Properties of Solitons

In Lecture 25 we saw two examples of distortion-free propagation of a pulse in a dispersive, nonlinear medium.

For an appropriate pulse shape, the nonlinearity leads to a kind of "bunching" of the pulse (as happens when a water wave "breaks"), that exactly counteracts the dispersion.

Many properties of solitons can be illustrated by considering a relatively simple one-dimensional wave equation called the KdV equation.

The KdV equation (Korteweg & deVries, Phil. Mag. 39 422 (1895).)

In a linear, nondispersive medium, the wave equation has the form

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0 \quad , \text{ where } c = \text{wave velocity.}$$

One-dimensional waves have the form $\psi = f(z \pm ct)$ where f can be any reasonable function. Since the pulse shape f does not change with time, one could say that solitons are included among the possible waves in this medium.

The name "soliton" includes the connotation of a solitary pulse, i.e. a unipolar pulse with a single maximum (or minimum).

Recall (Lecture 11, p126) that unipolar pulses cannot exist for 3-dimensional waves that are emitted by a bounded source. In 3-D, a pulse with a single maximum must be accompanied by a tails of the opposite sign:

Among the possible waves in a linear, nondispersive medium are monochromatic waves

$$\psi = \psi_0 \, e^{i(kz - \omega t)} \quad \text{where} \quad \omega = kc.$$

When we consider dispersive media, we have often started with monochromatic waves, noting that the frequency is a nonlinear function of the wave number:

$$\omega = \omega(k). \quad \text{(The dispersion relation.)}$$

WE NOW SEEK A SIMPLE WAVE EQUATION THAT INCLUDES DISPERSION AND NONLINEARITY.

WE START FROM THE DISPERSION-FREE RELATION $\omega = kc$, AND MODIFY IT TO

$$\omega = kc + \cdots$$

WE SUPPOSE THAT $\omega(-k) = -\omega(k)$, MOTIVATED BY THE REQUIREMENT THAT THE FOURIER SYNTHESIS

$$\psi = \int_{-\infty}^{\infty} \psi_k e^{i(kz - \omega(k)t)} dk$$

REPRESENT A REAL WAVE FUNCTION ψ. THEN, WRITING

$$\psi = \int_0^{\infty} \left[\psi_k e^{i(kz - \omega(k)t)} + \psi_{-k} e^{i(-kz - \omega(-k)t)} \right] dk$$

WE NEED $\psi_{-k} e^{i(-kz - \omega(-k)t)} = \left[\psi_k e^{i(kz - \omega(k)t)}\right]^* = \psi_k^* e^{-i(kz - \omega(k)t)}$

AND SO $\psi_{-k} = \psi_k^*$, $\omega(-k) = -\omega(k)$.

THUS, A POWER SERIES REPRESENTATION OF $\omega(k)$ CAN CONTAIN ONLY ODD POWERS OF k, AND THE SIMPLEST MODIFICATION IS

$$\omega(k) = kc - \alpha k^3.$$

[SOME PEOPLE CLAIM IT IS AGREEABLE THAT α SO DEFINED IS POSITIVE, AS THE EFFECT OF DISPERSION IS TO CAUSE A PHASE LAG IN $\phi = \omega t$ COMPARED TO kct.]

THEN, A WAVE $\psi = e^{i(kz - \omega t)} = e^{i(kz - kct + \alpha k^3 t)}$ OBEY

$$\dot\psi = (-ikc + i\alpha k^3)\psi, \quad \psi' = ik\psi, \quad \psi'' = -k^2\psi, \quad \psi''' = -ik^3\psi, \text{ AND}$$

SO $\dot\psi + c\psi' + \alpha\psi''' = 0$ IS A "WAVE EQUATION" FOR ψ.

AS WELL AS DISPERSION IN OUR WAVE EQUATION, WE WANT SOME NONLINEARITY. A SIMPLE WAY TO ADD THIS IS VIA A QUADRATIC TERM:

$$\dot\psi + c\psi' + \alpha\psi''' + \beta(\psi^2)' = 0.$$

WE CAN SCALE OUT THE COEFFICIENTS α, β & c VIA THE TRANSFORMATION

$z = \alpha^{1/3} x$ AND $\psi = \frac{1}{2\beta}(\alpha^{1/3} u - c)$

$$\frac{\partial}{\partial z} \to \frac{1}{\alpha^{1/3}}\frac{\partial}{\partial x}, \quad \frac{\partial^3}{\partial z^3} \to \frac{1}{\alpha}\frac{\partial^3}{\partial x^3}, \quad \phi^2 = \frac{1}{4\beta^2}\left(\alpha^{2/3}u^2 - 2\alpha^{1/3}uc + c^2\right),$$

AND THE WAVE EQUATION BECOMES

$$\frac{\alpha^{1/3}}{2\beta}\dot{u} + \frac{c\,\alpha^{1/3}u'}{2\beta\alpha^{1/3}} + \frac{\alpha\cdot\alpha^{1/3}u'''}{2\beta\alpha^{1/3}} + \frac{\beta}{4\beta^2\alpha^{1/3}}\left(2\alpha^{2/3}uu' - 2\alpha^{1/3}u'c\right) = 0$$

OR $\underline{\dot{u} + uu' + u''' = 0}$, THE KdV EQUATION

THIS EQUATION WAS ORIGINALLY DERIVED AS A DESCRIPTION OF WATER WAVES, FOR WHICH u HAS THE SIGNIFICANCE OF THE VELOCITY OF A PARTICLE AT THE SURFACE OF THE WAVE. THEN $\dot{u} + uu'$ IS INTERPRETED AS A CONVECTIVE DERIVATIVE: $\dfrac{Du}{Dt} = \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x}$

SO $\dfrac{Du}{Dt} + u''' = 0$, FOR WHAT IT'S WORTH.

