Princeton University

Ph501

Electrodynamics

Problem Set 5

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1. a) A charged particle moves in a plane perpendicular to a uniform magnetic field **B**. Show that if **B** changes slowly with time, the magnetic moment produced by the orbital motion of the charge remains constant. Show also that the magnetic flux through the orbit, $\Phi = \pi r^2 B$ is constant. These results are sometimes given the fancy name of **adiabatic invariants** of the motion.

   b) **The Magnetic Mirror**. Suppose instead, that the magnetic field is slightly non-uniform such that B_z increases with z. Then, if the charged particle has a small velocity in the z direction, it slowly moves into a stronger field. Again, we would expect the flux through the orbit to remain constant, which means that the orbital radius must decrease and the orbital velocity must increase. However, magnetic fields which are constant in *time* cannot change the magnitude of the velocity, therefore v_z must decrease. If B_z increases enough, v_z will go to zero, and the particle is "trapped" by the magnetic field. Write

$$v^2 = v_z^2 + v_\perp^2 = v_0^2 \; , \tag{1}$$

   where $v_\perp$ is the orbital velocity and v_0 is constant. Use the result of part a) to show that

$$v_z^2(z) \approx v_0^2 - v_\perp^2(0)\frac{B_z(z)}{B_z(0)} \; . \tag{2}$$

2. If one pitches a penny into a large magnet, eddy currents are induced in the penny, and their interaction with the magnetic field results in a repulsive force, according to Lenz' law. Estimate the minimum velocity needed for a penny to enter a long, 1-T solenoid magnet whose diameter is 10 cm.

   You may suppose that the penny moves so that its axis always coincides with that of the magnet, and that gravity may be ignored. The speed of the penny is low enough that the magnetic field caused by the eddy currents may be neglected compared to that of the solenoid. Equivalently, you may assume that the magnetic diffusion time is small.

3. a) **Diamagnetism**. We consider a model of an atom in which the distance r from the electron to the nucleus is somehow fixed, but the electron is free to orbit the nucleus. Then, if a field $\mathbf{B}$ is applied to the atom, an E.M.F. is induced around the orbit, while $\mathbf{B}$ is changing, which generates a magnetic dipole moment $\mathbf{m}$ via the resulting motion of the electron. Show that

$$\mathbf{m} = -\frac{e^2 r^2}{4mc^2}\mathbf{B} \ , \tag{3}$$

where e and m are the charge and mass of an electron, respectively. In bulk matter, with n atoms per unit volume, the magnetization $\mathbf{M}$ is then $\mathbf{M} = n\mathbf{m}$.

The magnetic susceptibility is defined by

$$\mathbf{M} = \chi_M \mathbf{H}. \tag{4}$$

Since $\mathbf{B} = \mu\mathbf{H}$, and also $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M} = (1 + 4\pi\chi_M)\mathbf{H}$, we see that the diamagnetic permeability obeys $\mu < 1$. Calculate $\chi_M = (\mu - 1)/4\pi$ for hydrogen gas at S.T.P. and compare with the measured value of -2.24×10^{-9}.

b) In materials where $\mathbf{B} = \mu\mathbf{H}$, we claim that the magnetic energy is

$$U_{\rm mag} = \frac{1}{8\pi}\int \mathbf{B}\cdot\mathbf{H}\ d\mathrm{Vol} = \frac{1}{8\pi}\int B^2\ d\mathrm{Vol} - \frac{1}{2}\int \mathbf{B}\cdot\mathbf{M}\ d\mathrm{Vol}\ . \tag{5}$$

Use your analysis from part a) to show that the last term is just the kinetic energy of the electron's motion induced by the field $\mathbf{B}$.

4. a) A **flip coil** is a practical device for measuring magnetic fields. A coil whose axis is the z-axis is flipped by 180° about the x-axis. The coil leads are connected to a charge integrator. Show that charge

$$Q = \frac{2\Phi}{R} \tag{6}$$

is collected in the flip, where Φ is the magnetic flux through the loop before (and after) flipping and R is the resistance of the integrator (plus coil).

b) A fancy flip coil is made by winding wire on the surface of a sphere such that the turns are distributed according to

$$dN \propto \sin\theta \; d\theta \; . \tag{7}$$

(Recall prob. of set 4.) All turns are parallel to the x-y plane. For this coil, show that

$$\Phi \propto \int B_z \; d\text{Vol}, \tag{8}$$

the integration being over the interior of the sphere.

c) The field component $B_z(r, \theta, \varphi)$ obeys $\nabla^2 B_z = 0$ inside the sphere, and so may be expanded in a series of Legendre functions. However, B_z is not necessarily azimuthally symmetric, so a slight generalization must be made:

$$B_z = \sum_{m,n} A_{m,n} r^n P_n^m(\cos\theta) e^{\pm im\varphi} \; , \tag{9}$$

where n and m are integers, and the P_n^m are the associated Legendre polynomials. Note that $P_n^0(\cos\theta) = P_n(\cos\theta)$, the ordinary Legendre polynomials. Using this, show that $\Phi \propto B_z(0, 0, 0)$, so that the $\sin\theta$ flip coil measures B_z at the center of the sphere, no matter how $\mathbf{B}$ varies over the sphere!

d) A $\sin\theta$ coil is hard to build. Suppose we try to make do with a simple cylindrical coil of radius a and height h. Show that if $h = \sqrt{3}a$, all effects of the first, second and third derivatives of the field vanish. With such a coil, accuracies of 1 in 10^4 may be achieved. Hint: Expand B_z in rectangular coordinates and note that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, $\boldsymbol{\nabla} \times \mathbf{B} = 0$ and hence $\nabla^2 \mathbf{B} = 0$.

5. A cylinder of dielectric constant ε rotates with constant angular velocity ω about its axis. A uniform magnetic field $\mathbf{B}$ is parallel to the axis, in the same sense as $\vec{\omega}$. Find the resulting dielectric polarization in the cylinder and the surface and volume charge densities, neglecting terms of order $(\omega a/c)^2$, where a is the radius of the cylinder.

