) - \frac{r^2}{2c^2}\left(\bar{a}_2 + (\bar{a}_2\cdot\hat{r}_{21})\hat{r}_{21}\right) + \frac{\bar{v}_1\times(\bar{v}_2\times\bar{r}_{21})}{c^2}\right\}$$

TO ORDER $1/c^2$.

NOW $\bar{r}_{12} = -\bar{r}_{21}$, SO THE NET FORCE IS

$$\bar{F}_{\text{TOTAL}} = \frac{e_1 e_2}{c^2 r^3}\left\{\frac{\bar{r}_{12}}{2}\left[v_1^2 - v_2^2 - 3(\hat{r}_{12}\cdot\bar{v}_1)^2 + 3(\hat{r}_{12}\cdot\bar{v}_2)^2\right] - \frac{r^2}{2}\left(\bar{a}_1 + \bar{a}_2 + ((\bar{a}_1+\bar{a}_2)\cdot\hat{r}_{12})\hat{r}_{12}\right) + (\bar{v}_2\times\bar{v}_1)\times\bar{r}_{12}\right\}$$

USING THE IDENTITY $\bar{a}\times(\bar{b}\times\bar{c}) - \bar{b}\times(\bar{a}\times\bar{c}) = (\bar{a}\times\bar{b})\times\bar{c}$

THIS IS OUR APPARENT VIOLATION OF NEWTON'S 3RD LAW. THE VIOLATION GOES LIKE $(v/c)^2$ OR a/c^2, AND COULD BE SAID TO BE A RELATIVISTIC EFFECT. THE ONLY SIMPLE CASE WHERE $\bar{F}_{\text{TOTAL}} = 0$ IS WHEN $\bar{v}_1 = \pm\bar{v}_2$, $\bar{a}_1 = -\bar{a}_2$.

Ph 206 LECTURE 24

FROM MECHANICS WE EXPECT $\bar{F}_{TOTAL} = \dfrac{d\bar{P}_{MECHANICAL}}{dt}$

WHERE $\bar{P}_{MECHANICAL} = m_1\bar{v}_1 + m_2\bar{v}_2$

(THE MASSES INCLUDE THE RELATIVISTIC MASS INCREASE $m = \gamma m^*$)

OUR CLAIM IS THAT $\dfrac{d\bar{P}_{FIELD}}{dt} = -\dfrac{d\bar{P}_{MECH}}{dt}$ SO $\bar{P}_{MECH} + \bar{P}_{FIELD} = \text{CONST.}$

FROM LECTURE 10, $\bar{P}_{FIELD} = \dfrac{\bar{E}\times\bar{B}}{4\pi c}$ IS THE MOMENTUM <u>DENSITY</u>

WHICH WE MUST INTEGRATE OVER ALL SPACE.

AT ANY POINT IN SPACE WE MAY WRITE $\bar{E} = \bar{E}_{FROM\,1} + \bar{E}_{FROM\,2}$

SO $$\bar{P}_F = \frac{1}{4\pi c}\left[(\bar{E}_1+\bar{E}_2)\times(\bar{B}_1+\bar{B}_2)\right] = \frac{1}{4\pi c}\left[\bar{E}_1\times\bar{B}_1 + \bar{E}_2\times\bar{B}_2 + \bar{E}_1\times\bar{B}_2 + \bar{E}_2\times\bar{B}_1\right]$$

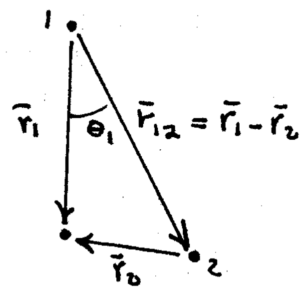
NOW AS OUR INTEGRATION APPROACHES POINT 1, $\bar{E}_1\times\bar{B}_1$ DIVERGES. THIS IS THE PROBLEM OF THE SELF-MOMENTUM OF A POINT CHARGE. WE TAKE THE ATTITUDE THAT THIS INTEGRAL $\int \dfrac{\bar{E}_1\times\bar{B}_1}{4\pi c}\,dvol$ IS ALREADY INCLUDED IN THE MECHANICAL MOMENTUM $m_1\bar{v}_1$.

BY THE FIELD MOMENTUM WE REALLY MEAN THAT EXTRA MOMENTUM DUE TO THE <u>INTERACTION</u> OF THE FIELDS FROM CHARGES 1 AND 2. THIS MOMENTUM IS FINITE.

$$\bar{P}_{FIELD} = \frac{1}{4\pi c}\int dvol\left[\bar{E}_1\times\bar{B}_2 + \bar{E}_2\times\bar{B}_1\right]$$

$$= \frac{e_1 e_2}{4\pi c^2}\int dvol\,\frac{\bar{r}_1\times(\bar{v}_2\times\bar{r}_2) + \bar{r}_2\times(\bar{v}_1\times\bar{r}_1)}{r_1^3\,r_2^3}$$ ACCURATE TO $(v/c)^2$

TO EVALUATE THIS INTEGRAL AT SOME FIXED TIME, WE CHOOSE OUR ORIGIN AT PARTICLE 1, AND TAKE THE z AXIS ALONG VECTOR $\bar{r}_{12}$ (WHICH IS A CONSTANT VECTOR DURING THE INTEGRATION)



THEN $dvol = r_1^2\,dr_1\,\sin\theta_1\,d\theta_1\,d\phi$

A CLEVER TRICK IS TO NOTE THAT

$$r_2^2 = r_1^2 + r^2 - 2rr_1\cos\theta_1 \qquad r \equiv |\bar{r}_{12}|$$

SO ON THE SPHERE $r_1 = \text{CONSTANT}$ WE ALSO HAVE

$$r_2\,dr_2 = r r_1 \sin\theta_1\,d\theta_1$$

THUS WE CAN WRITE $d\,vol = \frac{r_1 r_2}{r}\, dr_1\, dr_2\, d\phi$

THE LIMITS OF THE r_2 INTEGRATION DEPEND ON WHETHER r_1 IS GREATER OR LESS THAN r:

$$\int d\,vol = \frac{1}{r}\int_0^r r_1\,dr \int_{r-r_1}^{r+r_1} r_2\,dr_2 \int_0^{2\pi} d\phi + \frac{1}{r}\int_r^\infty r_1\,dr \int_{r_1-r}^{r_1+r} r_2\,dr \int_0^{2\pi} d\phi$$

THE NUMERATOR OF THE INTEGRAND IS $(\bar{r}_1\cdot\bar{r}_2)\bar{v}_2 - (\bar{r}_1\cdot\bar{v}_2)\bar{r}_2 + (\bar{r}_1\cdot\bar{r}_2)\bar{v}_1 - (\bar{r}_2\cdot\bar{v}_1)\bar{r}_1$

CONSIDER FIRST THE PIECE $(\bar{r}_1\cdot\bar{r}_2)(\bar{v}_1+\bar{v}_2) = \left(\frac{r_1^2+r_2^2-r^2}{2}\right)(\bar{v}_1+\bar{v}_2)$