THE KdV EQUATION DOES NOT HAVE sech SOLUTIONS; RATHER THEY DEPEND ON sech^2. WE TRY

$$u = A\,\text{sech}^2[B(x - vt)] \quad \text{WHERE } v = \text{PULSE VELOCITY}$$

THEN $\dot{u} = -vu'$, SO KdV $\Rightarrow$ $u''' = vu' - uu'$

NOW, $[\text{sech}\,x]' = -\text{sech}\,x\tanh x$, AND $[\tanh x]' = \text{sech}^2 x$, SO

$$u' = -2AB\tanh[B(x-vt)]\,\text{sech}^2[B(v-xt)]$$

$$u'' = -2AB^2\,\text{sech}^4[B(x-vt)] + 4AB^2\tanh^2[B(x-vt)]\,\text{sech}^2[B(x-vt)]$$

AND $u''' = -8AB^3\tanh[B(x-vt)]\,\text{sech}^2[B(x-vt)] + 24AB^3\tanh[B(x-vt)]\,\text{sech}^4[B(x-vt)]$

USING $\tanh^2 x = 1 - \text{sech}^2 x$

ALSO, $vu' - uu' = -2ABv\tanh[B(x-vt)]\,\text{sech}^2[B(x-vt)] + 2A^2B\tanh[B(x-vt)]\,\text{sech}^4[B(x-vt)]$

SO KdV IS SATISFIED IF $8AB^3 = 2ABv \Rightarrow B^2 = v/4$, $B = \dfrac{\sqrt{v}}{2}$

AND $24AB^3 = 2A^2B \Rightarrow A = 12B^2 = 3v$

$$\Rightarrow \quad u = 3v\,\text{sech}^2\left[\frac{\sqrt{v}}{2}(x - vt)\right]$$

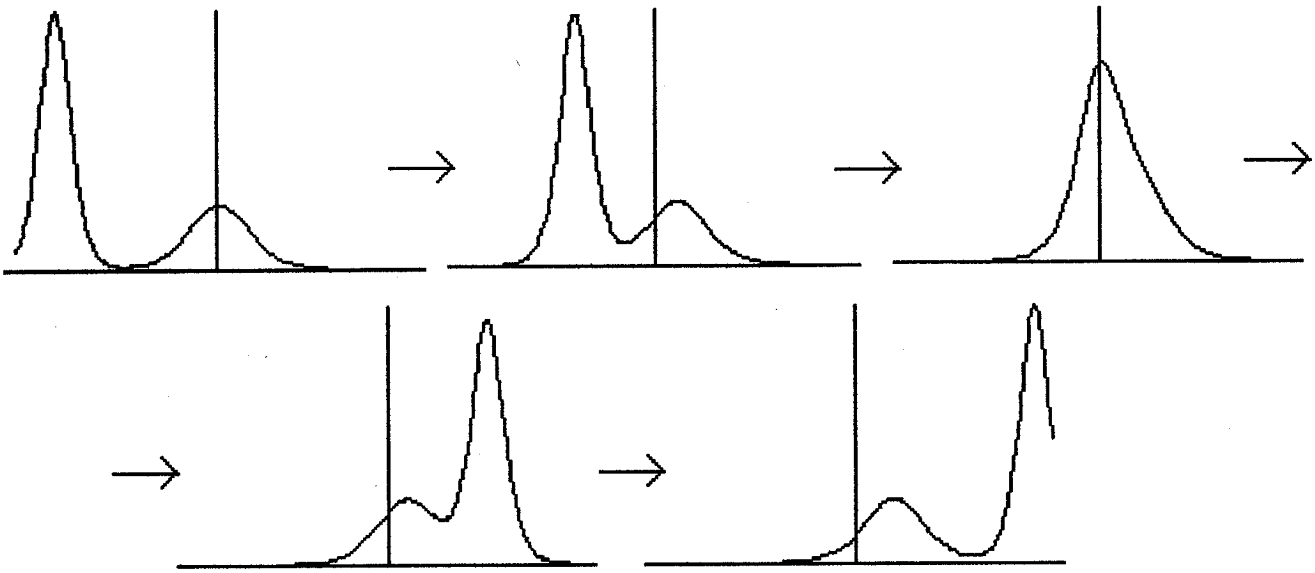
THE PULSE AMPLITUDE INCREASES LINEARLY WITH THE PULSE VELOCITY,

AND THE PULSE WIDTH $\Delta x \sim \dfrac{2}{\sqrt{v}}$ DECREASES WITH INCREASING VELOCITY.

EQUIVALENTLY, BIG PULSES TRAVEL FAST, AND ARE NARROW.

MULTIPLE SOLITONS

SOLITONS LANGUISHED FOR ~120 YEARS AS A CURIOSITY UNTIL KRUSKAL ET AL., PHYS. REV. LETT. 15, 240 (1965), DEMONSTRATED NUMERICAL THAT THEY ARE NEITHER SOLITARY NOR RARE. IN PARTICULAR, THEY FOUND THAT MANY NONLINEAR MEDIA SUPPORT SIMULTANEOUS MULTIPLE SOLITON PULSES WHOSE INTERACTIONS ARE VERY SLIGHT, AS IS THE CASE IN LINEAR MEDIA.



THE ORIGINAL NUMERICAL DEMONSTRATIONS WERE SOON FOLLOWED BY 3 TYPES OF ANALYTIC REPRESENTATION OF MULTIPLE SOLITONS:

1. THE "INVERSE SCATTERING" METHOD   GARDNER ET AL. PHYS. REV LETT. 19, 1905 (67)
2. THE BÄCKLUND TRANSFORMATION
3. THE "DIRECT" METHOD OF HIROTA, PHYS. REV. LETT. 27, 1192 (71)

WE FOLLOW THE "DIRECT" METHOD, BEING THE MOST STRAIGHTFORWARD, THO SOMEWHAT TEDIOUS ALGEBRAICALLY. THIS IS BASED ON REWRITING THE NONLINEAR KdV EQUATION AS A KIND OF PRODUCT OF LINEAR EQUATIONS — A BILINEAR DIFFERENTIAL EQUATION.