   Answer:

$$\mathbf{P} = \frac{\varepsilon - 1}{4\pi c\varepsilon}\omega B\mathbf{r} \tag{10}$$

   where $\mathbf{r}$ is the radial vector out from the axis.

   This problem can be conveniently analyzed by starting in the rotating frame. Consider also the electric displacement $\mathbf{D}$.

6. a) Show that the self- and mutual inductances of two circuits obey

$$L_{11}L_{22} \geq L_{12}^2 \qquad (11)$$

by considering the magnetic energy

$$U = \frac{1}{2}L_{11}I_1^2 + \frac{1}{2}L_{22}I_2^2 + L_{12}I_1I_2. \qquad (12)$$

b) A toroidal coil of N turns has a circular cross-section of radius a; the central radius of the coil is $b > a$. Show that the self-inductance is

$$L_{11} = \frac{8N^2}{c^2}(b - \sqrt{b^2 - a^2}) \sin^{-1} \frac{1}{\sqrt{1 + \frac{3b^2}{4a^2}}}. \qquad (13)$$

c) A second circuit in the form of a single loop of radius $> a$ links the toroid; the plane of the second circuit is the same as that of one of the turns of the toroid, and that turn is entirely inside the new circuit. Calculate the mutual inductance L_{12} between the toroid and the new circuit, and show that relation (11) is obeyed in this example.

7. a) A coaxial cable consists of a center wire of radius a surrounded by a thin conducting sheath of radius $b > a$. The region $a < r < b$ is vacuum. Consider a circuit formed by joining the two conductors at $\pm\infty$ to show that the self inductance per unit length is

$$L = \frac{2}{c^2}\left(\frac{1}{4} + \ln\frac{b}{a}\right) . \tag{14}$$

Assume the current is distributed uniformly within the center wire.

b) Suppose the axis of the sheath is a distance ϵ from the axis of the center wire. Calculate the self inductance accurate to terms in $(\epsilon/b)^2$.

8. a) A long cylinder of radius a has uniform magnetization $\mathbf{M}$ perpendicular to its axis. Find the magnetic fields $\mathbf{B}$ and $\mathbf{H}$ everywhere.

   Let $\hat{\mathbf{z}}$ be the axis of the cylinder and $\hat{\mathbf{x}}$ the direction of the magnetization.

   b) Suppose the cylinder is given a uniform velocity, $\mathbf{v} = v\hat{\mathbf{z}}$, along its axis. Find the resulting charge density and electric field everywhere. You may ignore effects of order $(v/c)^2$. You can check your result by noting that the Lorentz force on a charge at rest with respect to the cylinder should vanish.

9. An iron ring has a circular cross section of radius a, and average radius $b \gg a$. However, the ring has a narrow gap from azimuth $\theta = 0$ to $h/b \ll 1$; the gap width is w. A toroidal winding of N turns wraps around the ring.

   Calculate the stored magnetic energy as a function of the current I in the windings and the gap width w in a regime where the permeability of the iron is very large. Calculate the force needed to keep the gap from closing.

   Suppose the field in the gap were 15,000 Gauss, near the maximum that is readily achieved in an iron core magnet. Express the force/area that tends to close the gap in terms of atmospheric pressure.

10. Discuss the surface charges and flow of field energy in a cylindrical wire of radius a of conductivity σ that carries current I distributed uniformly within the wire.

For definiteness, assume the current returns in a hollow conducting cylinder of inner radius b and very large outer radius. Then, the current density $\mathbf{J}$ and electric field $\mathbf{E}$ are vanishingly small in the outer conductor, whose constant electrical potential may be taken as zero.

Steps in the discussion: Find the magnetic field $\mathbf{B}$ everywhere. Find the electric potential $\phi(r, \theta, z)$ and electric field $\mathbf{E}$ first for $r < a$, and then for $a < r < b$. Define $\phi(0, 0, 0) = 0$ at the center of the wire.

Answer:

$$\phi(a < r < b) = -\frac{Iz}{\pi a^2 \sigma} \frac{\ln(r/b)}{\ln(a/b)} . \tag{15}$$

Find the surface charge density at $r = a$ which is needed to shape the electric field inside the wire to be along $\mathbf{z}$. When the current first begins to flow, the electric field is not yet uniform and free charge heads for the surface of the wire until the desired static surface charge distribution is obtained.

A length l of the wire has resistance $R = l/\pi a^2 \sigma$ and consumes power at the rate $I^2 R$. Show that the Poynting vector $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{B}$ at the surface of the wire provides this power. Thus, according to Poynting, the power flows down the air gap and into the side of the wire.

As Sommerfeld says, "Electromagnetic energy is transported without losses only in nonconductors. 'Conductors' are nonconductors of energy, which is dissipated in Joule heating."

An alternative calculation of the surface charge density σ may be instructive. Consider first the question of how a tube of radius a of uniform axial electric field could be created in the absence of the wire. A capacitor consisting of a pair of circular plates of radius a has a very nonuniform field between the plates as their separation becomes large. We want the equipotentials to be perpendicular to the axis, and uniformly spaced, which could be approximately achieved by adding a set of conduting rings of radius a, spaced uniformly along the axis with potentials that vary linearly between the two end plates. The charge on a ring would be given by $Q = CV$, where C is he capacitance of a ring, and V is the desired potential of the ring.

The current-carrying wire is a kind of continuum limit of the above procedure. The desired potential inside the wire is $\phi(z) = -IRz$. For the coaxial geometry of the present problem, calculate the capacitance per unit length between the wire of radius a and the return conductor of radius b. Then calculate the charge per unit length, and the surface charge density, on the wire via $Q(z) = CV(z)$.