SINCE $\bar{r}_{12} = \bar{r}_1 - \bar{r}_2$

$$\frac{e_1 e_2(\bar{v}_1+\bar{v}_2)}{4\pi c^2}\,\frac{2\pi}{2r}\iint \frac{dr_1\,dr_2}{r_1^2 r_2^2}\left(r_2^2+r_1^2-r^2\right) = \frac{e_1 e_2}{2c^2}\,\frac{\bar{v}_1+\bar{v}_2}{r}$$

$$\underbrace{\int \frac{dr_1}{r_1^2}\left(r_2 - \frac{r_1^2-r^2}{r_2}\right)\Bigg|_{r_{2min}}^{r_{2max}}}_{2}$$

CONSIDER NEXT THE PIECE $-(\bar{r}_2\cdot\bar{v}_1)\bar{r}_1 = (\bar{r}_{12}\cdot\bar{v}_1)\bar{r}_1 - (\bar{r}_1\cdot\bar{v}_1)\bar{r}_1$

WE READILY SEE THAT UPON INTEGRATION, ONLY THE COMPONENT ALONG DIRECTION $\bar{r}_{12}$ WILL SURVIVE. FURTHERMORE, IF WE WRITE $\bar{v}_1 = \bar{v}_{1\parallel} + \bar{v}_{1\perp}$ WHERE $\parallel \Rightarrow \parallel$ TO $\bar{r}_{12}$

THEN ONLY THE PIECE $(\bar{r}_1\cdot\bar{v}_{1\parallel})\bar{r}_1$ WILL SURVIVE.

ALTOGETHER THE NON-TRIVIAL PART OF THE INTEGRAND IS

$$(\bar{r}_{12}\cdot\bar{v}_1)\; r_1\cos\theta_1\left(1-\frac{r_1}{r}\cos\theta_1\right)$$

NOW $\cos\theta_1 = \frac{r_1^2+r^2-r_2^2}{2rr_1}$ SO WE CAN CARRY OUT THE INTEGRAL. WE MERELY QUOTE THE RESULT

$$\bar{P}_{FIELD} = \frac{e_1 e_2}{2c^2 r}\left[\bar{v}_1+\bar{v}_2 + \left[(\bar{v}_1+\bar{v}_2)\cdot\hat{r}_{12}\right]\hat{r}_{12}\right]$$

IN CALCULATING $\frac{d\bar{P}_{FIELD}}{dt}$ WE NOTE THAT $\dot{\bar{r}}_{12} = \bar{v}_2 - \bar{v}_1$ DUE TO OUR GEOMETRICAL DEFINITIONS. BUT $\bar{a}_1 = \dot{\bar{v}}_1$, $\bar{a}_2 = \dot{\bar{v}}_2$.

THEN INDEED WE FIND THAT $\frac{d\bar{P}_{FIELD}}{dt} = -\frac{d\bar{P}_{MECH}}{dt}$!

LIKEWISE WE NOTE THAT THE FORCES $\bar{F}_1$ AND $\bar{F}_2$ AS ABOVE LEAD TO A NET TORQUE ABOUT AN ARBITRARY POINT. BUT IF WE CALCULATE THE RATE OF CHANGE OF THE FIELD ANGULAR MOMENTUM WE CONCLUDE THAT TOTAL ANGULAR MOMENTUM IS CONSERVED.

THE ARTICLE BY PAGE & ADAMS DISCUSSES SEVERAL OTHER EXAMPLES OF THIS NATURE.

BECKER SEC'S 64 & 91 DISCUSSES THE TROUTON-NOBLE EXPERIMENT - IN WHICH AN EVALUATION OF THE LORENTZ FORCE ON THE PLATES OF A MOVING CAPACITOR SUGGESTS THAT A TORQUE IS PRESENT. THE RESOLUTION OF THIS PARADOX DOES NOT INVOLVE FIELD ANGULAR MOMENTUM (THE VECTOR $\bar{r}_{12}$ BETWEEN THE PLATES IS CONSTANT EVEN FOR A MOVING CAPACITOR). RATHER THE MECHANICAL FORCES WHICH HOLD THE PLATES APART WILL PROVIDE A COUNTER TORQUE, ACCORDING TO THE LORENTZ TRANSFORMATION OF FORCE.

AT THE END OF LECTURE 10 WE GAVE A PARADOX ABOUT FIELD ANGULAR MOMENTUM. A FORMAL DISCUSSION OF THIS PARADOX CAN BE FOUND IN AM. J. PHYS 51, 213 (1983).

BINDING ENERGY AND ELECTROMAGNETIC MASS [T. H. BOYER, AM. J. PHYS 46, 383 (1978)

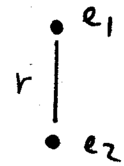
IN LECTURE 22 WE EXPLORED THE IDEA THAT THE ELECTROMAGNETIC FIELD ENERGY OF A POINT CHARGE CONTRIBUTES TO THE MASS OF THE CHARGE. WE DID NOT OBTAIN COMPLETE SATISFACTION, BUT WERE LEFT WITH THE NOTION OF 'MASS RENORMALIZATION': THE MASS OF A CHARGE IS THE SUM OF ELECTROMAGNETIC AND NON ELECTROMAGNETIC TERMS EACH OF WHICH IS INFINITE FOR A POINT CHARGE, BUT THE SUM IS FINITE.

WE DO NOT WISH TO LEAVE THE IMPRESSION THAT THE CONCEPT OF ELECTROMAGNETIC MASS IS COMPLETELY TIED TO THE IDEA OF RENORMALISATION. IF WE CONSIDER THE INTERACTION ENERGY BETWEEN TWO POINT CHARGES WE CALCULATE A FINITE RESULT (LECTURE 3), WHICH CORRESPONDS TO A FINITE ELECTROMAGNETIC MASS. WE CAN DEMONSTRATE THE CONSISTENCY OF THIS MASS CORRECTION (DUE TO THE 'BINDING ENERGY') WITH CLASSICAL MECHANICS.

(THIS WAS THE THEME OF PROBLEM 9, SET 11, ALSO.)

CONSIDER TWO POINT CHARGES SEPARATED BY DISTANCE r.

WHEN THE CHARGES ARE AT REST THEIR <u>INTERACTION</u> <u>ENERGY</u> IS $$U = \frac{e_1 e_2}{r}$$

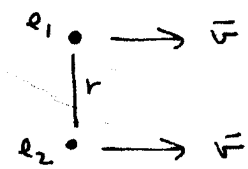


[THIS COMES FROM $U = \frac{1}{2}\int \rho\,\phi\, d\mathrm{vol}$, NOT $U = \frac{1}{8\pi}\int E^2 d\mathrm{vol}$ WHICH INCLUDES THE DIVERGENT SELF-ENERGY TERMS]

IF WE WRITE m_1^* AND m_2^* AS THE REST MASSES OF THE CHARGES IN ISOLATION FROM ONE ANOTHER, THEN WE WOULD SAY $m^* = m_1^* + m_2^* + \frac{e_1 e_2}{rc^2}$ = REST MASS OF THE WHOLE SYSTEM

$\frac{e_1 e_2}{rc^2}$ = CONTRIBUTION TO REST MASS FROM THE BINDING ENERGY.

WE NOW CONSIDER THE SYSTEM OF CHARGES IN MOTION AT RIGHT ANGLES TO THE LINE OF CENTERS. THE SEPARATION r BETWEEN THE CHARGES REMAINS CONSTANT – WHICH REQUIRES SOME NON-ELECTROMAGNETIC FORCE TO BE PRESENT.