WE START WE THE HYPERBOLIC SECANT PULSE, $u = 3v \operatorname{sech}^2\left[\frac{\sqrt{v}}{2}(x - vt)\right]$, WHOSE MERITS WERE KNOWN TO RAYLEIGH, PHIL. MAG. 1, 257 (1876).

THIS CAN BE WRITTEN AS $$u = 12 \frac{d^2}{dx^2} \ln \cosh\left[\frac{\sqrt{v}}{2}(x - vt)\right]$$

INSPIRED BY THIS FACT, WE SEEK MORE GENERAL SOLUTIONS TO THE KdV EQUATION OF THE FORM: $$u = 12 \frac{d^2}{dx^2} \ln F(x,t) = 12 \frac{d}{dx} \frac{F_x}{F} = \frac{12}{F^2}\left(F F_{xx} - F_x^2\right),$$

WHERE $F_x \equiv \frac{\partial F}{\partial x}$, ETC.

WE ALSO NEED $u'' = \frac{12}{F^4}\left(F^3 F_{xxxx} - 4F^2 F_x F_{xxx} - 3F^2 F_{xx}^2 + 12 F F_x^2 F_{xx} - 9F_x^4\right)$.

- THEN, THE KDV EQUATION IS $0 = \dot{u} + u u' + u''' = \dot{u} + \frac{1}{2}(u^2)' + (u'')'$.

$\dot{u}$ CAN BE WRITTEN AS $12 \frac{d^2}{dx\,dt} \frac{F_x}{F} = 12 \frac{d}{dx}\left(\frac{F_{xt}}{F} - \frac{F_x F_t}{F^2}\right)$, SO THAT

$$0 = 12\left(\frac{F_{xt}}{F} - \frac{F_x F_t}{F^2}\right)' + \left(\frac{u^2}{2}\right)' + (u'')'$$

WHICH INTEGRATES TO $$12\frac{F_{xt}}{F} - 12\frac{F_x F_t}{F^2} + \frac{u^2}{2} + u'' = K$$

FOR A PULSE THAT BEGINS AND ENDS AT $u = u' = u'' = 0$ AT AT FIXED x, WE HAVE $K=0$

EXPRESSING u^2 AND u'' IN TERMS OF F, WE FIND THAT

$$F F_{xt} - F_x F_t + 3F_{xx}^2 + F F_{xxxx} - 4F_x F_{xxx} = 0$$

WHILE THIS IS LENGTHY, NOTE THAT EACH TERM IS THE PRODUCT OF A PAIR OF DERIVATIVES OF $F(x,t)$; THIS IS A <u>BILINEAR</u> DIFFERENTIAL EQUATION.

- EVENTUALLY THIS SUGGESTED THE FOLLOWING TRICK: INTRODUCE THE OPERATOR

$$D_t^m D_x^n A\cdot B \equiv \lim_{\substack{x\to x'\\ t\to t'}} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n A(x,t)\, B(x',t')$$

FOR EXAMPLE, $D_x D_t\, F\cdot F = \lim\limits_{\substack{x\to x'\\ t\to t'}} \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)\left(F_t(x,t)\,F(x',t') - F(x,t)\,F_t(x',t')\right)$

$$= \lim_{\substack{x\to x'\\ t\to t'}} \left(F_{xt}\cdot F - F_x\cdot F_t - F_x\cdot F_t + F\cdot F_{xt}\right) \to 2FF_{xt} - 2F_xF_t$$

SIMILARLY, $D_x^4\, F\cdot F \to 2FF_{xxxx} - 8F_xF_{xxx} + 6F_{xx}^2$

THUS, $(D_xD_t + D_x^4)F\cdot F = 2\left(FF_{xt} - F_xF_t + FF_{xxxx} - 4F_xF_{xxx} + 3F_{xx}^2\right)$

COMPARING WITH THE TOP OF THE PAGE, WE SEE THAT IF

- $(D_xD_t + D_x^4)\,F\cdot F = 0$, THEN F WILL GENERATE A SOLUTION TO THE KDV EQUATION.

$F = 1 \Rightarrow u = 0$ = THE TRIVIAL "ZERO SOLITON" SOLUTION.

WE MIGHT SUPPOSE THAT $F = 1 + f_1$ CAN BE A ONE SOLITON SOLUTION

$$F = 1 + f_1 + f_2 = 2 \text{ SOLITON}$$

$$F = 1 + f_1 + f_1 + \cdots f_n = n \text{ SOLITONS.}$$

TO ORGANIZE OUR THINKING ABOUT THIS APPROACH WE WRITE

$$F = 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \cdots$$

WHERE WE EVENTUALLY TAKE $\epsilon = 1$, BUT CAN MEANWHILE SEEK SOLUTIONS "AT ORDER ϵ^n" TO THE BILINEAR FORM OF THE KdV EQUATION.

THUS, $0 = (D_x D_t + D_x^4) F \cdot F = (D_x D_t + D_x^4)(1 + \epsilon f_1 + \epsilon^2 f_2 + \ldots) \cdot (1 + \epsilon f_1 + \epsilon^2 f_2 + \cdots)$

$$= (D_x D_t + D_x^4)\left[1 \cdot 1 + \epsilon(1 \cdot f_1 + f_1 \cdot 1) + \epsilon^2 (1 \cdot f_2 + f_2 \cdot 1 + f_1 \cdot f_1) + \cdots\right.$$

AT "ORDER 1", $(D_x D_t + D_x^4)(1 \cdot 1) = 0$ IS CLEARLY TRUE.