11. Consider an air-core transformer in the form of two coaxial cylinders of length l and radii $r_1 < r_2 \ll l$. Each cylinder is wrapped with N_i turns, and the total resistance of coil i is R_i.

    a) Deduce the currents $I_1(t)$ and $I_2(t)$ in the coils when the primary coil 1 is driven by voltage $V_1(t) = V_0 \cos \omega t$. First, evaluate the self and mutual inductances, L_1, L_2 and M, and then solve the coupled circuit equations.

    Calculate the time-average power dissipated in coil 2.

    b) Evaluate the Poynting vector $\mathbf{S}$ to show that its time average is nonvanishing only for $r_1 < r < r_2$, and that the total Poynting flux $2\pi rl \langle S_r \rangle$ is just the power dissipated in coil 2. What is the direction of $\mathbf{S}$?

    c) Consider coil 2 as the primary driven by voltage $V_2(t) = V_0 \cos \omega t$, and discuss the relation between the Poynting vector and the power dissipated in coil 1.

12. **Feynman Disk Paradox.** Consider a small coil centered on the origin that carries a current which sets up a magnetic dipole moment $\mathbf{m} = m\hat{\mathbf{z}}$. A ring of radius a in the plane $z = 0$ has charge Q distributed uniformly on it. The ring is rigidly attached to the coil, but the assembly is free to rotate about the z axis.

a) Calculate the initial angular momentum $\mathbf{L}_{\rm EM}$ in the electromagnetic field.

Use the multipole expansion for the potential of a ring of charge, pp. 58-59, to show that

$$L_{{\rm EM},z} = \begin{cases} 2mQ/15ca, & r < a, \\ 13mQ/15ca, & r > a. \end{cases} \tag{16}$$

b) Now let the current in the coil decrease to zero. Calculate the field induced at the ring, and the resulting torque to show that

$$L_{{\rm mech},z} = \frac{mQ}{ca}\,, \tag{17}$$

once the moment m has vanished.

Hint: Since magnetic field lines always form loops, the flux through the ring is equal and opposite to that across the plane $z = 0$ outside the ring.

For yet another version of this problem, see
http://physics.princeton.edu/~mcdonald/examples/feynman_cylinder.pdf

13. Consider particle with charge e and momentum $\mathbf{P} = \mathbf{P}_z + \mathbf{P}_\perp$ ($P_\perp \neq 0$) that is moving on average in the z direction inside a solenoid magnet whose symmetry axis is the z axis and whose magnetic field strength is B_z. Inside the solenoid, the particle's trajectory is a helix of radius R, whose center is at distance R_0 from the magnet axis.

    The longitudinal momentum P_z is so large that when the particle reaches the end of the solenoid coil, it exits the field with little change in its transverse coordinates. This behavior is far from the adiabatic limit (*c.f.* Prob. 1) in which the trajectory spirals around a field line.

    When the particle exits the solenoid, the radial component of the magnetic "fringe" field exerts azimuthal forces on the particle, and, in general, leaves it with a nonzero azimuthal momentum, P_ϕ. Deduce a condition on the motion of the particle when within the solenoid, *i.e.*, on R, R_0, P_z, $P_\perp$, and B_z, such that the azimuthal momentum vanishes as the particle leaves the magnetic field region. Your result should be independent of the azimuthal phase of the trajectory when it reaches the end of the solenoid coil.

    Hint: Consider the canonical momentum and/or angular momentum.

Solutions

1. a) Since the particle moves in the plane perpendicular to the magnetic field , the velocity **v**, the field **B** and the force **F** on the particle of mass m and charge q are mutually orthogonal. The orbit is a circle of radius r related by

$$F = \frac{mv^2}{r} = \frac{qvB}{c}\,, \tag{18}$$

so long as B varies sufficiently slowly. Then,

$$r = \frac{mcv}{qB} \tag{19}$$

and the magnetic moment due to this orbit is of magnitude

$$\mu = \frac{\pi r^2 I}{c} = \frac{\pi r^2}{c}\frac{qv}{2\pi r} = \frac{qrv}{2c} = \frac{mv^2}{2B}\,. \tag{20}$$

The vector $\vec{\mu}$ is in the opposite direction to **B**, which can be considered as an example of Lenz' Law.

If the field B varies with time, then an electric field is induced around the particle's orbit as given by Faraday's Law:

$$\oint \mathbf{E}\cdot d\mathbf{l} = 2\pi r E = -\frac{1}{c}\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{S} = -\frac{\pi r^2 \dot{B}}{c}. \tag{21}$$

Thus,

$$E = \frac{r\dot{B}}{2c} = \frac{mv\dot{B}}{2qB}\,, \tag{22}$$

with **E** in the same direction as **v**. That is, if the magnetic field increases, the electric field causes the particle to accelerate,

$$\dot{v} = \frac{qE}{m} = \frac{v\dot{B}}{2B}\,. \tag{23}$$

The solution to this is

$$v \propto \sqrt{B}\,, \tag{24}$$

so that v^2/B is constant, and hence the magnetic moment (20) is constant. The flux linked by the orbit,

$$\Phi = \pi r^2 B = \frac{\pi m^2 c^2 v^2}{q^2 B}\,, \tag{25}$$

is also constant.

b) Suppose now that the magnetic field is constant in time, but varies in space. For example, consider a field that has azimuthal symmetry about the z axis, and B_z increasing with z. A charged particle with nonzero v_z, moves along a kind of helix in this field.

The magnetic field at the z coordinate of the particle then varies as

$$\frac{dB_z(z(t))}{dt} = \frac{dB_z}{dz} v_z. \tag{26}$$

If this change is slow, the analysis of part a) holds, and the particle's motion varies so as to keep $v_\perp^2(z)/B_z(z)$ constant, so that

$$v_\perp^2(z) \approx v_\perp^2(0)\frac{B_z(z)}{B_z(0)} . \tag{27}$$

In writing this, we recall from prob. 6, set 4 that for magnetic fields with azimuthal symmetry, $B_z(r,z) \approx B_z(0,z) - r^2 B_z''(0,z)/4 + ...$, and we ignore the radial dependence for orbits with small r.