FROM THE PRECEDING SECTION WE FIND THAT $\bar{P}_{FIELD} = \frac{e_1 e_2 \bar{v}}{rc^2}$ ACCURATE TO TERMS IN $(v/c)^2$. THUS THE TOTAL MOMENTUM OF THE SYSTEM IS $$\bar{P}_{TOT} = (m_1^* + m_2^*)\bar{v} + \frac{e_1 e_2 \bar{v}}{rc^2} = m^* \bar{v} \qquad \text{TO } \left(\frac{v}{c}\right)^2$$

WE CAN OBTAIN FURTHER EVIDENCE OF THE CONSISTENCY OF THE ELECTROMAGNETIC MASS CONTRIBUTION BY SUPPOSING THE SYSTEM UNDERGOES SMALL ACCELERATION $\bar{a} \parallel \bar{v}$.

[FOR $\bar{a}$ SMALL WE IGNORE ANY EFFECTS OF RADIATION, WHICH GO LIKE a^2]

TO ACCELERATE A SYSTEM OF REST MASS m^* SUCH THAT $\bar{a} \parallel \bar{v}$ WE MUST HAVE AN EXTERNAL FORCE

$$\bar{F}_{EXT} = \frac{d(\gamma m^* \bar{v})}{dt} = \gamma^3 m^* \bar{a}$$ [RECALL PROBLEM 3, SET 10]

ON THE OTHER HAND, THE LAWS OF MECHANICS WOULD SAY

$$\bar{F}_{TOT} = \bar{F}_{EXT} + \bar{F}_{1\,ON\,2} + \bar{F}_{2\,ON\,1} = \gamma^3 m^*_{MECH}\,\bar{a}$$

WHERE $m^*_{MECH} = m_1^* + m_2^*$ DOES NOT INCLUDE THE EFFECTIVE MASS OF THE INTERACTION ENERGY $U = e_1 e_2 / r$

FROM p. 287, WITH $\bar{v}_1 = v_2$, $\bar{a}_1 = \bar{a}_2 = a$ AND $\bar{r}_{12} = r \perp \bar{a}$

WE FIND $$\bar{F}_{1\,ON\,2} + \bar{F}_{2\,ON\,1} = -\frac{e_1 e_2 \bar{a}}{c^2 r}$$

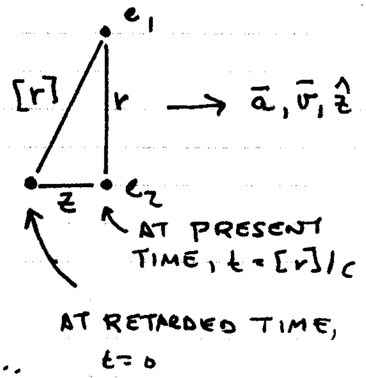
THUS $$\bar{F}_{EXT} = \left(\gamma^3 m^*_{MECH} + \frac{e_1 e_2}{c^2 r}\right)\bar{a} \approx \gamma^3\left(m^*_{MECH} + \frac{e_1 e_2}{c^2 r}\right)\bar{a} + O\left(\frac{1}{c^4}\right)$$
$$= \gamma^3 m^* \bar{a}$$

THUS THE FORCES THAT APPEAR TO VIOLATE NEWTON'S 3RD LAW CAN BE REINTERPRETED AS THE MASS OF THE FIELD TIMES ACCELERATION, WHILE NEWTON'S 2ND LAW NOW READS $\bar{F}_{EXT} = m_{TOT}\,\bar{a}$
$$= (m_{MECH} + m_{E\text{-}M})\,\bar{a}$$

RADIATION REACTION

THE PRECEDING ARGUMENTS IN THIS LECTURE HAVE IGNORED RADIATION. THE MODEL OF 2 CHARGES MOVING WITH $\bar{a} \parallel \bar{v} \perp \bar{r}_{12}$ CAN BE EXTENDED TO ILLUSTRATE HOW THE RADIATION REACTION ARISES. [D.J. GRIFFITHS & R.E. OWEN, AM. J. PHYS. 51, 1120 (1983)]

WE WILL USE THE LIENARD-WIECHERT FIELDS (p. 234) WHICH ARE SIMPLEST IF THE CHARGES ARE AT REST AT THE RETARDED TIME. THAT IS, IF $t=0$ IS THE MOMENT WHEN THE CHARGES ARE AT REST, WE CALCULATE THE FIELD AT PRESENT TIME $t = \frac{[r]}{c} > 0$.



AS BEFORE, r = SEPARATION OF CHARGES AT PRESENT TIME.

LET z = DISTANCE TRAVELLED $\parallel$ TO $\bar{a}$ DURING TIME $t = [r]/c$

THEN $[r] = \sqrt{r^2 + z^2} \approx r + \frac{z^2}{2r}$ SO $t = \frac{[r]}{c} \approx \frac{r}{c} + \frac{z^2}{2rc}$

WE EXPAND TO FIND $z(t) = z(0) + \dot{z}(0)\,t + \frac{1}{2}\ddot{z}(0)\,t^2 + \frac{1}{6}\dddot{z}(0)\,t^3 + \ldots$
$$= \frac{1}{2}[a]\,t^2 + \frac{1}{6}[\dot{a}]\,t^3 + \ldots \quad \text{WITH } [a] \text{ AT RETARDED TIME}$$

SO $$z \approx \frac{1}{2}[a]\left(\frac{r^2}{c^2} + \frac{z^2}{c^2}\right) + \frac{1}{6}[\dot{a}]\left(\frac{r^3}{c^3} + \frac{3}{2}\frac{r z^2}{c^3}\right) + \ldots$$

ITERATING, $z \approx \frac{1}{2}[a]\, r^2/c^2 + \frac{1}{6}[\dot{a}]\frac{r^3}{c^3} + O\left(\frac{r^4}{c^4}\right)$

WE CAN RELATE THE RETARDED ACCELERATION $[a]$ TO THE PRESENT ACCELERATION a VIA

$$[a] \approx a - \dot{a}t \approx a - \dot{a}\, r/c \quad ; \; [\dot{a}] \approx \dot{a} - \ddot{a}t \approx \dot{a}$$

THEN THE FIELD ON CHARGE 1 AT TIME t FROM 2 IS (p. 234)

$$\bar{E}_{ON\,1}(t) = e_2\left[\frac{\bar{r}}{r^3}\right] + \frac{e_2}{c^2}\left[\frac{\bar{r}\times(\bar{r}\times\bar{a})}{r^3}\right] = \frac{e_2}{[r^3]}\left[\bar{r} - \frac{r^2\bar{a}}{c^2} + \frac{(\bar{r}\cdot\bar{a})\bar{r}}{c^2}\right]$$

WE WILL NEED ONLY THE z-COMPONENT OF $\bar{E}$, SINCE THE TRANSVERSE COMPONENTS OF THE LORENTZ FORCE WILL CANCEL WHEN ADDING 1 & 2.

$$\text{So } E_{ON\,1,z}(t) = \frac{e_2}{[r^3]}\left(z - \frac{[a]}{c^2}\left([r^2]-z^2\right)\right) = \frac{e_2}{[r^3]}\left(z - \frac{[a]\,r^2}{c^2}\right) \quad \text{PRESENT } r$$

$$\approx \frac{e_2}{[r^3]}\left(-\frac{[a]r^2}{2c^2} + \frac{1}{6}[\dot{a}]\frac{r^3}{c^3}\right)$$

$$\approx \frac{e_2}{[r^3]}\left(-\frac{a r^2}{2c^3} + \frac{2}{3}\dot{a}\frac{r^3}{c^3}\right)$$

USING z & $[a]$ FROM ABOVE.