AT "ORDER ϵ": $(D_x D_t + D_x^4)(1 \cdot f_1 + f_1 \cdot 1) \rightarrow 2(f_{1xt} + f_{1xxxx})$

SETTING THIS TO ZERO, WE OBTAIN THE <u>LINEAR</u> DIFFERENTIAL EQUATION

$$f_{1xt} + f_{1xxxx} = 0$$

WE TRY A TRAVELLING WAVE SOLUTION: $f_1 = A e^{kx - \omega t}$

THEN $f_{1xt} = -k\omega f_1$, $f_{1xxxx} = k^4 f_1$

$$\Rightarrow 0 = (k^4 - k\omega) f_1 \quad \text{OR} \quad \omega = k^3 \quad \& \quad f_1 = A e^{kx - k^3 t} = e^{k(x - x_1 - vt)}$$

WHERE $v = k^2$ AND $kx_1 \equiv -\ln A$.

THIS f_1 IS NOT NECESSARILY A SOLUTION TO THE FULL KdV EQUATION UNLESS THE "HIGHER ORDER" TERMS $f_2, f_3 \ldots$ VANISH.

FOR EXAMPLE, DOES $f_2 = 0$ SATISFY THE "ORDER ϵ^2" RELATION

$$(D_x D_t + D_x^4)(1 \cdot f_2 + f_2 \cdot 1 + f_1 \cdot f_1) = 0 \ ?$$

IF $f_2 = 0$ THEN ~~WE~~ MUST HAVE $(D_x D_t + D_x^4) f_1 \cdot f_1 = 0$

BUT, WITH $f_1 = A e^{kx - k^3 t}$, $D_x f_1 \cdot f_1 = \lim_{x \to x'} (k f_1 \cdot f_1 - f_1 \cdot k f_1) = 0$

SO INDEED $(D_x D_t + D_x^4) f_1 \cdot f_1 = 0 \ldots$

SO $F = 1 + f_1 = 1 + e^{k(x - x_1 - vt)}$ WITH $v = k^2$ WILL GENERATE A SOLUTION TO THE KdV EQUATION. INDEED,

$$u = 12 \frac{d^2}{dx^2} \ln F = \frac{12}{F^2}\left(F F_{xx} - F_x^2\right) = 12k^2 \frac{(1+f_1)f_1 - f_1^2}{(1+f_1)^2} = 12k^2 \frac{f_1}{(1+f_1)^2}$$

$$= \frac{12k^2}{\left(\frac{1}{\sqrt{f_1}} + \sqrt{f_1}\right)^2} = \frac{12\,k^2}{\left(e^{\frac{k}{2}(x - x_1 - vt)} + e^{-\frac{k}{2}(x - x_1 - vt)}\right)^2} = 3v \,\mathrm{sech}^2 \frac{\sqrt{v}}{2}(x - x_1 - vt)$$

$\Rightarrow$ THE "ONE SOLITON" SOLUTION.

IT IS NOW NOT SUCH A BIG LEAP TO SUPPOSE THAT MULTIPLE SOLITONS SOLUTIONS CAN BE GENERATED BY ADDING MORE "PLANE WAVES" TO f_1

$\Rightarrow$ TRY $f_1 = e^{k_1(x - x_1) - \omega_1 t} + e^{k_2(x - x_2) - \omega_2 t} + \cdots e^{k_n(x - x_n) - \omega_n t}$

SINCE THE "ORDER ϵ" EQUATION FOR f_1 IS LINEAR, $f_{1xt} + f_{1xxxx} = 0$, OUR TRIAL SOLUTION IS OK HERE SO LONG AS EACH $\omega_i = k_i^3$.

WE WILL PURSUE THE 2 SOLITON SOLUTION THROUGH "ORDER ϵ^2" HERE.

IN GENERAL, WE DESIRE AT ORDER ϵ^2 THAT $0 = (D_x D_t + D_x^4)(1 \cdot f_2 + f_2 \cdot 1 + f_1 \cdot f_1)$

NOW $(D_x D_t + D_x^4)(1 \cdot f_2 + f_2 \cdot 1) = 2(f_{2xt} + f_{2xxxx})$ JUST AS FOR f_1.

SO f_2 MUST OBEY $f_{2xt} + f_{2xxxx} = -\frac{1}{2}(D_x D_t + D_x^4) f_1 \cdot f_1$

WE ABBREVIATE $f_1 = g_1 + g_2$ WHERE $g_i = e^{k_i(x - x_i) - \omega_i t}$

THEN, $f_1 \cdot f_1 = g_1 \cdot g_1 + g_1 \cdot g_2 + g_2 \cdot g_1 + g_2 \cdot g_2$

AND AFTER A BIT OF ALGEBRA, WE FIND THAT

$$f_{2xt} + f_{2xxxx} = (k_1 - k_2)\left[(\omega_1 - \omega_2) - (k_1 - k_2)^3\right] g_1 g_2$$

THIS FORM SUGGESTS THAT WE TRY $f_2 = B g_1 g_2$, FOR WHICH

$$f_{2xt} + f_{2xxxx} = B(k_1 + k_2)\left[-(\omega_1 + \omega_2) + (k_1 + k_2)^3\right]$$

NOTING THAT $\omega_1 = k_1^3$ AND $\omega_2 = k_2^3$, WE QUICKLY FIND THAT

$B = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2$. WE CLAIM WITHOUT PROOF THAT WE CAN SET $f_n = 0$, $n > 0$.