Of course, $v^2 = v_z^2 + v_\perp^2$ remains constant as well, so we have

$$v_z^2(z) = v_0^2 - v_\perp^2(0)\frac{B_z(z)}{B_z(0)} . \tag{28}$$

The particle stops moving forward in z at the plane where

$$B_z(z) = \frac{v_0^2}{v_\perp^2(0)} B_z(0) > B(0) . \tag{29}$$

Although the particle has $v_z = 0$ at this plane, its $v_\perp$ now equals $v(0)$, so still there is a large Lorentz force in the $-z$ direction, and the particle spirals its way back down the z axis. Hence the term "magnetic mirror".

A field configuration in which the axial field strength increases with $|z|$ can trap charged particles near the origin. This is no contradiction to Earnshaw's theorem, as a "magnetic bottle" has no static equilibrium point, but relies on electrodynamics to trap particles with constant, nonzero velocity.

2. The penny has radius a and thickness Δz. For the motion as stated in the problem, the eddy current will flow in concentric rings about the center of the disk. Therefore, we first examine a ring of radius r and radial extent Δr.

   The magnetic flux through the ring at position z is

$$\Phi \approx \pi r^2 B_z(0, z), \tag{30}$$

   whose time rate of change is

$$\dot{\Phi} = \pi r^2 \dot{B}_z = \pi r^2 B_z' v, \tag{31}$$

   where $\dot{}$ indicates differentiation with respect to time, $'$ is differentiation with respect to z, B_z stands for $B_z(0, z)$, and v is the velocity of the center of mass of the ring.

   The penny has electrical conductivity σ. Its resistance to currents around the ring is

$$R = \frac{2\pi r}{\sigma \Delta r \Delta z}, \tag{32}$$

   so the (absolute value of the) induced current is

$$I = \frac{\mathcal{E}}{R} = \frac{\dot{\Phi}}{cR} = \frac{\sigma r B_z' v \Delta r \Delta z}{2c}, \tag{33}$$

   using Faraday's law.

   The azimuthal eddy current interacts with the radial component of the magnetic field to produce the axial retarding force. Close to the magnetic axis, we estimate the radial field in term of the axial field according to

$$B_r(r, z) \approx r \frac{\partial B_r(0, z)}{\partial r} = -\frac{r}{2} \frac{\partial B_z(0, z)}{\partial z} \equiv -\frac{r B_z'}{2}, \tag{34}$$

   as can be deduced from the Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, noting that on the magnetic axis $\partial B_r/\partial r = \partial B_x/\partial x = \partial B_y/\partial y$. Then, the retarding force on the ring is

$$\Delta F_z = \frac{2\pi r B_r I}{c} = -\frac{\pi \sigma r^2 B_r B_z' v \Delta r \Delta z}{c^2} \approx -\frac{\pi \sigma r^3 (B_z')^2 v \Delta r \Delta z}{2c^2}. \tag{35}$$

   Alternatively, we note that the kinetic energy lost by the penny appears as Joule heating. Hence, for the ring analyzed above,

$$v \Delta F_z = \frac{dU}{dt} = -I^2 R = -\frac{\pi \sigma r^3 (B_z')^2 v^2 \Delta r \Delta z}{2c^2}, \tag{36}$$

   using eqs. (32) and (33), which agains leads to eq. (35).

   The equation of motion of the ring is

$$dF_z = -\frac{\pi \sigma r^3 (B_z')^2 v \Delta r \Delta z}{2c^2} = m\dot{v} = 2\pi \rho r \Delta r \Delta z \, v' v, \tag{37}$$

where ρ is the mass density of the metal. We integrate this equation with respect to radius to find

$$-\frac{\pi\sigma a^4 (B_z')^2 v \Delta z}{8c^2} = \pi\rho a^2 \Delta z\, v' v, \tag{38}$$

After dividing out the common factor $\pi a^2 \Delta z\, v$, we find

$$v' = -\frac{\sigma a^2 (B_z')^2}{8\rho c^2}\,. \tag{39}$$

For an estimate, we note that the peak gradient of the axial field of a solenoid of diameter D is about B_0/D, and the gradient is significant over a region $\Delta z \approx D$. Hence, on entering a solenoid the jet velocity is reduced by

$$\Delta v \approx \frac{\sigma a^2 B_0^2}{8c^2 \rho D}\,. \tag{40}$$

The penny must have initial velocity $v_0 > \Delta v$ to enter the magnet.

A copper penny has $a \approx 1$ cm, density $\rho \approx 10$ g/cm^3, electrical resistivity $\approx 10^{-6}$ Ω-cm, and therefore conductivity $\sigma \approx 9 \times 10^{17}$ Gaussian units. The minimum velocity to enter a 1-T $= 10^4$-G magnet with diameter $D = 10$ cm is then,

$$v_{\rm min} \approx \frac{9 \times 10^{17} \cdot (1)^2 \cdot (10^4)^2}{8 \cdot (3 \times 10^{10})^2 \cdot 10 \cdot 10} \approx 125 \text{ cm/s}. \tag{41}$$

The case of a sphere rather than a disk has been presented in J. Walker and W.H. Wells, *Drag Force on a Conducting Spherical Drop in a Nonuniform Magnetic Field*, ORNL/TN-6976 (Sept. 1979).

3. a) Assume that the electron of an atom is initially stationary and that the vector $\mathbf{r}$ between the electron and the nucleus is at a right angle to the direction of the increasing magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. As the magnetic field is applied, an electric field is induced around a loop of radius r according to

$$E = \frac{r\dot{B}}{2c}\,, \tag{42}$$

which, as Lenz' law decrees, will accelerate the electron to velocity v, given by

$$v = \int \dot{v}\; dt = \int \frac{eE}{m}\; dt = \int \frac{er\dot{B}}{2mc}\; dt = \frac{erB}{2mc}\,, \tag{43}$$

so that its magnetic dipole moment opposes the magnetic field. From the next to last equality in (20), we have

$$\mathbf{m} = -\frac{evr}{2c}\hat{\mathbf{z}} = -\frac{e^2r^2B}{4mc^2}\hat{\mathbf{z}} = -\frac{r_0r^2B}{4}\hat{\mathbf{z}}\;, \tag{44}$$

where $r_0 = e^2/mc^2 = 2.8 \times 10^{-13}$ cm is the classical electron radius.