FINALLY WE CAN NOW APPROXIMATE $[r^3]$ BY r^3 TO YIELD

$$E_{ON\,1,z} \approx e_2\left(\frac{-a}{2rc^2} + \frac{2}{3}\frac{\dot{a}}{c^3}\right)$$

THEN $F_{ON\,1} = e_1 E_{ON\,1}$, SO $F_{ON\,1} + F_{ON\,2} = -\frac{e_1 e_2 a}{r c^2} + \frac{4}{3}\frac{e_1 e_2 \dot{a}}{c^3} + \cdots$

THE FIRST TERM IS THE SAME FOUND ON THE TOP OF P. 292 BY A DIFFERENT APPROXIMATION; AGAIN WE REINTERPRET THIS FORCE AS THE REACTION AGAINST THE INTERTIA OF THE ELECTROMAGNETIC FIELD.

THE SECOND TERM IS (PART OF) THE RADIATION REACTION FORCE.

IF WE HAVE $e_1 = e_2 = e/2$ THE THE SECOND TERM IS $\frac{1}{3}\frac{e^2\dot{a}}{c^3}$)

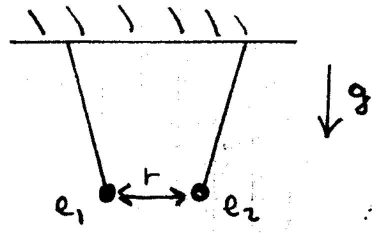
1/2 OF THE REACTION FORCE ON A CHARGE e AS FOUND ON P. 266.

OF COURSE, WE HAVE NOT CALCULATED THE RADIATION REACTION OF CHARGE e_1 ON ITSELF. THIS IS $\frac{2}{3}\frac{e_1^2\dot{a}}{c^3} = \frac{1}{6}\frac{e^2\dot{a}}{c^3}$ & LIKEWISE FOR e_2.

HOWEVER, THE PRESENT ARGUMENT CONFIRMS THE INTERPRETATION THAT THE RADIATION REACTION FORCE IS DUE TO THE EFFECT OF ONE PART OF A CHARGE DISTRIBUTION ON THE OTHER.

HOW MUCH DOES ELECTRICITY WEIGH? [T.H. BOYER, AM. J. PHYS 47, 129 (1979)]

SUPPOSE OUR SYSTEM OF TWO POINTS CHARGES WAS SUSPENDED FROM STRINGS, SUBJECT TO GRAVITY.



THE STRINGS WOULD NOT HANG VERTICALLY, DUE TO THE ELECTROSTATIC FORCE BETWEEN THE CHARGES (SHOWN AS ATTRACTIVE).

BUT WE WOULD SAY THAT THE WEIGHT OF THE SYSTEM IS EQUAL TO THE VERTICAL COMPONENT OF THE TOTAL FORCE IN THE STRINGS — ACCORDING TO MECHANICS.

ACCORDING TO OUR CONCEPTION OF ELECTROMAGNETIC MASS WE WOULD EXPECT

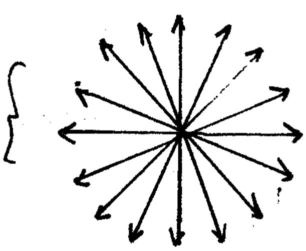
$$W = m^* g = m_1^* g + m_2^* g + \frac{e_1 e_2 g}{c^2 r}$$

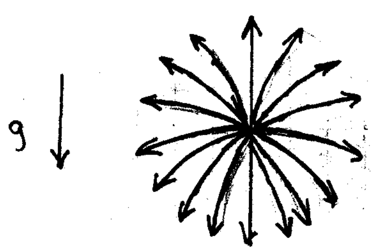
WE CAN SAY THAT $\frac{e_1 e_2 g}{c^2 r}$ IS THE WEIGHT OF THE ELECTRIC FIELD.

(OF COURSE, THE WEIGHT OF THE DIVERGENT SELF ENERGY IS INCLUDED IN $m_i^* g$. WHAT WE REALLY HAVE DISPLAYED IS THE WEIGHT OF THE ENERGY OF INTERACTION OF THE 2 CHARGES.)

NOW IF THE ELECTRIC FIELD HAS WEIGHT, AND THE FIELD ALSO BEHAVES SOMETHING LIKE AN ELASTIC SOLID AS DESCRIBED BY THE MAXWELL STRESS TENSOR, THEN WE EXPECT THE FIELD LINES TO SAG DUE TO GRAVITY. WE SKETCH THE FIELD OF A SINGLE CHARGE:

NO GRAVITY





FIELD WITH GRAVITY

MAXWELL WOULD HAVE US SOLVE AN ELASTICITY PROBLEM TO DEMONSTRATE THIS.

WE WILL FOLLOW A SUGGESTION OF EINSTEIN: THE PRINCIPLE OF EQUIVALENCE. ROUGHLY: THE PHYSICS OF A SYSTEM AT REST IN A GRAVITATIONAL FIELD IS THE SAME AS THAT FOR A UNIFORMLY ACCELERATED SYSTEM WITHOUT GRAVITY.

HENCE THE FIELD LINES OF A CHARGE AT REST ON THE END OF A STRING SHOULD BE THE SAME AS THOSE WE JUST CALCULATED FOR THE UNIFORMLY ACCELERATED CHARGE.

OF COURSE WE SHOULD KNOW THE FIELDS OF THE ACCELERATED CHARGE ACCORDING TO AN ACCELERATED OBSERVER TO HAVE THE FULL EQUIVALENCE TO THE STATIC GRAVITATIONAL CASE. OUR FIELDS IN THE PREVIOUS SECTION WERE FOR AN OBSERVER AT REST. SO IF WE CONSIDER THE FIELDS AT THE MOMENT $(t=0)$ WHEN THE ACCELERATING CHARGE IS AT REST WE SHOULD GET THE CORRECT PICTURE FOR A CHARGE AT REST IN GRAVITY - THANKS TO EINSTEIN.

REFERRING TO OUR FIELD EXPANSION ON P 287, WE SET $\bar{v}=0$ AND $\bar{a}=-\bar{g}$. (WE ACCELERATE UPWARDS IN AN ELEVATOR TO ADD TO OUR WEIGHT)

THEN $\underline{\bar{E} = \frac{e\bar{r}}{r^3} + \frac{e}{2c^2 r}\left(\bar{g} + (\bar{g}\cdot\hat{r})\hat{r}\right)} + O\left(\frac{1}{c^3}\right)$; $\bar{B}=0$

WHICH HAS THE QUALITATIVE SHAPE SKETCHED IN THE FIGURE!

IF WE HAVE TWO CHARGES AT THE SAME HEIGHT SEPARATED BY DISTANCE r, WE SEE THERE IS A DOWNWARDS ELECTRICAL FORCE DUE TO THE SAGGING FIELD LINES

$$\bar{F}_{on\,1}\Big|_{down} = \bar{F}_{on\,2}\Big|_{down} = \frac{e_1 e_2 g}{2c^2 r} \Rightarrow F_{TOTAL}\Big|_{down} = \frac{e_1 e_2 g}{c^2 r}$$

THIS 'UNEXPECTED' NET INTERNAL FORCE IS EXACTLY THE WEIGHT OF THE ELECTRIC FIELD $W = \frac{U}{c^2} g$!