THEN $F = 1 + f_1 + f_2 = 1 + g_1 + g_2 + B g_1 g_2$ $B = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2$

THIS GENERATES A SOLUTION TO THE KdV EQUATION VIA

$$\boxed{u = \frac{12}{F^2}(F F_{xx} - F_x^2) = \frac{12\left[k_1^2 g_1 + k_2^2 g_2 + (k_1 - k_2)^2 g_1 g_2 + B g_1 g_2 (k_2^2 g_1 + k_1^2 g_2)\right]}{(1 + g_1 + g_2 + B g_1 g_2)^2} \qquad g_1 = e^{k_1(x - x_1 - v_1 t)}, \quad g_2 = e^{k_2(x - x_2 - v_2 t)}, \quad v_i = k_i^2.}$$

THIS IS EASILY GRAPHED, AND MADE INTO A MOVIE — SOME OF WHICH CAN BE FOUND ON THE INTERNET. THE SEQUENCE OF PLOTS ON p. 321 SHOWS 2 SOLITONS THAT INTERACT, MERGE BRIEFLY INTO ONE PULSE, AND THEN SPLIT APART INTO 2 SOLITONS ESSENTIALLY IDENTICAL TO THE INITIAL STATES ...

ANALYTIC APPROXIMATIONS READILY ILLUSTRATE THE BEHAVIOR OF THE 2-SOLITON SOLUTION BEFORE & AFTER THE "COLLISION.

SET $x_1 = x_2 = 0$, SO NOMINALLY THE 2 PULSES WILL OVERLAP MAXIMALLY AT $x = t = 0$. ALSO, TAKE $k_1 > k_2$, SO $v_1 = k_1^2 > v_2 = k_2^2$

TO KEEP TRACK OF THE ASYMPTOTIC BEHAVIOR OF SOLITON 1 — GENERATED BY g_1, WE EMPHASIZE THE VARIABLE $\xi_1 = x - v_1 t$. IF SOLITON 1 STAYS TOGETHER, IT WILL BE BIG ONLY FOR SMALL ξ_1, EVEN AS $t \to \pm\infty$.

WE REWRITE: $g_1 = e^{k_1 \xi_1}$, $g_2 = e^{k_2(\xi_1 + (v_1 - v_2)t)}$

INITIALLY, WE EMPHASIZE ξ_1 SMALL, $t \to -\infty \Rightarrow g_2 \to 0$ (FOR SMALL ξ_1)

THEN $u(\xi_1 \text{ small}, t \to -\infty) \longrightarrow \dfrac{12 k_1^2 g_1}{1 + g_1^2} = 3 v_1 \operatorname{sech}^2\left[\frac{\sqrt{v_1}}{2}(x - v_1 t)\right]$,

WHICH IS THE EARLY STATE OF SOLITON 1, AS EXPECTED.

MUCH LATER, WITH ξ_1 STILL SMALL, BUT $t \to +\infty$, WE HAVE $g_2 \gg g_1$

THEN IN THE NUMERATOR OF u, ONLY THE TERM $B k_1^2 g_1 g_2^2$ MATTERS, WHILE THE DENOMINATOR GOES TO $g_2(1 + B g_1)$

THAT IS, $u(\xi_1 \text{ small}, t \to +\infty) \to \dfrac{12 k_1^2 B g_1}{(1 + B g_1)^2} = 3 v_1 \operatorname{sech}^2\left[\frac{\sqrt{v_1}}{2}(x - \Delta x_1 - v_1 t)\right]$

WHERE $\Delta x_1 = \dfrac{1}{k_1} \ln\left(\dfrac{k_1 + k_2}{k_1 - k_2}\right) > 0$

FOR LARGE t & SMALL ξ_1, SOLITON 1 HAS ITS ORIGINAL SHAPE, BUT HAS SKIPPED AHEAD BY AMOUNT $\Delta x_1 = \frac{1}{k_1}\ln\left(\frac{1}{B}\right) = \frac{1}{k}\ln\left(\frac{k_1+k_2}{k_1-k_2}\right)$

VIA A SIMILAR ANALYSIS THAT EMPHASIZES SOLITON 2 VIA SMALL VALUES OF $\xi_2 = x - v_2 t$ AS $t \to \pm\infty$, WE FIND THAT FOR $t \to +\infty$, SOLITON 2 PROPAGATES AS $u = 3v_2 \operatorname{sech}^2\left[\frac{\sqrt{v_2}}{2}(x - v_2 t)\right]$, BUT FOR $t \to -\infty$, IT BEHAVES AS $u = 3v_2 \operatorname{sech}^2\left[\frac{\sqrt{v_2}}{2}(x - \Delta x_2 - v_2 t)\right]$ WHERE $\Delta x_2 = \frac{1}{k_2}\ln\frac{k_1+k_2}{k_1-k_2}$

THAT IS, SOLITON 2 IS HELD BACK AS A RESULT OF ITS INTERACTION WITH SOLITON 1, WHILE SOLITON 1 JUMPS AHEAD.

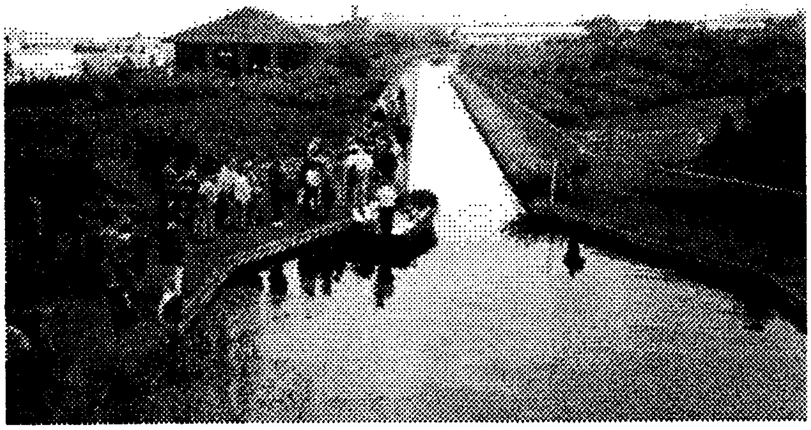
WITH $k_1 > k_2$, THE AMPLITUDE OF SOLITON 1, $3V_1$, IS BIGGER THAN THAT OF SOLITON 2.