With n atoms per unit volume, the magnetization is

$$\mathbf{M} = n\mathbf{m} = -\frac{nr_0r^2}{4}\mathbf{B}. \tag{45}$$

The magnetic susceptibility χ_M is related by

$$\mathbf{M} \equiv \chi_M\mathbf{H} = \frac{\chi_M}{1 + 4\pi\chi_M}\mathbf{B} \approx \chi_M B\;, \tag{46}$$

so

$$\chi_M \approx -\frac{nr_0r^2}{4}\;. \tag{47}$$

Hence, the permeability, $\mu = 1 + 4\pi\chi_M$ is less than one. For hydrogen, r is the Bohr radius, $a_0 = r_0/\alpha^2$, and, at S.T.P., $n \approx 5.4 \times 10^{19}/\mathrm{cm}^3$, so $\chi_M \approx -1.1 \times 10^{-10}$, which is of the same order of magnitude as the stated value of -2.24×10^{-9}.

b) In a volume V, there are $N = nV$ electrons, and their kinetic energy is

$$T = N\frac{1}{2}mv^2 = \frac{nVe^2r^2B^2}{8mc^2} = -\frac{V}{2}\mathbf{M}\cdot\mathbf{B}, \tag{48}$$

using (43) and (45), which is just the second term in the expression (5) for the magnetic energy.

4. a) As the coil flips, the total amount of flux cut by the coil is 2Φ. Then, since the E.M.F. $\mathcal{E}$ generated is proportional to the rate of change of flux, the charge integrated over time is

$$Q = \int I\ dt = \frac{1}{R}\int \mathcal{E}\ dt = \frac{1}{cR}\int \frac{d\Phi}{dt}\ dt = \frac{1}{cR}\int\ d\Phi = \frac{2\Phi}{cR}\ . \tag{49}$$

b) The total magnetic flux through the turns, which are perpendicular to the z axis, is

$$\Phi = \int\int\int \mathbf{B}\cdot(\hat{\mathbf{z}}\ dx\ dy)dN. \tag{50}$$

Since the density of turns obeys $dN \propto \sin\theta\ d\theta$, and $dz = a\sin\theta\ d\theta$, where a the radius of the sphere, we have $dN \propto dz$. Hence, (50) becomes

$$\Phi \propto \int B_z\ dxdydz = \int B_z\ d\text{Vol}. \tag{51}$$

c) Inserting the Legendre series (9) into (51), we have

$$\Phi \propto \int \sum_{m,n} A_{m,n} r^n P_n^m(\cos\theta) e^{\pm im\varphi} r^2\ dr\ d\cos\theta\ d\varphi. \tag{52}$$

The integral over the azimuthal angle is

$$\int_0^{2\pi} e^{\pm im\varphi}\ d\varphi = 2\pi\delta_{m0}. \tag{53}$$

The integral over the polar angle is then

$$\int_{-1}^{1} P_n(\cos\theta)\ d\cos\theta = \int_{-1}^{1} P_n(\cos\theta)P_0(\cos\theta)\ d\cos\theta = 2\delta_{n0}. \tag{54}$$

The radial integral is just

$$\int_0^a r^2\ dr = \frac{a^3}{3}\ . \tag{55}$$

Combining (52-55) then gives

$$\Phi \propto \frac{4\pi a^3}{3} A_{0,0} = B_z(0,0,0)\text{Vol}. \tag{56}$$

d) The total flux linked by the coil is again given by (50), where now the density of windings is $dN = ndz$. Thus,

$$\Phi = n\int B_z\ d\text{Vol}. \tag{57}$$

In a Taylor expansion of B_z in rectangular coordinates about the center of the coil, the integral of odd-order terms will vanish because the cylinder is symmetrical under reflections. Hence, up to third order, the only terms which survive are the zeroth-order term, $\pi a^2 h B_Z(0,0,0)$, and the second-order term,

$$\frac{1}{2}\sum_i \left.\frac{d^2 B_z}{dx_i^2}\right|_0 \int x_i^2\ d\text{Vol}. \tag{58}$$

In current-free regions and static situations,

$$\nabla^2 B_z = \sum_i \frac{d^2 B_z}{dx_i^2} = 0 \ , \tag{59}$$

so (58) will vanish if the three integrals in the sum are equal. The $x_1 = x$ and $x_2 = y$ integrals are automatically equal because of the symmetry of the cylinder, which means that for the term to vanish, we need

$$\int z^2 \ d\text{Vol} = \frac{1}{2} \int (x^2 + y^2) \ d\text{Vol} = \frac{1}{2} \int r^2 \ d\text{Vol} \tag{60}$$

$$\Rightarrow \quad \frac{\pi a^2 h^3}{12} = \frac{\pi a^4 h}{4} \ . \tag{61}$$

Hence, we require that $h = \sqrt{3}a$.

5. The $\mathbf{v}\times\mathbf{B}$ force on an atom in the rotating cylinder is radially outwards, and increasing linearly with radius, so we expect a positive radial polarization.

   We begin our analysis in the rotating frame, in which any polarization charge density is at rest and causes no additional magnetic field. Then, $\mathbf{P}' = \chi\mathbf{E}'$, where $\mathbf{E}'$ and $\mathbf{P}'$ are the electric field and dielectric polarization in the rotating frame. If $v = \omega r \ll c$, then the electric field in the rotating frame is related to lab frame quantities by

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \tag{62}$$

   where $\mathbf{E}$ is the electric field due to the polarization that we have yet to find. Since polarization is charge times distance, in the nonrelativistic limit the polarization is the same in the lab frame and the rotating frame: $\mathbf{P}' = \mathbf{P}$.

