LASERS FROM A CLASSICAL PERSPECTIVE

IN THE APPENDIX TO LECTURE 17 WE DISCUSSED THE FORM OF GAUSSIAN ELECTROMAGNETIC WAVES AS CAN BE PRODUCED BY LASERS.

NOW WE DISCUSS SOME ASPECTS OF THE PHYSICS OF A LASER MEDIUM, IN A CLASSICAL APPROXIMATION.

WE USE THE MODEL OF A DIELECTRIC MEDIUM AS CONSISTING OF POINT ELECTRONS TIED TO NUCLEI VIA SPRINGS (LECTURE 12).

THE INDEX OF REFRACTION $n = \sqrt{\epsilon}$, WHERE ϵ = DIELECTRIC CONSTANT.

THE SPRINGY MODEL IS THAT

$$\epsilon = 1 + \frac{4\pi N e^2}{m} \sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\gamma_i\omega} , \quad \sum_i f_i = 1$$

$$= 1 + \omega_p^2 \sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\gamma_i\omega}$$

WHERE ω_i = NATURAL FREQUENCY OF SPRING i (ATOMIC TRANSITION i)

$\gamma_i = \frac{1}{\tau_i}$ = DAMPING CONSTANT OF SPRING i ≈ LINEWIDTH ∝ 1/LIFETIME

f_i = OSCILLATOR STRENGTH

FOR A DILUTE MEDIUM, $n \approx 1$, AND WE CAN EXPAND n AS

$$n = \sqrt{\epsilon} \sim 1 + \frac{\omega_p^2}{2} \sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\gamma_i\omega} = 1 + \frac{\omega_p^2}{2} \sum_i \frac{f_i(\omega_i^2 - \omega^2 + i\gamma\omega)}{(\omega_i^2 - \omega^2)^2 + \gamma_i^2\omega^2}$$

IN ORDINARY MATERIALS, BOTH γ_i AND f_i ARE POSITIVE, SO $\mathrm{Im}\, n$ IS POSITIVE ALSO.

RECALL THAT A PLANE WAVE PROPAGATES IN THE MEDIA ACCORDING TO

$$\vec{E}(z,t) \sim E_0 e^{i(n\frac{\omega}{c}z - \omega t)} = E_0 e^{-\frac{\omega}{c}\mathrm{Im}\, n\, z} e^{i(\frac{\omega}{c}\mathrm{Re}\, n\, z - \omega t)},$$

SO $\mathrm{Im}\, n > 0 \Rightarrow$ ATTENUATION.

BUT IF $\mathrm{Im}\, n < 0$, THE FIELD STRENGTH WOULD GROW!

IN BRIEF, A CLASSICAL MODEL OF A LASER MEDIUM HAS $\mathrm{Im}\, n < 0$.

FOR SOME EXAMPLES IT SUFFICES TO DECLARE THAT $f_i < 0$, OR $\gamma_i < 0$ FOR ONE OR MORE SPECTRAL LINES i.

Ph 206 Lecture 25

For deeper understanding of the physics of lasers, one turns to quantum mechanics. But it turns out that the quantum mechanics of electric dipole transitions is very similar to a classical description of a medium that contains magnetic dipoles.

We sketch an argument first given by F. Bloch, Phys. Rev. 70, 460 (194

which set the stage for the eventual development of the laser.

Electromagnetic Waves in a Magnetic Medium

We consider a medium with dielectric constant $\epsilon = 1$, but with number density N of magnetic dipoles $\vec{m}$.

We ignore thermal effects.

Then if a static magnetic field $\vec{B}$ is applied to the medium, the dipoles line up with $\vec{B}$, as this is the lowest energy state of their interaction energy

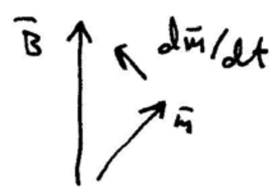
$$U = -\vec{m} \cdot \vec{B}.$$

If a dipole $\vec{m}$ is not initially aligned with $\vec{B}_{static}$, it does not immediately do so. Rather the alignment takes a characteristic time τ, during which the magnetic energy $U = -\vec{m}\cdot\vec{B}$ decreases via a lossy transfer of energy to nonmagnetic aspects of the medium.

We describe this via a damping constant γ, such that

$$\frac{d\vec{m}}{dt} = -\gamma m \left(\hat{m} - \hat{B}\right)$$

$$= -\gamma \vec{m} + \gamma m \hat{B}$$



If by some process the dipoles can be made to point away from $\vec{B}$, then magnetic energy will be stored in the medium. And, if a passing E&M wave can shove the dipoles back into alignment with $\vec{B}$, the wave will extract energy from the system — it will lase!

Heating the system is one way to dealign the dipoles — but this effect is too random to be of great use. We desire a coherent mechanism for rearranging the dipoles.

A useful trick, perfected by I. I. Rabi, is to apply a time-dependent magnetic field, typically at right angles to the static field.

RECALL (LECTURE 7, p. 87) THAT THE MAGNETIC MOMENT OF AN ELECTRON IN AN ATOM IS PROPORTIONAL TO ITS ANGULAR MOMENTUM,

$$\bar{m} = \Gamma \bar{L}, \qquad \Gamma \equiv \text{GYROMAGNETIC RATIO}, = \frac{e}{2mc}$$

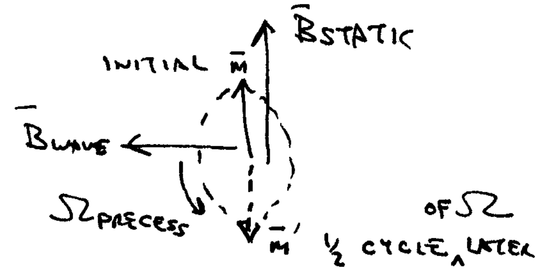
AND THAT THE TORQUE ON A MAGNETIC DIPOLE DUE TO A MAGNETIC FIELD IS

$$\text{TORQUE} = \bar{m} \times \bar{B} = \frac{d\bar{L}}{dt} = \frac{1}{\Gamma}\frac{d\bar{m}}{dt} \quad \text{(IGNORING DAMPING FOR NOW).}$$

HENCE, $\dfrac{d\bar{m}}{dt} = \Gamma \bar{m} \times \bar{B}$, AND $\dfrac{d\bar{M}}{dt} = \Gamma \bar{M} \times \bar{B}$, WHERE $\bar{M} = N\bar{m}$ IS THE MAGNETIZATION OF THE MEDIUM.

