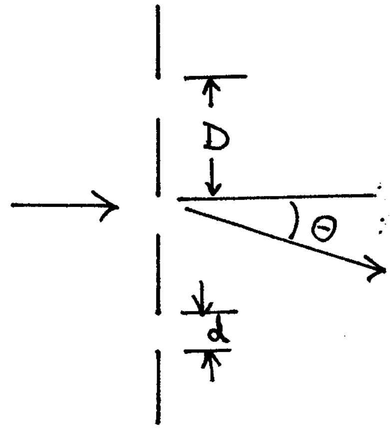
PH 206 PROBLEM SET 9

DUE: TUESDAY DEC. 10, 1996 ; MAXIMUM RECORDED SCORE = 80 POINTS

① MULTIPLE SLIT DIFFRACTION PATTERN



A FLAT SCREEN HAS N INFINITE SLITS EACH OF WIDTH d, SEPARATED BY DISTANCE D. LIGHT OF WAVELENGTH λ IS NORMALLY INCIDENT ON THE SCREEN.

SHOW THAT FOR FRAUNHOFER DIFFRACTION

$$I(\theta) = I_0 \left[\frac{\sin wd}{wd}\right]^2 \left[\frac{\sin NwD}{N \sin wD}\right]^2$$

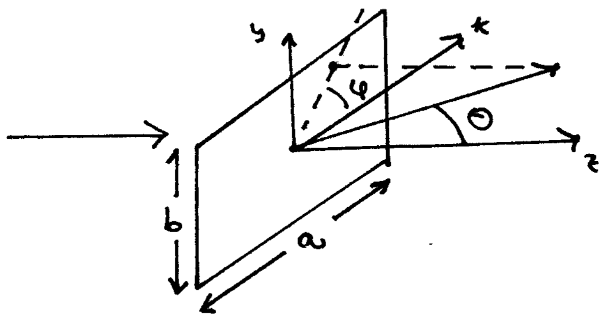
WHERE $w = \frac{\pi}{\lambda} \sin\theta$

SKETCH THIS FOR $N=4$, AND $D = 2d$

SHOW THAT THE ANGLE BETWEEN A PRINCIPAL MAXIMUM AND THE NEAREST MINIMUM IS $\Delta\theta \sim \frac{\lambda}{ND}$

IF THIS 'GRATING' IS USED TO RESOLVE SPECTRAL LINES OF DIFFERENT λ, SHOW THAT THE MAXIMUM 'RESOLVING POWER' IS $\frac{\lambda}{\Delta\lambda} = \frac{ND}{\lambda}$. REMEMBER, $\sin\theta \leq 1$

② RECTANGULAR APERTURE



SHOW THAT THE FRAUNHOFER DIFFRACTION PATTERN IS

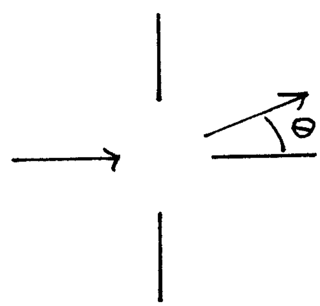
$$I = I_0 \left(\frac{\sin u}{u}\right)^2 \left(\frac{\sin v}{v}\right)^2$$

WHERE $u = \frac{\pi a}{\lambda} \sin\theta \cos\varphi$

$v = \frac{\pi b}{\lambda} \sin\theta \sin\varphi$

SUPPOSING PLANE WAVES NORMALLY INCIDENT ON THE APERTURE.

③ CIRCULAR APERTURE



A PLANE WAVE $\psi = A e^{i(kz - \omega t)}$ IS INCIDENT ON AN OPAQUE SCREEN AT $z=0$, WHICH HAS A CIRCULAR APERTURE OF RADIUS a CENTERED AT $x = y = 0$.

EXPAND THE FRAUNHOFER DIFFRACTION INTEGRAL IN A POWER SERIES AND EVALUATE TERM BY TERM TO SHOW

$$\psi_{OUT} = A \frac{k}{2\pi i} \pi a^2 \sum_{n=0}^{\infty} \frac{(-1)^n (w/2)^{2n}}{n!\,(n+1)!} \qquad \text{WHERE } w = ka \sin\theta$$

THIS TURNS OUT TO BE $\psi_{OUT} = A \frac{k}{2\pi i} \cdot 2\pi a^2 \frac{J_1(w)}{w}$

WHERE J_1 IS THE BESSEL FUNCTION OF ORDER 1.