   The velocity has magnitude $v = \omega r$, and is in the azimuthal direction. Thus, $\mathbf{v}\times\mathbf{B} = \omega B\mathbf{r}$, so that

$$\mathbf{P} = \chi\left(\mathbf{E} + \frac{\omega B}{c}\mathbf{r}\right) . \tag{63}$$

   There are no free charges, so the electric displacement is zero:

$$\mathbf{D} = 0 = \mathbf{E} + 4\pi\mathbf{P}. \tag{64}$$

   Thus, $\mathbf{E} = -4\pi\mathbf{P}$. Recalling that $\chi = (\varepsilon - 1)/4\pi$, (63) leads to

$$\mathbf{P} = \frac{\varepsilon - 1}{4\pi c\varepsilon}\omega B\mathbf{r}. \tag{65}$$

   The surface charge density is

$$\sigma_{\rm pol} = \mathbf{P}(a)\cdot\hat{\mathbf{r}} = \frac{\varepsilon - 1}{4\pi c\varepsilon}\omega B a, \tag{66}$$

   where a is the radius of the cylinder. As well as this surface charge density, there is a volume charge density,

$$\rho_{\rm pol} = -\boldsymbol{\nabla}\cdot\mathbf{P} = -\frac{1}{r}\frac{\partial r P_r}{\partial r} = -\frac{\varepsilon - 1}{2\pi c\varepsilon}\omega B, \tag{67}$$

   so that the cylinder remains neutral over all.

   Both the surface and volume charge densities are proportional to $v(r)/c$, and are moving at velocity $v(r)$. Hence, the magnetic field created by these charges is of order v^2/c^2, and we neglect it in this analysis.

   This example is perhaps noteworthy in that a nonvanishing, static volume charge density arises in a charge-free, linear dielectric material. In pure electrostatics this cannot happen, since $\mathbf{P} = \chi\mathbf{E}$ together with $\boldsymbol{\nabla}\cdot\mathbf{D} = 0 = \boldsymbol{\nabla}\cdot\mathbf{E} + 4\pi\boldsymbol{\nabla}\cdot\mathbf{P}$ imply that $\rho_{\rm pol} = -\boldsymbol{\nabla}\cdot\mathbf{P} = 0$.

We also offer an iterative solution. The axial magnetic field acts on the rotating molecules to cause a $\mathbf{v} \times \mathbf{B}$ force radially outwards. This can be described by an effective electric field

$$\mathbf{E}_0 = \frac{\omega B}{c}\mathbf{r}. \tag{68}$$

This field causes polarization

$$\mathbf{P}_0 = \chi \mathbf{E}_0 = \chi \frac{\omega B}{c}\mathbf{r}. \tag{69}$$

Associated with this is the uniform volume charge density

$$\rho_0 = -\boldsymbol{\nabla} \cdot \mathbf{P}_0 = -2\chi\omega B. \tag{70}$$

According to Gauss' Law, this charge density sets up a radial electric field

$$\mathbf{E}_1 = 2\pi\rho_0 \mathbf{r} = -4\pi\chi\omega B\mathbf{r}. \tag{71}$$

At the next iteration, the total polarization is

$$\mathbf{P}_1 = \chi(\mathbf{E}_0 + \mathbf{E}_1) = \chi(1 - 4\pi\chi)\frac{\omega B}{c}\mathbf{r}. \tag{72}$$

This causes additional charge density ρ_2, which leads to additional electric field $\mathbf{E}_2$, ... At the nth iteration, the polarization will have the form

$$\mathbf{P}_n = k_n \frac{\omega B}{c}\mathbf{r}. \tag{73}$$

Then,

$$\rho_n = -\boldsymbol{\nabla} \cdot \mathbf{P}_n = -2k_n\omega B, \tag{74}$$

and

$$\mathbf{E}_{n+1} = 2\pi\rho_n \mathbf{r} = -4\pi k_n \omega B\mathbf{r}. \tag{75}$$

The effective electric field at iteration $n+1$ is the sum of $\mathbf{E}_0$ due to the $\mathbf{v} \times \mathbf{B}$ force and $\mathbf{E}_{n+1}$ due to the polarization charge. Thus,

$$\mathbf{P}_{n+1} = \chi(\mathbf{E}_0 + \mathbf{E}_{n+1}) = \chi(1 - \pi k_n)\frac{\omega B}{c}\mathbf{r}. \tag{76}$$

But by definition,

$$\mathbf{P}_{n+1} = k_{n+1}\frac{\omega B}{c}\mathbf{r}. \tag{77}$$

Hence,

$$k_{n+1} = \chi(1 - 4\pi k_n). \tag{78}$$

If this series converges to the value k, then we must have

$$k = \chi(1 - 4\pi k), \tag{79}$$

so that

$$k = \frac{\chi}{1 + 4\pi\chi} = \frac{\varepsilon - 1}{4\pi\varepsilon}, \tag{80}$$

which again gives (10) for the polarization.