THEN IF $\bar{B}_{WAVE}$ IS $\perp$ TO $\bar{B}_{STATIC}$, THE DIPOLES WILL PRECESS ABOUT $\bar{B}_{WAVE}$, AND SPEND PART OF THEIR TIME ANTIALIGNED WITH B_{STATIC} — IN A HIGH ENERGY STATE.

THE PRECESSION FREQUENCY IS $\Omega = \Gamma B = \dfrac{eB}{2m_e c}$



$$\frac{\Omega}{\omega_i} = \frac{eB}{2m_e c\,\omega_i} \approx \frac{e\hbar}{m_e^2 c^3} \cdot B \cdot \frac{m_e c}{\hbar} \cdot \frac{c}{\omega_i} = \frac{B}{B_{crit}} \cdot \frac{\lambda_i}{\lambda_c}$$

WHERE $B_{crit} = \dfrac{m_e^2 c^3}{e\hbar} \sim 4 \times 10^{13}$ GAUSS IS THE QED CRITICAL FIELD

$\lambda_{COMPTON} = \dfrac{\hbar}{m_e c} = 4 \times 10^{-11}$ cm

$\lambda_i \approx 10^{-4}$ cm FOR OPTICAL WAVELENGTHS

$$\text{SO } \frac{\Omega}{\omega_i} \sim \frac{B\,[\text{GAUSS}]}{4 \times 10^{13}} \cdot \frac{10^{-4}}{4 \times 10^{-11}} \approx 10^{-7} \cdot B_{WAVE} \text{ IN GAUSS} \ll 1$$

HENCE, THE DIPOLES SPEND A LARGE NUMBER OF WAVE CYCLES ANTIALIGNED, AND THERE IS TIME TO TAKE ADVANTAGE OF THE RESULTING 'INVERTED' POPULATION TO SEND IN A PROBE PULSE THAT CAN EXTRACT ENERGY FROM THE SYSTEM $\Rightarrow$ LASER!

WE SUPPOSE THAT DAMPING IS SLOW ENOUGH THAT IT CAN BE IGNORED DURING A CYCLE OF Ω. THEN THE COMBINED EQUATION OF MOTION IS

$$\frac{d\bar{M}}{dt} + \gamma \bar{M} = \Gamma \bar{M} \times \bar{B}_{TOTAL} + \gamma M \hat{B}_{STATIC}.$$

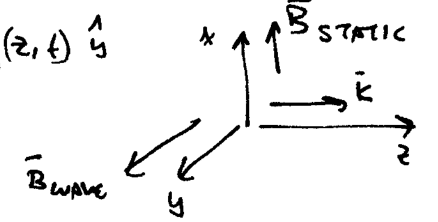
BEFORE EXAMINING THE FULL EQUATION OF MOTION FOR $\bar{M}$ IN DETAIL, WE REMIND OURSELVES OF THE BROADER CONTEXT OF WAVE PROPAGATION IN A MAGNETIC MEDIUM.

IN A LINEAR MEDIUM, WE CAN DEFINE A PERMEABILITY μ SUCH THAT $\bar{B} = \mu \bar{H} = \bar{H} + 4\pi \bar{M}$. THUS IF $\bar{B} = B_y \hat{y}$, $B_y = \mu H_y = H_y \left(1 + 4\pi \frac{M_y}{H_y}\right)$

AND $\mu = 1 + 4\pi \frac{M_y}{H_y}$

THE INDEX OF REFRACTION IS RELATED BY $n = \sqrt{\mu} \sim 1 + 2\pi \frac{M_y}{H_y}$ IF $\frac{M_y}{H_y}$ SMALL

LET'S TRY THIS OUT FOR A WAVE WITH $\bar{B}$ ALONG $\hat{y}$, MOVING ALONG $\hat{z}$, WITH A STATIC FIELD ALONG $\hat{x}$: $\bar{B} = B_x \hat{x} + B_y(z,t)\hat{y}$



THE COMPONENTS OF THE EQ. OF MOTION ARE:

$$\frac{dM_x}{dt} + \gamma M_x = \gamma M - \Gamma M_z B_y$$

$$\frac{dM_y}{dt} + \gamma M_y = \Gamma M_z B_x$$

$$\frac{dM_z}{dt} + \gamma M_z = \Gamma (M_x B_y - M_y B_x)$$

WE ARE PARTICULARLY INTERESTED IN M_y, SINCE $\bar{B}_{WAVE} = B_y \hat{y} = (H_y + 4\pi M_y)\hat{y}$

TRICK: TAKE THE DERIVATIVE:

$$\frac{d^2 M_y}{dt^2} + \gamma \frac{dM_y}{dt} = \Gamma \frac{dM_z}{dt} B_x = \Gamma B_x \left[\Gamma(M_x B_y - M_y B_x) - \gamma M_z\right]$$

$$= \Gamma^2 B_x (M_x H_y - M_y H_x) - \gamma\left(\frac{dM_y}{dt} + \gamma M_y\right)$$

OR $$\frac{d^2 M_y}{dt^2} + 2\gamma \frac{dM_y}{dt} + (\gamma^2 + \Gamma^2 B_x H_x) M_y = \Gamma^2 B_x H_x \frac{M_x}{H_x} \cdot H_y$$

LET $\omega_0^2 = \Gamma^2 B_x H_x$ - A CONSTANT

THEN $$\frac{d^2 M_y}{dt^2} + 2\gamma \frac{dM_y}{dt} + (\gamma^2 + \omega_0^2) M_y = \omega_0^2 \frac{M_x}{H_x} H_y$$

THIS IS A DRIVEN OSCILLATOR EQUATION FOR M_y, WHERE THE DRIVING TERM IS PROPORTIONAL TO H_y, BUT WITH A COEFFICIENT $\frac{M_x}{H_x}$ WHICH IS NEGATIVE DURING 1/2 OF THE PRECESSION CYCLE.