HOW BIG IS THE SAG IN THE FIELD LINES?

FOR LINES WHICH START OUT HORIZONTAL, THE RATIO OF THE TRANSVERSE COMPONENT TO THE RADIAL COMPONENT OF $\bar{E}$

IS $\frac{gr}{2c^2}$. FOR $r \sim 1$ METER, RATIO $\sim \frac{10^3 \cdot 10^2}{2\cdot(3\times10^{10})^2} \sim 10^{-16}$

NEEDLESS TO SAY, THIS EFFECT HAS NEVER BEEN OBSERVED EXPERIMENTALLY?

CAN YOU SUGGEST A WAY TO MEASURE THIS ??

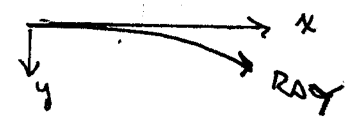
REMARK: IF $\bar{r}$ IS TAKEN UPWARDS THEN $\bar{E}=0$ FOR $r \geq c^2/g \approx 1$ LIGHT YEAR! GRAVITY, IF UNIFORM, WOULD PULL BACK ANY ELECTRIC FIELD $\Rightarrow$ "EVENT HORIZON"!
[THIS WOULD NOT OCCUR IF THE $\bar{a}$ DEPENDENCE OF $\bar{E}$ (P. 287) WERE $\bar{a}-(\bar{a}\cdot\hat{r})\hat{r}$.]

HOW HEAVY IS LIGHT?

IF WE ACCEPT NEWTON'S CORPUSCULAR VIEW OF LIGHT THEN WE EXPECT LIGHT 'RAYS' TO BE DEFLECTED BY GRAVITY.

LIKEWISE ONCE WE REALIZE THAT THE ENERGY DENSITY OF ELECTROMAGNETIC FIELDS BEHAVES LIKE $M_{EFF.} = U/c^2$, THEN WE EXPECT CONSISTENCY WITH THE NEWTONIAN DEFLECTION

NEWTON: $\left.\begin{array}{l} x = ct \\ y = \frac{1}{2} g t^2 \end{array}\right\} \quad y = \dfrac{g x^2}{2c^2}$



THE ANGLE OF DEFLECTION IS $\tan\theta = y' = \dfrac{g x}{c^2}$

(NOTE THAT OUR ARGUMENT IMPLIES THAT THE SPEED OF LIGHT VARIES IN A GRAVITATIONAL FIELD!)

THIS RESULT IS VERY SIMILAR TO THAT FOUND FOR THE SAG OF THE ELECTROSTATIC FIELD IN GRAVITY!

THIS IS THE <u>SAME</u> DEFLECTION AS WILL BE CALCULATED IN GENERAL RELATIVITY (TO ORDER $1/c^2$), SO LONG AS THE ACCELERATION DUE TO GRAVITY IS CONSTANT.

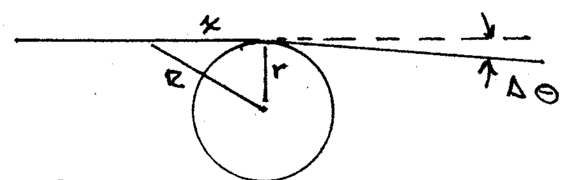
AGAIN DUE TO THE SMALLNESS OF THE DEFLECTION IT HAS NEVER BEEN OBSERVED IN THE LABORATORY.

IN 1911 EINSTEIN REALIZED THAT AN OBSERVABLE DEFLECTION WOULD OCCUR FOR LIGHT JUST GRAZING THE SURFACE OF THE SUN.

A NEWTONIAN DERIVATION AGREES WITH EINSTEIN'S 1911 CALCULATION.

FOR LIGHT WAVES THE MOMENTUM DENSITY $\bar{P} = \dfrac{U}{c}\hat{k}$

THE FORCE OF GRAVITY ON THE ENERGY DENSITY IS $\bar{F} = \dfrac{d\bar{P}}{dt} = -\dfrac{G M U/c^2}{R^2}\hat{R}$



SO THE TOTAL TRANSVERSE MOMENTUM CHANGE AS THE LIGHT PASSES THE STAR IS

$$\Delta P_\perp = \int F_\perp dt = \int \frac{dx}{c} \cdot \frac{G M U/c^2}{R^2} \frac{r}{R} = \frac{G M U r}{c^3} \int_{-\infty}^{\infty} \frac{dx}{(\sqrt{x^2+r^2})^3}$$

$$= \frac{2 G M}{r c^2} U/c$$

SO $\Delta\theta = \Delta P_\perp / P = \dfrac{2GM}{rc^2} \sim 0.9''$ (SECONDS OF ARC) FOR THE SUN.

By 1915 Einstein found that the Newtonian argument requires significant modification for non-uniform gravitational fields. For example consider an observer rotating about the origin with angular velocity ω. The effective gravitational field has strength $g = \omega^2 r$ outwards. The observer measures the radius r the same as an inertial observer, but due to the Lorentz contraction (s)he measures the diameter as $d = 2\pi r / \sqrt{1 - v^2/c^2} \approx 2\pi r \left(1 + \frac{gr}{2c^2}\right)$. In such cases we say that 'space is curved'! This results in a doubling of the deflection of light grazing the sun. However in the case of a uniform gravitational field, space is not curved, and the Newtonian deflection should be correct! (This has never been studied experimentally.)

The Gravitational Red Shift

$\Phi = gh$ • 2

$\downarrow g$

$\Phi = 0$ • 1

As another example of the interplay between electromagnetism and mechanics, we discuss how the color of a light wave changes as it moves thru a changing gravitational potential.

The result is readily obtained if one accepts Einstein's 1905 relation between the energy and frequency of the quantum of light

$$E = h\nu$$

Suppose the photon moves upwards in a uniform gravitational field. Then conservation of energy tells us

$$E_1 = E_2 + m_{eff\,2}\Delta\Phi = E_2 + \frac{E_2}{c^2}\Delta\Phi$$

$$\text{so } \nu_2 = \frac{\nu_1}{1 + \frac{\Delta\Phi}{c^2}} < \nu_1 \quad \text{for } \Delta\Phi > 0$$

The frequency is shifted towards red (from blue) as the light climbs upwards.

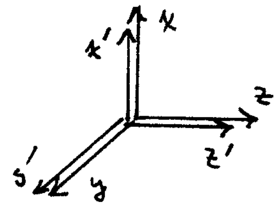
Einstein never used the quantum argument as given here, but preferred to derive the red shift by 'classical' arguments of the sort used in the theory of special relativity.

We sketch here Einstein's argument of 1907.

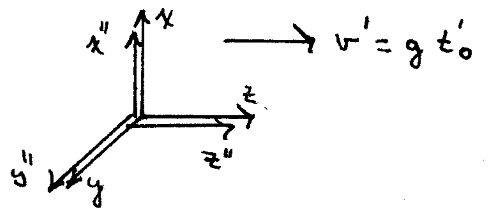
By the principle of equivalence the physics in a uniform gravitational field should be the same as that reported by observers in a uniformly accelerated frame without gravity.

The goal of the argument is to show that clocks run at different rates at different places in an accelerated frame.