SUMMARY: THE BIGGER SOLITON (1) OVERTAKES THE SMALLER, SLOWER SOLITON (2), INTERACTS BRIEFLY WITH IT; THEN THE 2 SOLITON RETURN TO THEIR ORIGINAL SHAPES, WITH THE BIG SOLITON APPARENTLY SHIFTED FORWARD, AND THE SMALL SOLITON HELD BACK.

THESE FEATURES ARE READILY SEEN IN THE INTERNET MOVIES OF A PAIR OF SOLITONS...

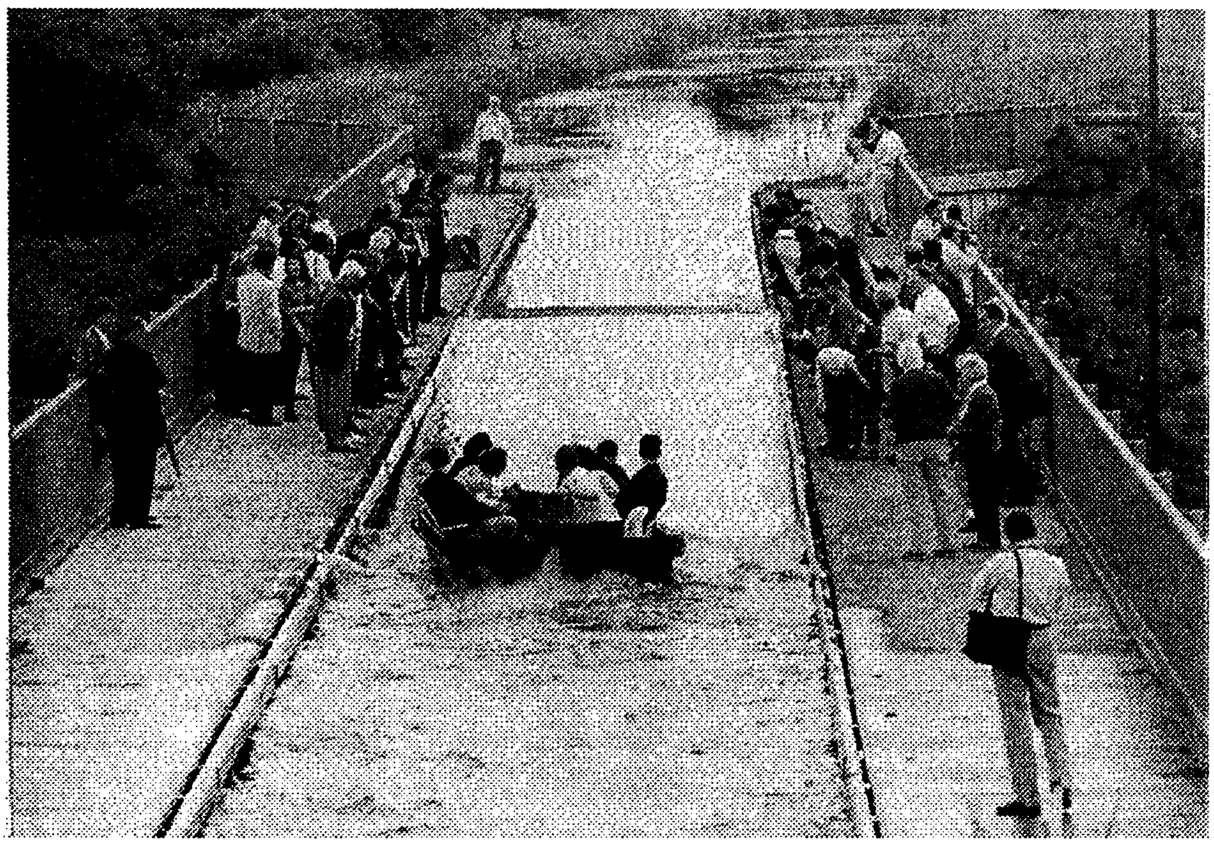
THE "DIRECT METHOD" CAN BE GENERALIZED TO GIVE A SOLUTION CONTAINING N SOLITONS THAT COALESCE & REFORM...

THE FIRST RECORDED OBSERVATION OF A SOLITON WAS IN 1834 BY SCOTT RUSSELL ON THE UNION CANAL NEAR EDINBURGH.



The Scott Russell Aqueduct on the Union Canal near Heriot-Watt University, 12 July 1995.

For the technically minded, the aqueduct is 89.3 m long, 4.13m wide, and 1.52m deep.



Soliton on the Scott Russell Aqueduct on the Union Canal near Heriot-Watt University, 12 July 1995.