THEN, THE OSCILLATIONS IN M_y ARE OUT OF PHASE WITH H_y, WHICH PERMITS THE FIELD H_y TO ABSORB ENERGY FROM THE MEDIUM (RATHER THAN TRANSFER ENERGY INTO THE MEDIUM AS WHEN M_y & H_y ARE IN PHASE).

THUS, IF $H_y = H_y e^{-i\omega t}$ WE TRY $M_y = M_y e^{-i\omega t}$ AND FIND THAT

$$M_y = \frac{M_x}{H_x} \frac{\omega_0^2 H_y}{\omega_0^2 - \omega^2 + \gamma^2 - 2i\gamma\omega}$$

REMARK: FOR M_x TO PRECESS ABOUT B_y IN THE X-Z PLANE, WE NEED $M_y H_x \ll M_x H_y$ IN dM_z/dt. IF WE SET $\omega = 2M_{x0} H_x/\hbar = 2\Gamma H_x \approx 2\omega_0$, AS SUGGESTED BY QM, THIS HOLDS....

INDEX $n \sim 1 + 2\pi \frac{M_y}{H_y} = 1 + 2\pi \frac{M_x}{H_x} \frac{\omega_0^2}{\omega_0^2 - \omega^2 + \gamma^2 - 2i\gamma\omega}$

$$= 1 + 2\pi \frac{M_x}{H_x} \frac{\omega_0^2(\omega_0^2 - \omega^2 + \gamma^2 + 2i\gamma\omega)}{(\omega_0^2 - \omega^2 + \gamma^2)^2 + 4\gamma^2\omega^2}$$

$\mathrm{Im}(n) \sim \frac{M_x}{H_x} < 0$ FOR THE 'OPTICALLY ACTIVE' MEDIUM DURING THE TIMES WHEN M_x IS OPPOSITE TO H_x.

SUPERRADIANCE

THE BLOCH FORMALISM PRESENTED ABOVE WAS NOT INITIALLY APPLIED TO A DESCRIPTION OF THE OPTICAL STATE OF THE MEDIUM AS A WHOLE.

IT WAS R.H. DICKE, PHYS. REV. 93, 99 (1954), WHO MAY HAVE BEEN THE FIRST TO REALIZE THAT THE PRECESSING MAGNETIZED MEDIUM COULD GIVE ENERGY AWAY DURING PART OF THE SPIN CYCLE.

IN PARTICULAR, SUPPOSE THE WAVE WERE TURNED OFF JUST WHEN M_x WAS AT ITS MOST NEGATIVE VALUE, AND THE WAVE FREQUENCY ω HAD BEEN A NATURAL FREQUENCY OF THE MEDIUM. THEN, WE WOULD NOT SEE SLOW RELAXATION OF M_x TO REALIGNMENT WITH H_x, BUT RATHER, A BURST OF RADIATION AS THE DIPOLES MAKE A QUANTUM JUMP BACK INTO ALIGNMENT.

OBSERVATION OF THIS EFFECT IN THE MID 1950'S WAS THE FIRST CONTROLLED LASING ACTION.

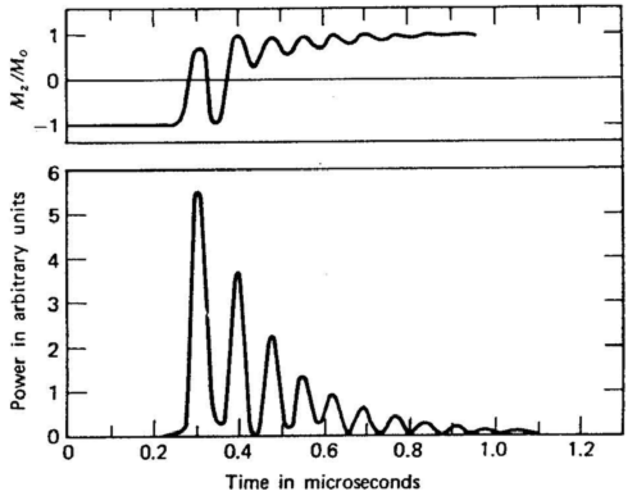


FIGURE 15.6 Superradiant emission of an inverted spin system. Upper trace: magnetization. Lower trace: radiated power.

A. YARIV. J. APPL. PHYS. 31, 740 (1960).

SPIN ECHO ; PHOTON ECHO

AN EARLY EXAMPLE OF THE MANIPULATION OF POPULATIONS OF MAGNETIC DIPOLES IS THE SPIN ECHO EFFECT OF E.L. HAHN, PHYS. REV. 80, 580 (1950).

IN THIS A COLLECTION OF MAGNETIC MOMENTS INITIALLY RESIDES IN A NEARLY UNIFORM FIELD $H_z \hat{z}$. THE MOMENTS ARE THEREFORE INITIALLY ALL ALONG THE +Z AXIS.

IF THE MOMENTS WERE, HOWEVER, NOT ALONG THE Z AXIS, THEY WOULD PRECESS ABOUT THE Z AXIS AT ANGULAR VELOCITY

$$\Omega = \Gamma H_z \quad , \quad \text{SINCE } \frac{d\bar{M}}{dt} = \Gamma \bar{M} \times \bar{H} \text{ , IGNORING DAMPING, AND } \bar{M} \ll \bar{H}$$

LET H_0 = CENTRAL VALUE OF H_z OVER THE SAMPLE VOLUME, AND $\Omega_0 = \Gamma H_0$

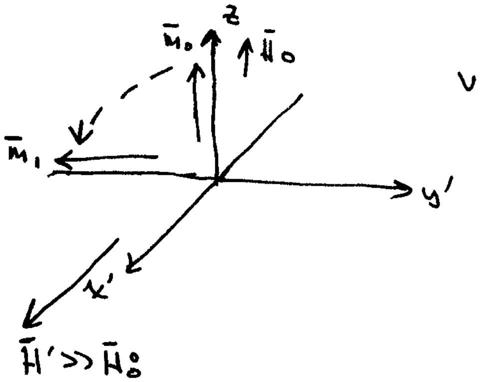
AT $t=0$, APPLY A MAGNETIC FIELD H' THAT LIES IN THE X-Y PLANE AND ROTATES ABOUT THE Z AXIS AT ANGULAR VELOCITY Ω_0.