Ph 206 Lecture 24



The frames A and I' at $t = t' = 0$



The frames A and I'' at time t_0'. Both A and I'' have velocity $v' = g t_0'$ rel. to I'.

A = accelerated frame.
I', I'' = inertial frames

A has acceleration g rel. to I'
I'' has velocity $v' = g t_0'$ rel to I'

We label an event in A as (x, y, z, t)
" I' as (x', y', z', t')
I'' as (x'', y'', z'', t'')

1. At $t' = 0$ in the inertial frame I' the accelerated frame A has zero velocity. Thereafter A has velocity $v' = g t'$ rel. to I'.

2. At $t' = 0$ all clocks throughout frames A and I' are set to zero. We presume all clocks are built so as to run at the same rate <u>if</u> they are under the same conditions.

3. At some time t_0' according to inertial frame I' the accelerated frame A has velocity $v' = g t_0'$ rel. to I'. All points in A have moved distance $\Delta z' = \frac{1}{2} g t_0'^2$ rel. to I'. The clocks in A have all advanced by the same amount and read t_0 according to observers <u>in I'</u>. Because of time dilation effects $t_0 \neq t_0'$. But the time dilation cannot be bigger than if frame A always had velocity v'!

$$\text{So} \quad t_0' \leq \frac{t_0}{\sqrt{1 - v'^2/c^2}} \approx t_0\left(1 + \frac{1}{2}\frac{v'^2}{c^2}\right) = t_0\left(1 + \frac{g^2 t_0'^2}{2c^2}\right) \Rightarrow t_0' = t_0 + O\left(\frac{g^2}{c^2}\right)$$

Thus for small accelerations, the clocks in A read the same as the clocks in frame I'

4. According to an observer in frame I', in one clock in frame I' reads t_0', then all clocks in I' must read I'. This is the notion of <u>simultaneity</u>. We must recall that this concept is established by means of light signals. We say that events at positions $\bar{x}_1'$ and $\bar{x}_2'$ are simultaneous if an observer at $\bar{x}_1$ who 'looks' at the clock at $\bar{x}_2$ by means of light rays, then the time read off clock 2 is actually earlier by $\Delta t = \frac{|\bar{x}_1 - \bar{x}_2|}{c}$ than the reading on clock 1.

5. BUT THE CONCEPT OF SIMULTANEITY IS RELATIVE! AN OBSERVER IN FRAME A WHO LOOKS AT CLOCKS IN FRAME I' AT SOME MOMENT t_0 WILL NOT FIND THE SAME READING ON ALL CLOCKS IN I'. BUT IN STEP 4 WE FOUND THAT THE CLOCKS IN A AND I' READ THE SAME, ACCORDING TO OBSERVERS IN I'. THE ONLY WAY TO RESOLVE THIS APPARENT CONTRADICTION IS FOR CLOCKS IN FRAME A TO RUN A DIFFERENT RATES IN DIFFERENT PLACES! THIS COULD NOT HAPPEN IN AN INERTIAL FRAME – BUT A IS NOT INERTIAL.

6. TO BE QUANTITATIVE ABOUT THE VARIATION OF CLOCK RATE IN FRAME A, EINSTEIN INTRODUCES A THIRD FRAME I''. THIS INERTIAL FRAME HAS VELOCITY $v' = g t_0'$ ALWAYS, REL TO I', AND ITS AXES COINCIDE WITH THOSE OF FRAME A AT TIME t_0'. WE CALL THIS THE INSTANTANEOUS LORENTZ FRAME CORRESPONDING TO FRAME A AT TIME t_0'. THE IDEA IS THAT FOR A VERY SHORT TIME AFTER t_0', PHYSICS APPEARS THE SAME IN BOTH FRAMES A AND I''. SINCE I'' IS AN INERTIAL FRAME THIS PHYSICS SHOULD BE WELL UNDERSTOOD! FOR EXAMPLE THE RATE OF CLOCKS IN FRAMES A AND I'' MUST BE THE SAME AT TIME t_0' – ALTHOUGH THE READINGS OF THE CLOCKS CANNOT BE THE SAME (DUE TO THE RELATIVITY OF SIMULTANEITY BETWEEN I' AND I''). MOST IMPORTANTLY, THE CONCEPT OF SIMULTANEITY IS THE SAME IN FRAMES A AND I'' AT TIME t_0' – BECAUSE SIMULTANEITY CAN BE ESTABLISHED BY SENDING SIGNALS BETWEEN CLOSE NEIGHBORS OVER SHORT PATHS ⇔ SMALL TIMES.

7. WE CONSIDER TWO SIMULTANEOUS EVENTS IN FRAME I'' AT TIME t_0'' = TIME ON CLOCK AT THE ORIGIN OF I'' AT TIME t_0'.

SAY (x'', y'', z'', t_0'') AND $(0, 0, 0, t_0'')$

THE CORRESPONDING EVENTS IN FRAME A ARE THEN SIMULTANEOUS IN FRAME A. NAMELY (x, y, z, t) AND $(0, 0, 0, t_0)$

IN FACT $x = x''$, $y = y''$, $z = z''$, BUT $t \neq t_0''$!

WE FIND THE CLOCK READING t BY TRANSFORMING TO FRAME I' – FOR WHICH CLOCKS IN A READ THE SAME AS IN I'.

THE TIME TRANSFORMATION BETWEEN I' AND I'' IS

$$t' = \gamma\left(t'' + \frac{v'}{c^2} z''\right) \approx t'' + \frac{g t_0'}{c^2} z \quad + O\left(g^2/c^2\right)$$

SO THE CLOCK AT z IN FRAME A READS

$$t = t' \sim t_0'' + \frac{g t_0'}{c^2} z$$

WHILE THE CLOCK AT THE ORIGIN IN FRAME A READ

$$t_0 = t_0' \sim t_0'' \quad \text{TO ORDER } g^2/c^2$$

THUS $\underline{t = t_0\left(1 + \frac{g z}{c^2}\right)}$ IS THE VARIATION OF CLOCK RATE WITH POSITION IN THE ACCELERATED FRAME A!

8. BY THE PRINCIPLE OF EQUIVALENCE, CLOCKS IN A UNIFORM GRAVITATIONAL FIELD RUN AT DIFFERENT RATES

$$t = t_0\left(1 + \frac{\Phi}{c^2}\right) \quad \text{WHERE } \Phi = g z = \text{GRAVITATIONAL POTENTIAL}: \vec{g} = -\vec{\nabla}\Phi$$

THIS READILY EXPLAINS THE GRAVITATIONAL RED SHIFT. IF A WAVE IS EMITTED AT $z = 0$ WITH FREQUENCY ν_0, THEN IN TIME t_0, $N = \nu_0 t_0$ CRESTS ARE SENT OUT. THESE ARE RECEIVED AT z IN A TIME $t = t_0\left(1 + \frac{\Phi}{c^2}\right)$ ACCORDING THE THE RECEIVING OBSERVER. THIS OBSERVER THEN MEASURES THE FREQUENCY AS

$$\nu = \frac{N}{t} = \frac{\nu_0 t_0}{t_0(1 + \Phi/c^2)} = \frac{\nu_0}{1 + \Phi/c^2} \quad \text{AS BEFORE!}$$

NOW $\nu \sim \nu_0(1 - \Phi/c^2)$, SO $\frac{\Delta\nu}{\nu} = \frac{\Phi}{c^2} = \frac{g z}{c^2} \sim 10^{-15}$ FOR $z = 10$ m.