GIVEN THAT THE FIRST ZERO OF $J_1(w)$ IS AT $w = 3.83$, WHAT IS THE MINIMUM ANGLE BETWEEN SOURCES WHICH CAN BE RESOLVED, IF VIEWED THRU THE CIRCULAR APERTURE (A LENS?)

ANS: $\Delta\theta_{MIN} = 1.22 \frac{\lambda}{D}$ ACCORDING TO THE RAYLEIGH CRITERION.

(4) PARTIALLY OPAQUE DISK SUPPOSE A CIRCULAR DISK OF RADIUS a ABSORBS ONLY A FRACTION η OF THE AMPLITUDE OF INCIDENT RADIATION.

CONSIDER FRAUNHOFER DIFFRACTION OF NORMALLY INCIDENT PLANE WAVES OF WAVE NUMBER k.

CALCULATE $\sigma_{SCATTER}$, $\sigma_{ABSORBTION}$, σ_{TOTAL} AND $f(0)$

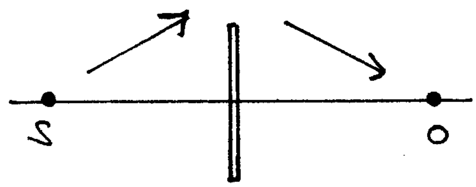
[$f(\theta)$ = RELATIVE SCATTERING AMPLITUDE AT ANGLE θ]

WHEN CALCULATION $\sigma_{ABSORBTION}$, NOTE THAT THE ABSORBED INTENSITY IS THAT WHICH IS NOT TRANSMITTED!

SHOW $\sigma_{TOT} = \frac{4\pi}{k} \operatorname{Im} f(0) = 2\pi a^2 \eta$ SO THAT THE 'OPTICAL THEOREM' HOLDS HERE.

(5) FRESNEL DIFFRACTION BY AN OPAQUE CIRCULAR DISK

A SOURCE AND OBSERVER BOTH LIE ON THE AXIS OF AN OPAQUE CIRCULAR DISK OF RADIUS a.



QUALITATIVELY, DO YOU EXPECT THE OBSERVER TO SEE MORE OR LESS LIGHT THAN IF (S)HE WERE SLIGHTLY OFF AXIS? AS THE OBSERVER MOVES AWAY FROM THE DISK, BUT REMAINING ON THE AXIS, WILL THE INTENSITY INCREASE OR DECREASE?

THINK OF THE CORNU SPIRAL.

TO BE MORE QUANTITATIVE, SUPPOSE BOTH S AND O ARE DISTANCE R FROM THE DISK. INCLUDE THE OBLIQUITY FACTOR

$$\frac{\cos\theta_s + \cos\theta_0}{2}$$

IN THE FRESNEL DIFFRACTION INTEGRAL. SHOW BY APPROPRIATE MANIPULATION OF THE INTEGRAL INTO A POWER SERIES THAT

$$I(R) \sim \frac{I_0}{4}\,\frac{R^2}{R^2+a^2} + O\left(\frac{1}{R}\right)$$

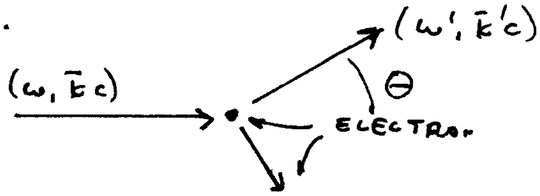
WHERE I_0 IS THE INTENSITY JUST AT THE EDGE OF THE DISK.

(6) THE COMPTON EFFECT IN 1905 EINSTEIN SUGGESTED THAT LIGHT CAN BE THOUGHT OF AS CONSISTING OF 'QUANTA' OF ENERGY $E = \hbar\omega$, MOMENTUM $\bar{p} = \hbar\bar{k}$, WHERE $\hbar = h/2\pi$ = PLANCK'S CONSTANT. THUS $\hbar c$ x WAVE 4-VECTOR

$$= (\hbar\omega, \hbar\bar{k}c) = (E, \bar{p}c)$$

= ENERGY-MOMENTUM 4-VECTOR.