WE SUPPOSE THAT $H' \gg H_0$, SO THAT NOW THE DIPOLES PRECESS ABOUT $\bar{H}'$

IT'S MUCH EASIER TO KEEP TRACK OF THINGS IN A FRAME THAT ROTATES ABOUT THE Z AXIS AT ANGULAR VELOCITY Ω_0.

IN THIS FRAME, $\bar{H}'$ IS AT REST, SAY, ALONG THE x' AXIS.

WE LEAVE FIELD H' ON JUST LONG ENOUGH TO ROTATE THE DIPOLES BY 90°, SAY FOR $0 < t < t_1$.



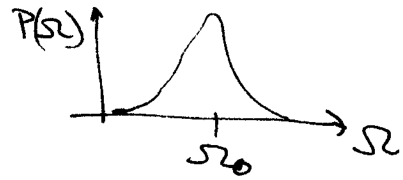
VIEW IN THE ROTATING FRAME FOR $0 < t < t_1$

ALL DIPOLES ARE NOW ALONG THE $-y'$ AXIS.

ONCE WE TURN OFF $\bar{H}'$ AT TIME t_1, THE DIPOLES WOULD PRECESS SLOWLY ABOUT THE z AXIS AT ANGULAR VELOCITY Ω_0 IF H_z VERY UNIFORM.

BUT IN PRACTICE, NON UNIFORMITIES IN $H_z(x,y,z)$ RESULT IN A SPECTRUM OF PRECESSION FREQUENCIES $\Omega(x,y,z) = \Gamma H_z(x,y,z)$

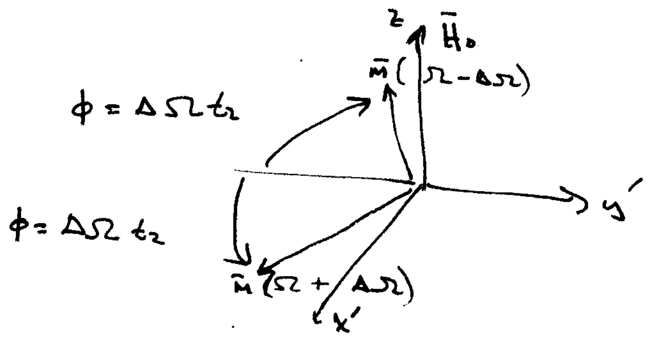
THIS SPECTRUM PEAKS AT Ω_0:



FOR SIMPLICITY, WE ONLY CONSIDER A PAIR OF MOMENTS WHOSE CORRESPONDING PRECESSION FREQUENCIES ARE $\Omega_0 \pm \Delta\Omega$.

IN THE ROTATING FRAME THE DIPOLE WITH $\Omega_0 + \Delta\Omega$ PRECESSES COUNTERCLOCKWISE, WHILE THAT AT $\Omega_0 - \Delta\Omega$ GOES CLOCKWISE.

AFTER SOME TIME $t_2 >> t_1$, THE PICTURE IS

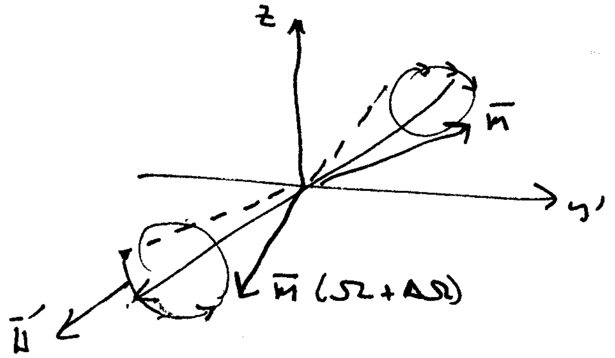


FOR t_2 BIG ENOUGH, THE WHOLE SPECTRUM OF DIPOLES WILL BE UNIFORMLY SPREAD AROUND THE x'-y' PLANE - AND THE NET MAGNETIZATION WILL BE ZERO.

NOW TURN $\bar{H}'$ BACK ON FOR A TIME INTERVAL $2t_1$. THE DIPOLES AGAIN PRECESS ABOUT THE x' AXIS,

BUT THIS TIME BY $180°$

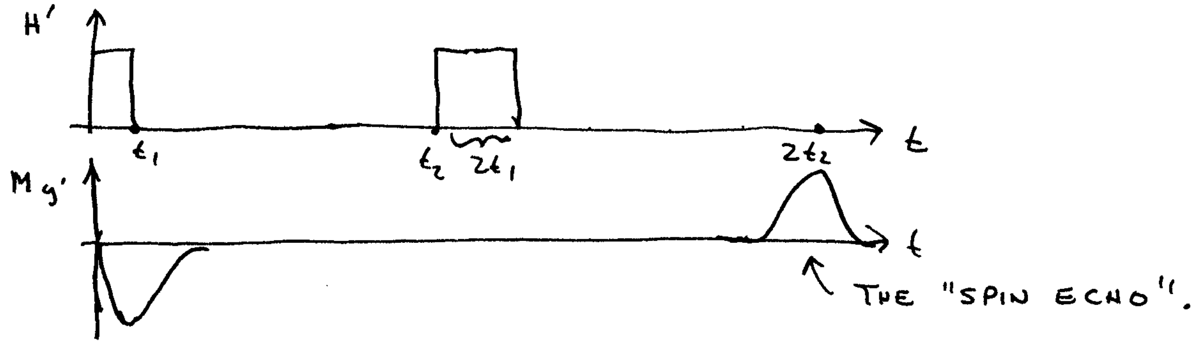
THE NET MAGNETIZATION IS STILL ZERO, HOWEVER.



$\bar{H}'$ OFF, AND WAIT ANOTHER TIME INTERVAL t_2.

THE DIPOLES AGAIN PRECESS IN THE x'-y' PLANE ABOUT THE z AXIS, AND BY THE SAME AMOUNT AS DURING THE FIRST INTERVAL t_2

THIS BRINGS ALL DIPOLES INTO ALIGNMENT WITH THE $+y'$ AXIS, AND THE NET MAGNETIZATION IS <u>BIG</u> AGAIN.