AT THE SURFACE OF THE EARTH. IT IS EASIER TO MEASURE SMALL FREQUENCY SHIFTS THAN SMALL DEFLECTIONS, AND THIS EFFECT HAS BEEN OBSERVED ON THE EARTH! CF. POUND & REBKA PHYS. REV. LETT. 4, 337 (1960)

OF COURSE, THE RED SHIFT FROM STARS IS QUITE SIGNIFICANT.

A PICTURESQUE WAY OF DESCRIBING THE FACT THAT CLOCK RATES VARY WITH POSITION IN A GRAVITATIONAL FIELD IS TO SAY "TIME IS CURVED". AS REMARKED EARLIER, IN THIS EXAMPLE, SPACE IS STILL "FLAT"! NOTE THAT IN THE EXAMPLE OF THE ROTATING FRAME, THE CLOCK RATE VARIES AS $t = t_0\sqrt{1 - v^2/c^2} \sim t_0\left(1 - \frac{g r}{2c^2}\right) = t_0\left(1 + \frac{\Phi}{c^2}\right)$

EINSTEIN'S RED SHIFT ARGUMENT IS SOMEWHAT COMPLICATED, AND IS ONLY APPROXIMATE. BUT IT ALLOWS US A GLIMPSE INTO THE BIRTH OF GENERAL RELATIVITY.

AN EXPOSITION OF THE CLASSIC 3 TESTS OF GENERAL RELATIVITY, WITHOUT USE OF THE FULL TENSOR FORMALISM IS GIVEN IN SOMMERFELD'S 'ELECTRODYNAMICS' SECTION 38.

IF YOU WISH TO BEGIN READING FULL SCALE TEXTS ON G.R.T. YOU MIGHT TRY WEINBERG, OR LANDAU & LIFSHITZ, CLASSICAL THEORY OF FIELDS.

ELECTRICITY AND COSMOLOGY (R.H. DICKE)

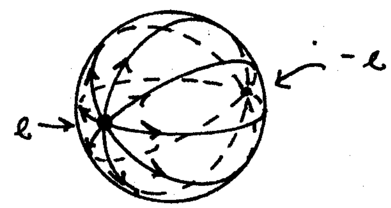
AS A FINAL TICKLE WE CONSIDER WHAT MAXWELL HAS TO SAY ABOUT A COSMIC QUESTION - IS THE TOTAL CHARGE IN THE UNIVERSE ZERO OR NON-ZERO?

a) IF THE CHARGE IN THE UNIVERSE WERE CREATED AT SOME MOMENT (OR MOMENTS) THE NET CHARGE MUST BE ZERO. THIS FOLLOWS AT ONCE FROM GAUSS' LAW.

b) SUPPOSE THE UNIVERSE IS "CLOSED." THIS MEANS THAT 3-SPACE IS CURVED - IN A MANNER ANALAGOUS TO THE CURVED 2-DIMENSIONAL SURFACE OF A SPHERE.

THEN IF THERE WERE JUST A SINGLE CHARGE IN THE UNIVERSE, THE FIELD LINES EMANATING FROM THIS CHARGE CONVERGE ON A POINT AT THE "OPPOSITE END OF THE UNIVERSE."

AGAIN ACCORDING TO GAUSS' LAW THIS IS IMPOSSIBLE - UNLESS AN 'ANTI-CHARGE' IS LOCATED AT THE OPPOSITE END OF THE UNIVERSE.



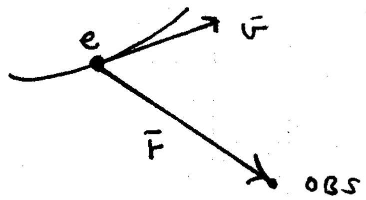
IF YOU DON'T LIKE THE IDEA OF THE 'ANTI-GALAXIES..." THEN WE MUST HAVE NET CHARGE = ZERO LOCALLY, SO THAT NO FIELD LINES ESCAPE TO "∞".

HAVE FUN AND LIVE BETTER ELECTRICALLY!

Appendix: Fields of a Moving Charge to order $(1/c^2)$

We start with the Lienard-Wiechert potentials (Lectures 18-19)

$$\phi = \left[\frac{e}{r - \frac{\bar{r}\cdot\bar{v}}{c}}\right] \qquad \bar{A} = \left[\frac{e\bar{v}/c}{r - \frac{\bar{r}\cdot\bar{v}}{c}}\right]$$



where the $[f]$ means $f(t' = t - r/c) \Rightarrow$ evaluation at the retarded time.

We first express ϕ and $\bar{A}$ in a power series involving only quantities evaluated at the present time. We will keep terms only up to order $(1/c^2)$.

Then we calculate $\bar{E} = -\bar{\nabla}\phi - \frac{1}{c}\frac{\partial \bar{A}}{\partial t}$ $\qquad \bar{B} = \bar{\nabla}\times\bar{A}$

We first deal with $[r] = r\left(t' = t - \frac{[r]}{c}\right)$

By a Taylor series: $$[r] = r - \frac{[r]}{c}\dot{r} + \frac{1}{2}\frac{[r]^2}{c^2}\ddot{r} + O\left(\frac{1}{c^3}\right)$$

Now we can iterate: $$[r] = r - \frac{\dot{r}}{c}\left\{r - \frac{[r]}{c}\dot{r}\right\} + \frac{1}{2}\frac{\ddot{r}}{c^2}\left\{r - \frac{[r]\dot{r}}{c}\right\}^2 + \dots$$

$$= r - \frac{r\dot{r}}{c} + \frac{r\dot{r}^2}{c^2} + \frac{1}{2}\frac{r^2\ddot{r}}{c^2} + O(1/c^3)$$

so $$[r] = r - \frac{r\dot{r}}{c} + \frac{1}{2c^2}\frac{d}{dt}(r^2\dot{r}) + O(1/c^3)$$

With this, we can expand any retarded function $[f]$:

$$[f] = f - \frac{[r]}{c}\dot{f} + \frac{1}{2}\frac{[r]^2}{c^2}\ddot{f} + O(1/c^3)$$

$$= f - \frac{r\dot{f}}{c} + \frac{r\dot{r}\dot{f}}{c^2} + \frac{1}{2}\frac{r^2}{c^2}\ddot{f}$$

so $$[f] = f - \frac{r\dot{f}}{c} + \frac{1}{2c^2}\frac{d}{dt}(r^2\dot{f}) + O(1/c^3)$$

NEXT WE NOTE THAT $\bar{r}\cdot\bar{v}$ CAN BE REWRITTEN AS

$-\bar{r}\cdot\dot{\bar{r}}$ SINCE $\bar{v} = -\dot{\bar{r}}$ ACCORDING TO THE GEOMETRY

BUT $\bar{r}\cdot\dot{\bar{r}} = \frac{1}{2}\frac{d}{dt}(\bar{r}\cdot\bar{r}) = \frac{1}{2}\frac{dr^2}{dt} = r\dot{r}$

SO $\underline{\dot{r} = -\frac{\bar{r}\cdot\bar{v}}{r}}$ WHICH WE WILL ALSO USE LATER