A STRIKING EXAMPLE OF THE CONSEQUENCE OF THIS HYPOTHESIS WAS GIVEN BY A.H. COMPTON IN 1923. THE SCATTERING OF VERY SHORT WAVELENGTH LIGHT OFF FREE ELECTRONS.



SUPPOSE A SINGLE QUANTUM OF LIGHT WITH 4-VECTOR $\hbar(\omega, \bar{k}c)$ STRIKES AN ELECTRON AT REST, AND SCATTERS THRU ANGLE θ. APPLY ENERGY AND MOMENTUM CONSERVATION TO THE 4-VECTORS INVOLVED TO SHOW THAT

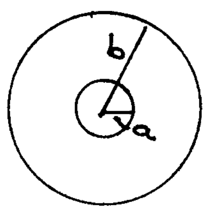
$$\omega' = \omega \frac{1}{1 + \frac{\hbar\omega}{mc^2}(1-\cos\theta)}$$

(m = ELECTRON REST MASS)

NOTE THAT $\omega' < \omega$ WHICH IS NOT EXPECTED IN CLASSICAL ANALYSIS OF LIGHT SCATTERING.

IT WAS THE DETAILED EXPERIMENTAL VERIFICATION OF THE COMPTON FORMULA WHICH FINALLY CONVINCED MOST PHYSICISTS (INCLUDING NIELS BOHR!) TO TAKE THE LIGHT QUANTUM CONCEPT SERIOUSLY.

(7) A COAXIAL CABLE HAS INNER CONDUCTOR OF RADIUS a, OUTER CONDUCTOR OF RADIUS b. A FIXED D.C. VOLTAGE IS MAINTAINED BETWEEN THE TWO CONDUCTORS, AND CURRENT I FLOWS IN THE CENTER, CURRENT $-I$ IN THE OUTER CONDUCTOR.



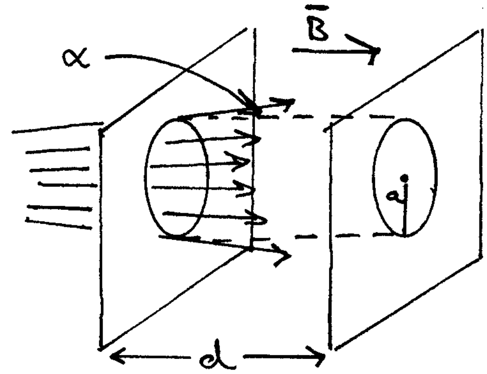
ELECTRONS LEAVE THE CENTRAL CONDUCTOR WITH NEGLIGIBLE VELOCITY (THERMIONIC EMISSION) AND ARE ATTRACTED TO THE OUTER CONDUCTOR. SHOW THAT THE ELECTRONS CAN NEVER REACH THE OUTER CONDUCTOR IF

$$I > \frac{c}{2e\ln b/a}\sqrt{e^2V^2 + 2eVmc^2}$$

(e = ELECTRON CHARGE, m = ELECTRON MASS.)

HINT: THIS PROBLEM IS A BIT TRICKY. IT HELPS TO TRANSFORM TO A FRAME IN WHICH ONE OF $\bar{E}$ OR $\bar{B}$ VANISHES .... SKETCH THE ELECTRON'S TRAJECTORY.

(8) MAGNETIC LENS



A BEAM OF CHARGED PARTICLES ALL OF THE SAME ENERGY PASSES THRU A HOLE OF RADIUS a IN A FLAT SCREEN. THE BEAM DIVERGES SLIGHTLY, WITH ANGLE α AS THE MAXIMUM DEPARTURE FROM THE NORMAL.

WE WISH TO SQUEEZE ALL OF THIS BEAM THRU ANOTHER CIRCULAR HOLE OF RADIUS a AT DISTANCE d AWAY FROM THE FIRST HOLE. (THE TWO HOLES ARE CONCENTRIC). TO DO THIS A UNIFORM MAGNETIC FIELD $\bar{B}$ IS APPLIED ALONG THE AXIS OF THE BEAM.