IF AT TIME $2t_2$, $\bar{H}'$ WERE TURNED BACK ON FOR A TIME t_1, THE MOMENTS WOULD PRECESS BACK UP INTO ALIGNMENT WITH THE z AXIS, ETC.

THESE EXAMPLES DESCRIBE CLASSICAL BEHAVIOR OF MAGNETIC DIPOLE MOMENTS IN A STATIC BACKGROUND MAGNETIC FIELD (+ A TIME DEPENDENT MAGNETIC FIELD TO INDUCE SOME DESIRED CHANGE).

THE QUANTUM DESCRIPTION OF THE PROBABILITY AMPLITUDES FOR ATOMIC LEVELS THAT CAN BE LINKED BY AN ELECTRIC DIPOLE PHOTON IS EXTREMELY SIMILAR.

HOWEVER, NO BACKGROUND STATIC ELECTRIC FIELD IS NEEDED.

AMPLITUDES EVOLVE LIKE $e^{i\frac{\Delta U t}{\hbar}}$, WHICH CAN BE THOUGHT OF AS A KIND OF PRECESSION AT ANGULAR VELOCITY $\Omega = \frac{\Delta U}{\hbar}$.

NOTE THAT IN OUR CLASSICAL MAGNETIC SYSTEM, THE ENERGY DIFFERENCE BETWEEN ALIGNED AND ANTIALIGNED DIPOLES IS $\Delta U = 2MB$

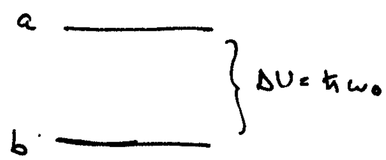
SO WE CAN WRITE $\Omega_{precess} = \Gamma B = \frac{\Delta U}{2M/\Gamma}$

WE ALSO KNOW THAT FOR AN ELECTRON IN A CLASSICAL ORBIT, $\bar{M} = \Gamma L$, WHERE L = ANGULAR MOMENTUM. WITH THE QUANTUM CONDITION THAT $L = \hbar$ FOR THE SIMPLEST NON TRIVIAL ORBIT, WE GET $\Omega_{precess} = \frac{\Delta U}{2\hbar}$

THUS THE CLASSICAL MAGNETIC MOMENT MODEL TRANSLATES READILY INTO Q.M.

OUR BRIEF SKETCH OF HOW THE QUANTUM MECHANICS OF ELECTRIC DIPOLE TRANSITION BETWEEN 2 LEVELS CAN ALWAYS BE PUT IN A FORM ANALAGOUS TO PRECESSION OF CLASSICAL MAGNETIC DIPOLES HAS BEEN PUT IN AN ELEGANT FORM BY R.P. FEYNMAN, F.L. VERNON & R.W. HELLWARTH, J. APPL. PHYS. 28, 49 (1957)

SOME HIGHLIGHTS:



$$\psi(t) = a(t)\,\psi_a + b(t)\,\psi(b)$$

a, b COMPLEX, $|a|^2 + |b|^2 =$ CONSTANT.

LET $\bar{r}$ BE DEFINED BY
$$r_1 = ab^* + ba^*$$
$$r_2 = i(ab^* - ba^*)$$
$$r_3 = aa^* - bb^*$$

THEN $|\bar{r}|^2 = |a|^2 + |b|^2 =$ CONST.

IF HAMILTONIAN $H = H_0 + V$ WITH $H_0 = \begin{pmatrix} E_0 + \frac{\hbar\omega}{2} & 0 \\ 0 & E_0 - \frac{\hbar\omega_0}{2} \end{pmatrix}$

AND $V = \begin{pmatrix} 0 & V_{ab} \\ V_{ba} & 0 \end{pmatrix}$

THEN SCHRÖDINGER'S EQ FOR $\psi(t)$ REDUCES TO

$$\frac{d\bar{r}}{dt} = \bar{\omega} \times \bar{r} \quad \text{WHERE} \quad \omega_1 = (V_{ab} + V_{ba})/\hbar$$
$$\omega_2 = i(V_{ab} - V_{ba})/\hbar$$
$$\omega_3 = \Delta U/\hbar = \omega_0$$

FOR AN ELECTRIC DIPOLE TRANSITION DUE TO A LINEARLY POLARIZED FIELD E AT FREQUENCY ω_0, $V_{ab} = V_{ba} = -\mu_{ab}\cdot E$, $\mu_{ab} = \langle \psi_b | e\bar{x}\cdot\hat{E} | \psi_a \rangle$

AND $\omega_1 = -\dfrac{2\mu_{ab}}{\hbar}E$, $\omega_2 = 0$, $\omega_3 = \omega_0$

IF THE ELECTRIC FIELD IS CIRCULARLY POLARIZED, WITH $\bar{E}$ IN THE X-Y PLANE,

$V_{ab} = -\frac{\gamma}{2}E(\cos\omega_0 t \mp i\sin\omega_0 t)$ FOR $\bar{E} = E_0(\hat{x}\cos\omega_0 t \pm \hat{y}\sin\omega_0 t)$

$V_{ba} = -\frac{\gamma}{2}E(\cos\omega_0 t \pm i\sin\omega_0 t$ AND $\gamma = \langle\psi_b | e(\hat{x} + i\hat{y}) | \psi_a\rangle$ IS REAL

THEN $\omega_1 = -\frac{\gamma E}{\hbar}\cos\omega_0 t$, $\omega_2 = -\frac{\gamma E}{\hbar}\sin\omega_0 t$, $\omega_3 = \omega_0$.

ETC.

Self-Induced Transparency

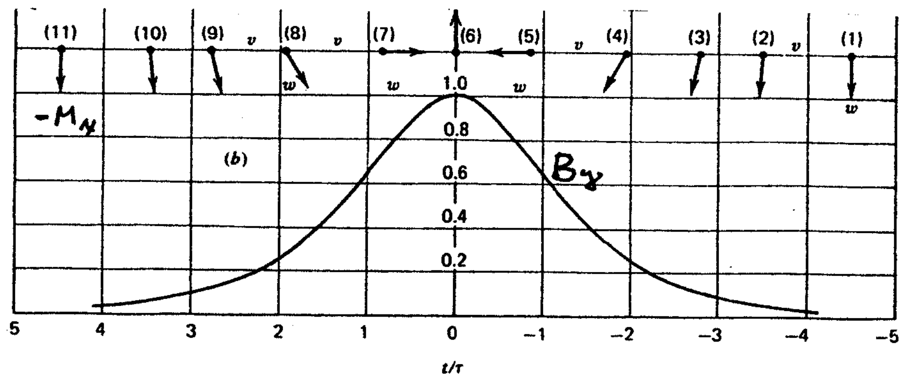
One of the most interesting phenomenon in our magnetic medium is solitons – pulses that propagate without changing their shape despite the dispersion.