6. a) In the expression (12) for the total magnetic energy in the circuits, let

$$I_1 = -\frac{L_{12}}{L_{11}} I_2. \tag{81}$$

The total energy is then

$$U = \frac{1}{2}\left(L_{22} - \frac{L_{12}^2}{L_{11}}\right) I_2^2. \tag{82}$$

Since the magnetic energy U is also given by $\int B^2\, d\mathrm{vol}/8\pi$, it must be non-negative. Hence, the factor in parentheses in (82) must be non-negative, and

$$L_{11}L_{22} \geq L_{12}^2. \tag{83}$$

b) The magnetic field due to current I in a toroid of N windings is azimuthal, and is confined to the interior. Ampère's Law gives the magnitude as

$$B(r) = \frac{2NI}{cr}\ , \tag{84}$$

where r is the perpendicular distance from the axis. The self inductance L_{11} is related by $N\Phi_1/cI$, where Φ_1 is the flux linked by one turn. Thus, for a toroid of central radius b whose cross section is a circle of radius a,

$$L_{11} = \frac{2N^2}{c^2}\int_{-a}^{a} \frac{2dx\sqrt{a^2-x^2}}{x+b}. \tag{85}$$

Substituting $y = x + b$,

$$\begin{aligned} L_{11} &= \frac{4N^2}{c^2}\int_{b-a}^{b+a} \frac{dy\sqrt{-y^2+2by+a^2-b^2}}{y}. \\ &= \frac{4N^2}{c^2}\left[\sqrt{-y^2+2by+a^2-b^2} - b\sin^{-1}\frac{2(b-y)}{\sqrt{4a^2+3b^2}}\right. \\ &\qquad \left. -\sqrt{b^2-a^2}\sin^{-1}\frac{2by+2a^2-2b^2}{\sqrt{4a^2+3b^2}}\right]_{b-a}^{b+a} \\ &= \frac{8N^2}{c^2}(b-\sqrt{b^2-a^2})\sin^{-1}\frac{1}{\sqrt{1+\frac{3b^2}{4a^2}}}\ . \end{aligned} \tag{86}$$

c) To calculate the mutual inductance between the two circuits, we note that the second loop links all the flux of the toroidal field, which we called Φ_1 above. Hence,

$$L_{12} = \frac{\Phi_1}{cI} = \frac{L_{11}}{N}\ . \tag{87}$$

If the second circuit has radius R, and is made of a wire of radius r_0, then its self inductance is

$$L_{22} = \frac{4\pi R}{c^2}\left(\ln\frac{8R}{r_0} - \frac{7}{4}\right), \tag{88}$$

from p. 115b of the Notes. Hence, for the system of loop plus toroid,

$$\frac{L_{11}L_{22}}{L_{12}^2} = \frac{N^2 L_{22}}{L_{11}} = \frac{\pi R\left(\ln\frac{8R}{r_0} - \frac{7}{4}\right)}{2(b - \sqrt{b^2 - a^2})\sin^{-1}\frac{1}{\sqrt{1+\frac{3b^2}{4a^2}}}}. \tag{89}$$

The numerator is smallest when $R = a$, the minimum for which the second loop fully links the toroid. The denominator is largest when $b = a$ and the toroid looks like a donut whose hole has shrunk to zero. Then,

$$\left.\frac{L_{11}L_{22}}{L_{12}^2}\right|_{\min} = \frac{\pi\left(\ln\frac{8a}{r_0} - \frac{7}{4}\right)}{2\sin^{-1}\sqrt{\frac{4}{7}}}. \tag{90}$$

This expression equals unity when $a = 1.06r_0$, *i.e.*, when the second loop is also essentially a donut with no hole. However, the expression (88) for the self inductance of a loop was deduced supposing that $R \gg r_0$. Since the general restriction (11) is satisfied using (88) for any $R > 1.06r_0$, we infer that (88) is still reasonably accurate for R only a few times r_0.

7. a) In cylindrical coordinates (r, θ, z), the magnetic field is azimuthal in a coaxial cable whose axis is the z axis. When current I flows in the cable, whose solid inner conductor has radius a and whose outer conductor is a cylindrical shell of radius b, the field strength follows from Ampère's law as

$$B_\theta(r) = \begin{cases} 2Ir/a^2c, & r \le a, \\ 2I/cr, & a \le r \le b, \\ 0, & r > b. \end{cases} \tag{91}$$

The energy per unit length along the cable of this magnetic field is

$$U = \frac{1}{8\pi}\int B^2 \, d\text{Area} = \frac{I^2}{2\pi c^2}\left(\int_0^a \frac{r^2}{a^4} 2\pi r \; dr + \int_a^b \frac{2\pi r \; dr}{r^2}\right) = \frac{I^2}{c^2}\left(\frac{1}{4} + \ln\frac{b}{a}\right) \tag{92}$$

Since the energy can be expressed in terms of the self inductance L as $U = \frac{1}{2}LI^2$, we obtain the result (14).

Alternatively, we can evaluate the self inductance as $L = \Phi/cI$, where Φ is the magnetic flux per unit length linked by the circuit. The flux linked for $a < r < b$ is clearly

$$\Phi(a < r < b) = \int_a^b B_\theta \, dr = \frac{2I}{c}\int_a^b \frac{dr}{r} = \frac{2I}{c}\ln\frac{b}{a}\,. \tag{93}$$

More care is required when discussing the region $r < a$. On p. 115a of the Notes we saw that a consistent procedure for an extended current distribution is to average the flux linked by the various filamentary currents. In the present case, consider first a filament of area $r' dr' d\theta$ at (r', θ). We can define the surface through which the flux is to be calculated as that portion of the shell of radius r' that connects (r', θ) with the point $(r', 0)$, plus the plane $\theta = 0$ between r' and a. Since the field is azimuthal, no flux is linked on the shell; all filaments on the same shell link the same flux. Thus,

$$\Phi(r < a) = \frac{2I}{c}\frac{1}{\pi a^2}\int_0^a 2\pi r' \; dr' \int_{r'}^a \; dr \frac{r}{a^2} = \frac{2I}{a^2 c}\int_0^a r' \; dr' \left(1 - \frac{r'^2}{a^2}\right) = \frac{2I}{4c} \tag{94}$$

Combining (93-94) and dividing by cI, we again arrive at (14).

b) It appears impossible to make an accurate estimate of the self inductance when the outer cylinder is off center by either of the methods used in part a). The reason is that the currents are no longer uniformly distributed over the surfaces of the cylinders, so it is hard to calculate the magnetic field properly.