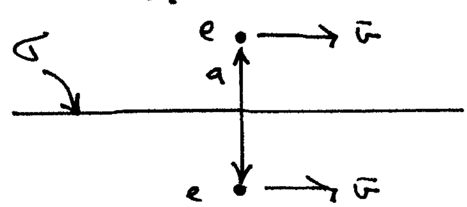
SHOW THAT FOR THIS, $B_{MIN} \approx \frac{\pi pc}{ed}\left(1 + \frac{2\alpha d}{\pi^2 a}\right)$ IF $\alpha \ll 1$

p = MOMENTUM.

IF B IS INCREASED FROM B_{MIN} THERE COMES A POINT WHEN THE 'FOCUSSING' NO LONGER WORKS. WHAT IS THIS B_{MAX}? (INCREASING B STILL FURTHER, ONE FINDS ADDITIONAL REGIMES IN WHICH THE FOCUSSING WORKS...)

(9)

TWO ELECTRONS WITH EQUAL VELOCITIES $\bar{v}$ MOVE SIDE BY SIDE A DISTANCE a APART. MIDWAY BETWEEN THEM IS AN INFINITE SHEET OF FIXED POSITIVE CHARGE WITH SURFACE CHARGE DENSITY σ. WHAT MUST σ BE IN ORDER THAT THE ELECTRONS MAINTAIN CONSTANT SEPARATION a?



SOLVE THIS PROBLEM BOTH IN THE REST FRAME OF THE SHEET, AND IN THE ELECTRON'S REST FRAME.

Ph 206 SET 9

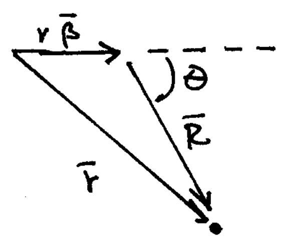
(10) OBTAIN THE ELECTRIC AND MAGNETIC FIELDS OF A CHARGE e MOVING WITH UNIFORM VELOCITY $\bar{v}$ BY A LORENTZ TRANSFORMATION.

SHOW $$\bar{E} = \frac{e\,(\bar{r} - r\bar{\beta})}{\gamma^2 (r - \bar{r}\cdot\bar{\beta})^3} = \frac{e\,\bar{R}}{\gamma^2 R^3 \,(1-\beta^2 \sin^2\theta)^{3/2}}$$

AND $\bar{B} = \bar{\beta} \times \bar{E}$ WHERE $\bar{\beta} = \frac{\bar{v}}{c}$ $\quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$

$\bar{r}$ = VECTOR FROM THE RETARDED POSITION

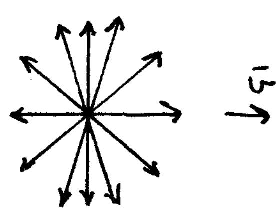
$\bar{R}$ = VECTOR FROM THE PRESENT POSITION



THUS $\bar{E}$ IS RADIAL WITH RESPECT TO THE CHARGES PRESENT POSITION.

$\bar{E}(\theta)$ IS A MINIMUM AT $\theta = 0$, A MAX AT $\theta = 90°$

THE LINES OF $\bar{E}$ ARE "SQUEEZED" TOWARDS THE PLANE $\theta = 90°$



$$|\bar{E}(0°)| = \frac{e}{\gamma^2 R^2} < \frac{e}{R^2}$$

HOW CAN THIS BE CONSISTENT WITH THE FIELD TRANSFORMATION

$$E'_{\parallel} = E_{\parallel} \; ?$$

(11) Laser Acceleration? Can a focussed laser beam accelerate an isolated free charge? In general, no!

Consider a particle of charge q that moves with velocity $\approx c$ along the line $z = x\,\theta$ for a small angle θ, passing thru the origin at $t = 0$.
A Gaussian laser beam propagates along the $+z$ axis, with its focus at $z = 0$. Use the expansion of the Gaussian field to first order in parameter $\theta_0 = w_0/z_0$ to show that the force on the charge can be derived from a potential

$$U = -iq\,\frac{\theta_0^2}{\theta}\,E_0\,e^{-if\varsigma\,\theta^2/\theta_0^2}$$

where $\varsigma = z/z_0$, $f = i/i-\varsigma$. Then since $U(\varsigma=-\infty) = U(\varsigma=+\infty)$ the charge extracts no net energy from the laser beam as it passes across the latter $\Rightarrow$ no net acceleration.

[Comment: 2/98. The above argument is perhaps too idealized, and some acceleration is possible...]