This is a subtle consequence of the nonlinear character of the equations of motion of p. 307. Recall that $\bar{B} = \mu \bar{H} = \mu(\bar{B} + 4\pi \bar{M})$, so $\bar{B} = \frac{4\pi \bar{M}}{1-\mu}$ ....

We start with a magnetic medium with a static field B_x. All the moments initially point in the $+x$ direction.

Then we send in a pulse along the z axis, with magnetic field B_y. If the pulse shape is just right, the dipoles precess about the y axis by $180°$ during the time up to the peak of the pulse. Then after the pulse passes, the dipoles precess another $180°$ back to their original position.

The medium is left in its original state, and the pulse does not deform.



Although the algebraic details are a bit messy, it may be instructive to go through some of them.

We suppose the characteristic time τ of the pulse is short compared to the damping time — and we ignore damping.

The equations of motion from p. 307 are then

$$\dot{M}_x = -\Gamma B_y M_z$$

$$\dot{M}_y = \omega_0 M_z \qquad \text{where } \omega_0 = \Gamma B_x$$

$$\dot{M}_z = \Gamma B_y M_x - \omega_0 M_y$$

We first note a formal solution when $\omega_0 = 0$, based on the

pulse area $\quad A(z,t) = \int_{-\infty}^{t} \Gamma B_y(z,t')\,dt' \Rightarrow \dot{A} = \Gamma B_y\ ,\ \ddot{A} = \Gamma \dot{B}_y$

WE SEEK SOLUTIONS TO: $\dot{M}_x = -\Gamma B_y M_z$, $\dot{M}_z = \Gamma B_y M_x$ (M_y = CONST)

WE READILY SEE THAT $M_x = \sin A$, $M_z = -\cos A$

AND ALSO $M_x = \cos A$, $M_z = \sin A$ ARE SOLUTIONS.

WE ARE INTERESTED IN A <u>PULSE</u> OF B_y, SO $A(t=-\infty) = 0$. WE WANT $M_x = 1$ AND $M_z = 0$ INITIALLY. HENCE, THE SECOND SOLUTION SET IS MORE LIKE WHAT WE DESIRE.

WE NOW RETURN ω_0 TO ITS NONZERO VALUE, AND MAKE A <u>GUESS</u> THAT THE SOLUTION HERE HAVE SIMPLE RELATIONS TO THOSE FOR $\omega_0 = 0$.

IN PARTICULAR, WE TRY $M_z = F(\omega_0) \sin A$ WHERE $F(0) \equiv 1$.

WE ALSO GUESS THAT M_y WILL INVOLVE $F(\omega_0) \cos A$ - BUT THIS CAN'T BE QUITE RIGHT, SINCE WE WANT $M_x = 1$ WHEN $A = 0$. WE FIX THINGS UP AS

$$M_x = F(\omega_0)[\cos A - 1] + 1$$

TO SEE IF THIS WORKS, WE DIFFERENTIATE THE EQUATION FOR $\dot{M}_z$ AND USE THE EQUATIONS FOR $\dot{M}_x$ & $\dot{M}_y$:

$$\ddot{M}_z = \Gamma \dot{B}_y M_x + \Gamma B_y \dot{M}_x - \omega_0 \dot{M}_y$$

$$\frac{d}{dt}\dot{M}_z = \frac{d}{dt}(F\dot{A}\cos A) = \ddot{A} M_x - \Gamma^2 B_y^2 M_z - \omega_0^2 M_z$$

$$F\ddot{A}\cos A - F\dot{A}^2 \sin A = \ddot{A}[F(\cos A - 1) + 1] - \dot{A}^2 F \sin A - \omega_0^2 F \sin A$$

$$\Rightarrow \ddot{A} = \frac{\omega_0^2 F}{\underbrace{1-F}_{\text{A CONSTANT WITH DIMENSIONS } 1/t^2}} \sin A$$

WE DEFINE $\frac{1}{\tau^2} = \frac{\omega_0^2 F}{1-F} \Rightarrow F = \frac{1}{1+\omega_0^2\tau^2}$. τ WILL PROVE TO BE THE PULSE WIDTH.

WE ARE LEFT WITH THE DIF. EQ. $\ddot{A} = \frac{1}{\tau^2}\sin A$

AS t GOES FROM $-\infty$ TO $+\infty$, A GOES FROM 0 TO A_{MAX} WHICH WE TAKE AS 2π.

EXPERTS RECOGNIZE THAT THIS MATHIEU EQUATION HAS A PARTICULAR SOLUTION

$A = 4\tan^{-1} f$ WHERE $f = e^{\frac{t - z/v}{\tau}}$, v = PULSE VELOCITY.

A derivation: $\ddot{A} = \frac{1}{\tau^2} \sin A \Rightarrow \dot{A}\ddot{A} = \frac{\dot{A} \sin A}{\tau^2}$.

This integrates to $\frac{\dot{A}^2}{2} = K - \frac{\cos A}{\tau^2}$.

A is the area of the pulse up to time t (at some point z), so $A(-\infty) = 0$, and also $\dot{A}(-\infty) = 0$. Hence, $K = 1/\tau^2$, and

$$\frac{\dot{A}^2}{2} = \frac{1 - \cos A}{\tau^2} = \frac{2}{\tau^2} \sin^2 \frac{A}{2}.$$

Take square root: $\frac{\dot{A}}{2} = \frac{1}{\tau} \sin A/2$,

$$\text{or} \quad \frac{d\,A/2}{\sin A/2} = \frac{dt}{\tau}.$$

This integrates to $\ln \tan \frac{A}{4} = \frac{t}{\tau} + K$.