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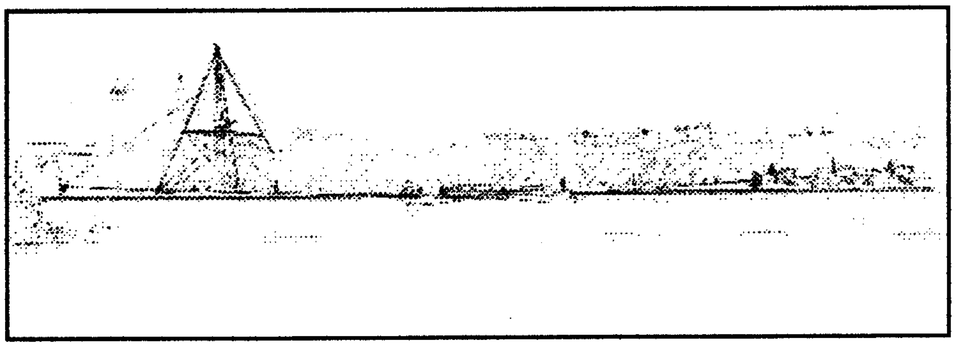
John Scott Russell and the solitary wave

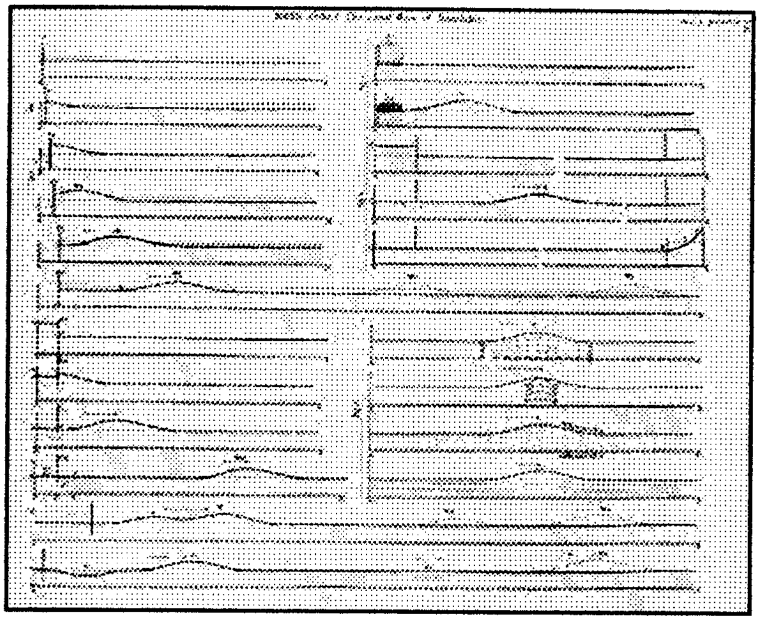
Over one hundred and fifty years ago, while conducting experiments to determine the most efficient design for canal boats, a young Scottish engineer named John Scott Russell (1808-1882) made a remarkable scientific discovery. As he described it in his "Report on Waves": (Report of the fourteenth meeting of the British Association for the Advancement of Science, York, September 1844 (London 1845), pp 311-390, Plates XLVII-LVII).

``I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".
(Cet passage en francais)

This event took place on the Union Canal at Hermiston, very close to the Riccarton campus of Heriot-Watt University, Edinburgh.



Following this discovery, Scott Russell built a 30' wave tank in his back garden and made further important observations of the properties of the solitary wave.



Throughout his life Russell remained convinced that his solitary wave (the ``Wave of Translation") was of fundamental importance, but ninteenth and early twentieth century scientists thought otherwise. His fame has rested on other achievements. To mention some of his many and varied activities, he developed the "wave line" system of hull construction which revolutionized ninteenth century naval architecture, and was awarded the gold medal of the Royal Society of Edinburgh in 1837. He began steam carriage service between Glasgow and Paisley in 1834, and made the first experimental observation of the "Doppler shift" of sound frequency as a train passes. He reorganized the Royal Society of Arts, founded the Institution of Naval Architects and in 1849 was elected Fellow of the Royal Society of London. He designed (with Brunel) the "Great Eastern" and built it; he designed the Vienna Rotunda and helped to design Britain's first armoured warship (the "Warrior"). He developed a curriculum for technical education in Britain, and it has recently become known that he attempted to negotiate peace during the American Civil War.

It was not until the mid 1960's when applied scientists began to use modern digital computers to study nonlinear wave propagation that the soundness of Russell's early ideas began to be appreciated. He viewed the solitary wave as a self-sufficient dynamic entity, a "thing" displaying many properties of a particle. From the modern perspective it is used as a constructive element to formulate the complex dynamical behaviour of wave systems throughout science: from hydrodynamics to nonlinear optics, from plasmas to shock waves, from tornados to the Great Red Spot of Jupiter, from the elementary particles of matter to the elementary particles of thought.

For a more detailed and technical account of the solitary wave, see R K Bullough, *"The Wave" "par excellence", the solitary, progressive great wave of equilibrium of the fluid - an early history of the solitary wave*, in **Solitons**, ed. M Lakshmanan, Springer Series in Nonlinear Dynamics, 1988, 150-281.

-->Solitons home page
-->Re-creation of the soliton, 12 July 1995

Last updated 2 Nov 1998 (JCE)
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Homemade Soliton Model

Alex Kasman

The point of the "homemade soliton model" shown on my homepage is to explain the existence, but NOT the dynamics of solitons. In particular, after the discovery of "solitary non-linear waves" and before the modern understanding of solitons, it was argued that solitary waves would be very RARE. The argument was that their existence required a perfect balance of the distortion from the nonlinear terms and the dispersive terms in the equation, which would "obviously" hardly ever occur. In fact, as we now know, solitons are NOT rare, and the model is intended to show the way in which these two different effects on the wave can balance themselves automatically; i.e. there is a coupling between distortion and dispersion.

The model is very simple: take a long rod, and hang free swinging pedula from the rod at regular intervals. It is important that these pendula can swing around the rod freely, but do not move sideways. Put fixed weights at the end of the rods and connect them with rubber bands which are tight when the pendula all hang straight down. The precise weight and strength of the rubber bands is not important...and that is the point.

If the rotation of the rods is unaffected by friction (you can attempt to approximate this with lubricant) then the motion of the pendula is approximated by the discrete Sine-Gordon equation. The case in which all of the rods are hanging down (so there is not much tension on the rubberbands) is the zero solution. (If you have access to a Macintosh computer then I strongly recommend that you download the program "3D-filmstrip" by R. Palais at Brandeis University. It will show you an animation of an ideal model of this sort along with a description.)