THUS $\phi = \left[\frac{e}{r + \frac{r\dot{r}}{c}}\right] = \left[\frac{e}{r(1+\dot{r}/c)}\right] = \left[\frac{e}{r}\left(1 - \frac{\dot{r}}{c} + \left(\frac{\dot{r}}{c}\right)^2 + O\frac{1}{c^3}\right)\right]$

$$\bar{A} = \left[\frac{e\bar{v}}{cr(1+\dot{r}/c)}\right] = \left[\frac{e\bar{v}}{cr}\left(1 - \frac{\dot{r}}{c} + \left(\frac{\dot{r}}{c}\right)^2 + \cdots\right)\right]$$

BOTH OF THESE HAVE THE FORM $\left[f\left(1 - \dot{r}/c + (\dot{r}/c)^2 + \cdots\right)\right]$

WHICH WE EXPAND INTO PRESENT QUANTITIES AS ON THE PREVIOUS PAGE

$$f - \frac{r\dot{f}}{c} + \frac{1}{2c^2}\frac{d}{dt}(r^2\dot{f})$$

$$-\frac{\dot{r}f}{c} + \frac{r}{c}\frac{d}{dt}\left(\frac{f\dot{r}}{c}\right)$$

$$+ \frac{f\dot{r}^2}{c^2} + O(1/c^3)$$

$$= f - \frac{1}{c}\frac{d}{dt}(rf) + \frac{1}{2c^2}\frac{d^2}{dt^2}(r^2 f) + O(1/c^3)$$

FOR THE SCALAR POTENTIAL ϕ, $f = e/r$

SO $\phi = e/r + \frac{e\ddot{r}}{2c^2} + O(1/c^3)$

FOR THE VECTOR POTENTIAL, $f = \frac{e\bar{v}}{cr}$

SO $\bar{A} = \frac{e\bar{v}}{cr} - \frac{e\bar{a}}{c^2} + O(1/c^3)$ $(\bar{a} = \dot{\bar{v}}$ HOLDS$)$

$$\bar{E} = -\bar{\nabla}\phi - \frac{1}{c}\frac{\partial\bar{A}}{\partial t} = -e\bar{\nabla}\left(\frac{1}{r}\right) - \frac{e}{2c^2}(\bar{\nabla}\ddot{r}) - \frac{e\bar{a}}{c^2 r} + \frac{e\bar{v}\dot{r}}{c^2 r^2} + O(1/c^3)$$

OF COURSE, $\dot{r} = -\frac{(\bar{r}\cdot\bar{v})}{r}$; $\bar{\nabla}r = \frac{\bar{r}}{r}$; $\bar{\nabla}\left(\frac{1}{r}\right) = -\frac{\bar{\nabla}r}{r^2} = -\frac{\bar{r}}{r^3}$

NOW $\ddot{r} = -\frac{d}{dt}\frac{\bar{r}\cdot\bar{v}}{r} = \frac{v^2}{r} - \frac{\bar{r}\cdot\bar{a}}{r} + \frac{(\bar{r}\cdot\bar{v})\dot{r}}{r^2} = \frac{v^2}{r} - \frac{\bar{r}\cdot\bar{a}}{r} - \frac{(\bar{r}\cdot\bar{v})^2}{r^3}$

$$(\ddot{\bar{\nabla}r}) = \frac{d^2}{dt^2}\left(\frac{\bar{r}}{r}\right) = \frac{d}{dt}\left(\frac{-\bar{v}}{r} - \frac{\bar{r}\dot{r}}{r^2}\right) = \frac{-\bar{a}}{r} - \frac{\bar{v}\dot{r}}{r^2} - \frac{\bar{v}\dot{r}}{r^2} - \frac{\bar{r}\ddot{r}}{r^2} + \frac{2\bar{r}\dot{r}^2}{r^3}$$

$$= -\frac{\bar{a}}{r} + \frac{2\bar{v}(\bar{r}\cdot\bar{v})}{r^3} - \frac{\bar{r}v^2}{r^3} + \frac{\bar{r}(\bar{r}\cdot\bar{a})}{r^3} + \underbrace{\frac{\bar{r}(\bar{r}\cdot\bar{v})^2}{r^5} + \frac{2\bar{r}(\bar{r}\cdot\bar{v})^2}{r^5}}_{\frac{3\bar{r}(\bar{r}\cdot\bar{v})^2}{r^5}}.$$

$$\bar{E} = \frac{e\bar{r}}{r^3} - \frac{e}{2c^2}\left\{-\frac{\bar{a}}{r} + \frac{2\bar{v}(\bar{r}\cdot\bar{v})}{r^3} - \frac{\bar{r}v^2}{r^3} + \frac{\bar{r}(\bar{r}\cdot\bar{a})}{r^3} + \frac{3\bar{r}(\bar{r}\cdot\bar{v})^2}{r^5}\right\} - \frac{e\bar{a}}{c^2 r} - \frac{e\bar{v}(\bar{r}\cdot\bar{v})}{c^2 r^3}$$

$$\boxed{\bar{E} = \frac{e\bar{r}}{r^3}\left(1 + \frac{v^2}{2c^2} - \frac{3(\bar{r}\cdot\bar{v})^2}{2c^2r^2}\right) - \frac{e}{2c^2 r}\left(\bar{a} + \frac{(\bar{a}\cdot\bar{r})\bar{r}}{r^3}\right) + O\left(\tfrac{1}{c^3}\right)}$$

THE FIRST TERM IS READILY VERIFIED TO BE THE EXPANSION OF THE FIELD OF A UNIFORMLY MOVING CHARGE WHICH YOU FOUND ON PROBLEM 10, SET 9:

$$\bar{E} = \frac{e\bar{r}}{r^3}\frac{(1 - v^2/c^2)}{\left(1 - v^2/c^2 + \frac{(\bar{r}\cdot\bar{v})^2}{r^2c^2}\right)^{3/2}}$$

TO USE OUR PRESENT NOTATION.

THE SECOND TERM IS $\bar{E}_{RAD} = -\frac{e}{c^2}\left[\frac{\bar{a}_\perp}{r}\right]$ CONVERTED TO PRESENT QUANTITIES. THE CONVERSION IS NOT COMPLETELY INTUITIVE.

FINALLY $\bar{B} = \bar{\nabla}\times\bar{A} = \bar{\nabla}\times\left(\frac{e\bar{v}}{cr} - \frac{e\bar{a}}{c^2}\right)$

THE $\bar{\nabla}$ OPERATES ONLY ON THE $1/r$, SO $\bar{B} = -\frac{e\bar{v}}{c}\times\bar{\nabla}\left(\frac{1}{r}\right)$

$$\boxed{\bar{B} = \frac{e\bar{v}\times\bar{r}}{c r^3} + O\left(\tfrac{1}{c^3}\right)}$$

THERE IS SOMETHING ODD ABOUT THIS RESULT. WHERE IS THE RADIATION PIECE OF THE MAGNETIC FIELD? IT VANISHES TO $O(1/c^3)$. HOWEVER IT REAPPEARS IN 4RTH ORDER, IN A TERM LIKE ea^2/c^4

THE POYNTING VECTOR FOR THE RADIATION IS THEN MADE OF THE 1ST ORDER FIELD IN $\bar{E}$ x 4RTH ORDER IN $\bar{B}$: $\bar{S} \sim c\,\bar{E}_1\times\bar{B}_4 \sim c\,\frac{e}{r^2}\cdot\frac{ea^2}{c^4} = \frac{e^2a^2}{c^3r^2}$

IT'S CORRECT, BUT VERY ODD!