A cute trick: evaluate this at time t_0 such that $\frac{1}{2}$ of the pulse has arrived: $A(t_0) = \frac{A_{MAX}}{2}$

If we define $A_{MAX} = 2\pi$ things are very nice:

$A(t_0) = \pi$, $\tan \frac{A(t_0)}{4} = \tan \frac{\pi}{4} = 1$

$$\ln \tan \frac{A(t_0)}{4} = 0 = \frac{t_0}{\tau} + K$$

$$\text{so } K = -\frac{t_0}{\tau}$$

$$\ln \tan \frac{A}{4} = \frac{t - t_0}{\tau}$$

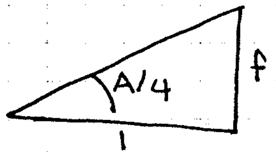
$$\tan \frac{A}{4} = e^{\frac{t-t_0}{\tau}}$$

$$A = 4 \tan^{-1}\left(e^{\frac{t-t_0}{\tau}}\right)$$

Since we desire a travelling wave pulse, with velocity v, we identify $t_0 = \frac{z}{v}$

$$A = 4 \tan^{-1} f, \quad f = e^{\frac{t - z/v}{\tau}}$$ as claimed.

To get the components M_i, we need $\sin A$ & $\cos A$, which will also permit a direct confirmation that our solution obeys the diff. equation.



$$\cos^2\frac{A}{4} = \frac{1}{1+f^2}\ ,\quad \sin^2\frac{A}{4} = \frac{f^2}{1+f^2}$$

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = 4\sin\frac{A}{4}\cos\frac{A}{4}\left(\cos^2\frac{A}{4} - \sin^2\frac{A}{4}\right) = 4\sqrt{\frac{f^2}{1+f^2}\cdot\frac{1}{1+f^2}}\left(\frac{1-f^2}{1+f^2}\right)$$

$$= 4\frac{f(1-f^2)}{(1+f^2)^2} = 4\frac{f^2\left(\frac{1}{f}-f\right)}{f^2\left(\frac{1}{f}+f\right)^2} = -2\tanh\left(\frac{t-z/v}{\tau}\right)\operatorname{sech}\left(\frac{t-z/v}{\tau}\right)$$

WE WILL ALSO NEED $\cos A = 1 - 2\sin^2\frac{A}{2} = 1 - 8\sin^2\frac{A}{4}\cos^2\frac{A}{4} = 1 - \frac{8f^2}{(1+f^2)^2} = 1 - \frac{8}{\left(\frac{1}{f}+f\right)^2}$

$$= 1 - 2\operatorname{sech}^2\left(\frac{t-z/v}{\tau}\right)$$

ALSO, $\dot{A} = \frac{4}{1+f^2}\dot{f} = \frac{4}{\tau}\frac{f}{1+f^2}$ NOTING THAT $\dot{f} = \frac{f}{\tau}$

$$\Rightarrow\ \ddot{A} = \frac{4}{\tau}\left(\frac{\dot{f}}{1+f^2} - \frac{2f^2\dot{f}}{(1+f^2)^2}\right) = \frac{4f}{\tau^2}\frac{1-f^2}{(1+f^2)^2} = \frac{\sin A}{\tau^2}$$ AS DESIRED!

THE MAGNETIZATION COMPONENTS ARE THEN

$$M_x = 1 + F(\cos A - 1) = 1 - 2F\operatorname{sech}^2\left(\frac{t-z/v}{\tau}\right)$$ SO $M_x(\pm\infty) = 1$, $M_x(0) \approx -1$ IF $\omega_0\tau \ll 1$.

$$M_z = F\sin A = -2F\tanh\left(\frac{t-z/v}{\tau}\right)\operatorname{sech}\left(\frac{t-z/v}{\tau}\right)$$, A BIT MESSY.

TO GET M_y, WE AGAIN USE THE EQ. FOR $\dot{M}_z$, WHICH TELLS US THAT

$$\omega_0 M_y = \Gamma B_y M_x - \dot{M}_z = \Gamma(1-F)B_y = \Gamma\omega_0^2\tau^2 B_y F \quad \text{SINCE } F = \frac{1}{1+\omega_0^2\tau^2}.$$

$$= \omega_0^2\tau^2\dot{A}F = 4\omega_0^2\tau\frac{Ff}{1+f^2} = 4\frac{\omega_0^2\tau F}{\frac{1}{f}+f} = 2\omega_0^2\tau F\operatorname{sech}\frac{t-z/v}{\tau}.$$

SO $M_y = 2\omega_0\tau F\operatorname{sech}\frac{t-z/v}{\tau}$,

AND $B_y = \frac{2}{\Gamma\omega_0\tau}\operatorname{sech}\left(\frac{t-z/v}{\tau}\right)$ IS THE MAGNETIC FIELD PULSE.

THE ORIGINAL WORK IS BY S.L. McCALL & E.L. HAHN, PRL 18, 908 (1967); PHYS. REV. A 2, 861 (1970). SEE ALSO, G.L. LAMB, REV. MOD. PHYS. 43, 99 (1971).

REMARK: IF THERE ARE NO LOSSES, $v = \frac{c}{n} = c/\sqrt{\mu}$. BUT FOR A DIAMAGNETIC MATERIAL, AS HERE, $\mu < 1 \Rightarrow v > c$. TO KEEP $v < c$, MUST CONSIDER LOSSES...

Self Focusing

Another type of soliton behavior in electromagnetism is the self focusing of a laser beam in a nonlinear medium, first reported by R.Y. Chiao, E. Garmire & C.H. Townes, Phys. Rev. Lett. 13, 479 (1964)

Recall that in a linear dielectric, $\bar{P} = \chi_0 \bar{E}$ relates the polarization to the electric field. In a strong electric field nonlinear corrections arise:

$$\bar{P} = \chi_0 \bar{E} + \chi_2 E^2 \bar{E} + \dots$$

(For $\bar{P}$ to follow $\bar{E}$, can't have coefs that are odd powers of $|E|$.)