To generate a single kink-soliton, start with the zero solution and take all of the pendula to the left of the center point and pull them over the top of the rod. You will get the one-soliton shape that is shown in the photograph because the rubberbands will pull the pendula near the center point together so that they stand up. Now we see the distortion and the dispersion! Gravity is pulling DOWN on the weights, attempting to make the soliton more narrow and "sharp", but the rubberbands are trying to pull the pendula together and "flatten" it out. The coupling is evident from the following observations: if you take the model from the first floor of a building up to the 20th floor, the strength of the gravitational pull on the model has changed. However, the solitary wave shape does not collapse! Similarly, you could have used stronger weights or weaker rubberbands, but everything will still work. Why? Because the more you pull on the rubberband the more it pulls back. So, it eventually finds an equilibrium point...and you see the solitary wave shape.

To generate a soliton/anti-soliton pair, start with the zero solution and pull JUST the center pendulum over the top of the rod until it is pointing straight down on the other side. You will have two "humps"...if your rod is long enough you can push these humps apart and they will stay where they are. But, if you let them come together they cancel each other and return to the vacuum solution. This is a good model for realizing the creation of an electron/positron pair since the continuous limit of this model, the Sine-Gordon equation, describes an electro-magnetic field under proper assumptions with the solitons playing the role of the particles through "bosonization".

I hope that this description is adequate for your interests. If I have not been clear or if you have any further questions, please let me know.

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Permanent Exhibition

Switch to Japanese

Many Faces of Solitons

Entrance -- KdV equation -- Modified KdV equation -- Sine-Gordon equation

Entrance of This Exhibition

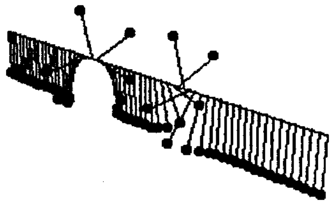
Sine-Gordon equation $u_{tt} - u_{xx} + \sin u = 0$

The term "sine-Gordon equation" is presumably a kind of joke, obviously originating in the name of the "Klein-Gordon equation" in relativistic field theories. As this name shows, this equation is a relativistic nonlinear equation in 1+1 dimensional space-time. Its precursor, just the KdV equation, can be found in the 19th century mathematics (Darboux's work on surface geometry). The sine-Gordon equation, too, has a wide range of applications in physics, not only in relativistic field theories but in solid-state physics, nonlinear optics, etc.

In order to visualize solutions, we use a coupled pendulum model. This is a mechanical model consisting of an elastic wire (or, rather, a straight spring) attached with perpendicular rods in an equal spacing. The rods behave as a pendulum receiving an angular force from the two neighboring rods through twist of the wire. (One can do a cheaper experiment using a rubber string and needles in place of a wire and rods.) In the limit as the spacing of rods tends to zero, this mechanical system approaches the sine-Gordon equation.

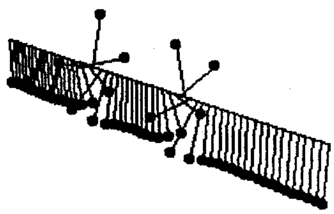
Soliton solutions of the sine-Gordon equation are far richer than those of the KdV and modified KdV equations. Even the 1-solution soliton solution consists of two different cases -- "kink" and "anti-kink". A kink is a solution whose boundary values at the left infinity is 0 and at the right infinity is 2pi; the boundary values of an anti-kink is 0 and -2pi, respectively. More intuitively, the chain of pendulums, in both cases, winds up once around the wire, but in an opposite direction. Similarly, 2-soliton solutions can be classified into several distinct cases -- collision of two "kinks", collision of two "anti-kinks", collision of a "kink" and "anti-kink", and a kind of "bound state" called "breather solution". The last one is rather hard to explain, but the animation will clearly show how it behaves.

- **kink-antikink collision** [mpeg| movie| gif]
  Here is an animation of kink-anti-kink collision. The kink and the anti-kink are given the same speed and proceed in an opposite direction. Note that the twist of the pendulum chain disappears at a moment (t = 0) during collision.



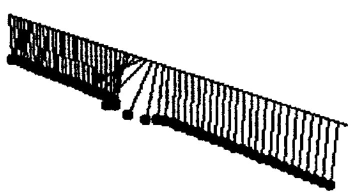
[kink-antikink solution before collision (coupled pendulum model)]
solution: u = 4*Arctan[(p*Sinh[Sqrt[p^2 - 1]*t]) / (Sqrt[p^2 - 1]*Cosh[p*x])].
parameters: p = 2.

- **kink-kink collision** [mpeg| movie| gif]
  Here is an animation of kink-kink collision. Since the kinks are twisted in the same direction, the pendulum chain remains twisted (twice) during collision. It is interesting that the two kinks look like "repelled" rather than collide. (I never imagined this phenomena until I made this animation!)



[kink-kink solution before collision (coupled pendulum model)]
solution: u = 4*Arctan[(Sqrt[p^2 -1]*Sinh[p*x]) / (p*Cosh[Sqrt[p^2 - 1]*t])].
parameters: p = 2.

- **breather solution** [mpeg| movie| gif]
  Here is an animation of the "breather solution". This name originates in the behavior of its profile, which repeats regularly oscillating upwards and downwards, thereby looking like breathing. In the pendulum model, this behavior is nothing but a localized collective oscillation, as you see in the animation.



[breather solution (coupled pendulum model)]
solution: u = 4*Arctan[(p*Sin[Sqrt[p^2 +1]*t]) / (Sqrt[p^2 + 1]*Cosh[p*x])].
parameters: p = 2.

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