For example, we can expect that when the field energy in one atomic volume equals $\hbar\omega_0$, for a resonance ω_0 of the atom, the nonlinear term is comparable to the linear term in the susceptibility χ:

$$\frac{\chi_2 E^2}{\chi_0} \sim \frac{E^2 \cdot \mathrm{vol}/8\pi}{\hbar\omega_0}$$

Now, $\mathrm{Vol}_{\mathrm{atom}} \sim 1/N \sim 10^{-21}\ \mathrm{cm}^3/\mathrm{atom}$ for a gas

Also $n_0 = \sqrt{\epsilon_0} = \sqrt{1+4\pi\chi_0} \sim 1 + 2\pi\chi_0$,

so $\chi_0 \sim \frac{n_0 - 1}{2\pi} \sim 10^{-4}$ for a gas

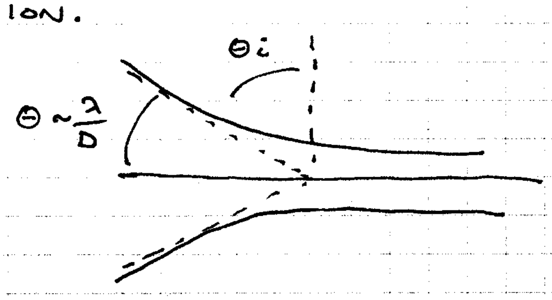
$\hbar\omega_0 \sim 1\,\mathrm{eV} = 10^{-19}\,\mathrm{J} = 10^{-12}\,\mathrm{erg}$

$$\text{so } \chi_2 \sim \frac{\chi_0}{8\pi N \hbar\omega_0} \sim \frac{10^{-4} \cdot 10^{-21}}{10 \cdot 10^{-12}} \sim \underline{\underline{10^{-14}}}$$

Since the index of refraction is a function of laser intensity, a laser beam creates a kind of "light pipe" inside a dielectric medium, in which the index is higher closer to the axis.

This is the condition needed for transverse confinement of the beam by total internal reflection.

Example: A beam that converges at an $\Theta \sim \lambda/D$ in a nonlinear dielectric will be trapped in a tube of constant radius if $n_{max} \sin\Theta_i = n_{max} \cos\Theta \gtrsim 1$.



$$\left[1 + 2\pi(\chi_0 + \chi_2 E^2)\right]\left[1 - \frac{\Theta^2}{2}\right] \sim 1$$

$\Rightarrow$ need $E^2 \sim \frac{\Theta^2}{4\pi\chi_2}$. The required beam power is given by

$$P \sim SD^2 = \frac{cD^2E^2}{4\pi} = \frac{c\,D^2(\lambda/D)^2}{(4\pi)^2\chi_2} = \frac{c}{\chi_2}\left(\frac{\lambda}{4\pi}\right)^2 \sim \frac{10^{10}\cdot(10^{-5})^2}{10^{-14}} \sim 10^{14}\ \mathrm{erg} \sim \underline{10^7\ \mathrm{watts}}$$

This is, for example, 1 Joule in 100 ns, which is readily achieved...

Ph 206 Lecture 16

We consider a pulse that propagates with constant transverse profile in such a nonlinear medium:

$$\bar{E} = \bar{E}_\perp(x,y)\, e^{i(k_z z - \omega t)}.$$

The wave equation is $\nabla^2 \bar{E} = \frac{n^2}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2} = \frac{\epsilon_0 + \epsilon_2 E^2}{c^2}\left(-\omega^2 \bar{E}\right)$

Now $\nabla^2 \bar{E} = \nabla_\perp^2 \bar{E} - k_z^2 \bar{E}$; and we suppose that only the time average of $\epsilon_0 + \epsilon_2 E^2$ is important for the description of $\bar{E}_\perp$.

We write $K = \frac{\omega}{c}$ $\quad K_0 = n_0 \frac{\omega}{c} = \sqrt{\epsilon_0}\,\frac{\omega}{c}$, and $\Gamma^2 = k_z^2 - k_0^2$.

Then the eq. for $\bar{E}_\perp$ is $\quad \nabla_\perp^2 \bar{E}_\perp - \Gamma^2 \bar{E}_\perp + \epsilon_2 K^2 \langle E^2 \rangle \bar{E}_\perp = 0$

Example: A 2-d sheet beam, with $\bar{E}_\perp = E_x(x)\,\hat{x}$, so $\langle E^2 \rangle = \frac{E_x^2}{2}$

The eq. for E_x is $\quad \frac{d^2 E_x}{dx^2} = \Gamma^2 E_x - \frac{\epsilon_2 k^2}{2} E_x^3 \equiv \Gamma^2\left(E_x - \frac{2}{a^2} E_x^3\right)$

where $a = \frac{2\Gamma}{\sqrt{\epsilon_2}\, k}$ $\quad$ [assuming $\Gamma^2 > 0$].

To integrate, multiply by E_x' : $\quad E_x' E_x'' = \Gamma^2\left(E_x' E_x - \frac{2}{a^2} E_x' E_x^3\right)$

$$\Rightarrow \frac{E_x'^2}{2} = \Gamma^2\left(\frac{E_x^2}{2} - \frac{E_x^4}{2a^2}\right) + K$$

For a <u>pulse</u>, we want $E_x(\pm\infty) = 0 = E_x'(\pm\infty) \Rightarrow K = 0$

Then, $\quad E_x' = \frac{\Gamma}{a} E_x \sqrt{a^2 - E_x^2}$

so $\quad \frac{\Gamma}{a} x + \frac{K}{a} = \int \frac{dE_x}{E_x\sqrt{a^2 - E_x^2}} = -\frac{1}{a}\,\mathrm{sech}^{-1}\frac{E_x}{a}$

$\Rightarrow E_x = a\,\mathrm{sech}(\Gamma x + K)$ $\quad$ and $K = 0$ for a pulse symmetric about $x = 0$.

or $\quad E_x = \dfrac{2\Gamma}{\sqrt{\epsilon_2}\,k}\,\mathrm{sech}(\Gamma x)$

Again, we find the hyperbolic secant to be the pulse shape that propagates without distortion in a nonlinear medium.

The wave velocity is $v = \frac{\omega}{k_z}$, and since $\Gamma^2 = k_z^2 - k_0^2 > 0$, $k_z = \frac{n\omega}{c}$ for $n > n_0$

and $v = \frac{c}{n} < c$. $\quad$ [If $\Gamma^2 < 0$, the solution for E_x blows up at $x = 0$. ...]