A solution can be given for the closely related problem in which the inner conductor, as well as the outer conductor, is a cylindrical shell. Then, we know from transmission line analysis (Lecture 13) that $LC = 1/c^2$, where C is the capacitance per unit length. With some effort we then find that

$$L = \frac{2}{c^2}\left(\ln\frac{b}{a} - \frac{\epsilon^2}{b^2 - a^2}\right), \tag{95}$$

to $\mathcal{O}(\epsilon^2/b^2)$. See my note *An Off-Center "Coaxial" Cable* (Nov. 21, 1999). http://physics.princeton.edu/~mcdonald/examples/coax.pdf

Here we illustrate what happens if we follow the approachs of part a), assuming the currents are uniformly distributed over the two cylinders.

If the center of the outer cylinder is at $(r, \theta) = (\epsilon, 0)$, then the surface of that cylinder follows

$$b^2 = r^2 + \epsilon^2 - 2\epsilon r \cos\theta, \tag{96}$$

or

$$r(\theta) = \epsilon\cos\theta + \sqrt{b^2 - \epsilon^2 \sin^2\theta} \approx b + \epsilon\cos\theta - \frac{\epsilon^2}{2b}\sin^2\theta. \tag{97}$$

We first calculate the self inductance via the energy method. Inside the outer cylinder the magnetic field is still given by the first two lines of eq. (91), but with $r = b$ replaced by $r(\theta)$ from eq. (97). Outside the cylinder the field is not quite zero because the magnetic field vectors from the currents in the inner and outer cylinders have slightly different magnitudes and directions. The vector from the center of the outer cylinder, $(\epsilon, 0)$ to a point (r, θ) has magnitude $r' \approx r - \epsilon\cos\theta$, and makes angle $\approx (\epsilon/r)\sin\theta$ to $\mathbf{r}$. Hence, the magnetic field from the current in the outer cylinder is

$$\mathbf{B} \approx \frac{2I}{cr}\left(\frac{\epsilon\sin\theta}{r}, -1 - \frac{\epsilon\cos\theta}{r}\right), \tag{98}$$

and the total magnetic field outside the outer cylinder is

$$\mathbf{B}_{\rm outside} \approx \frac{2I\epsilon}{cr^2}(\sin\theta, -\cos\theta), \tag{99}$$

so its magnitude is $B_{\rm outside} \approx 2I\epsilon/cr^2$.

The magnetic field energy per unit length along the axis is now

$$\begin{aligned} U &= \frac{1}{8\pi}\int B^2\ d\text{Area} = \frac{I^2}{2\pi c^2}\left(\int_0^a \frac{r^2}{a^4}2\pi r\ dr + \int_0^{2\pi} d\theta \int_a^{r(\theta)} \frac{r\ dr}{r^2} + \int_0^{2\pi} d\theta \int_{r(\theta)}^\infty \frac{\epsilon^2 r\ dr}{r^4}\right) \\ &\approx \frac{I^2}{2\pi c^2}\left(\frac{\pi}{2} + \int_0^{2\pi} d\theta\ \ln\left[\frac{b}{a}\left(1 + \frac{\epsilon}{b}\cos\theta - \frac{\epsilon^2}{2b^2}\sin^2\theta\right)\right] + \frac{\epsilon^2}{2}\int_0^{2\pi}\frac{d\theta}{b^2}\right) \\ &\approx \frac{I^2}{2\pi c^2}\left(\frac{\pi}{2} + \int_0^{2\pi} d\theta\left(\ln\frac{b}{a} + \frac{\epsilon}{b}\cos\theta - \frac{\epsilon^2}{2b^2}\sin^2\theta - \frac{\epsilon^2}{2b^2}\cos^2\theta\right) + \frac{\pi\epsilon^2}{b^2}\right) \\ &= \frac{I^2}{c^2}\left(\frac{1}{4} + \ln\frac{b}{a}\right) = \frac{1}{2}LI^2. \end{aligned} \tag{100}$$

Hence, we would conclude from the energy method that there is no change in the inductance to second order.

We contrast this with a calculation of the flux linked by the off-center coax. The contribution for $r < a$ is again given by (94). For $r > a$ but inside the off-center outer cylinder, the magnetic field is still $\mathbf{B} = (0, 2I/cr)$. The flux through the region

$r(\theta) > r > a$ varies with azimuth, so we average over filaments on the outer cylinder:

$$\begin{aligned}\Phi(r(\theta) > r > a) &= \frac{2I}{c}\frac{1}{2\pi}\int_0^{2\pi} d\theta \int_a^{r(\theta)} \frac{dr}{r} = \frac{2I}{2\pi c}\int_0^{2\pi} d\theta \, \ln\left[\frac{b}{a}\left(1 + \frac{\epsilon}{b}\cos\theta - \frac{\epsilon^2}{2b^2}\sin^2\theta\right)\right] \\ &\approx \frac{2I}{2\pi c}\int_0^{2\pi} d\theta \left(\ln\frac{b}{a} + \frac{\epsilon}{b}\cos\theta - \frac{\epsilon^2}{2b^2}\sin^2\theta - \frac{\epsilon^2}{2b^2}\cos^2\theta\right) \\ &= \frac{2I}{c}\left(\ln\frac{b}{a} - \frac{\epsilon^2}{2b^2}\right). \end{aligned} \tag{101}$$

Combining (94) and (101), we find that the self inductance is now

$$L = \frac{2}{c^2}\left(\frac{1}{4} + \ln\frac{b}{a} - \frac{\epsilon^2}{2b^2}\right), \tag{102}$$

to $\mathcal{O}(\epsilon^2/b^2)$.

Comparing with the result (95), we infer that the calculation via the linked flux is more accurate than that via the energy method when we use the incorrect assumption of uniform current distributions.

8. a) Since there are no free currents in the problem, $\boldsymbol{\nabla} \times \mathbf{H} = 0$ and we can define a magnetic scalar potential such that $\mathbf{H} = -\boldsymbol{\nabla}\phi$. As the cylinder is very long, we approximate the problem as 2-dimensional: $\phi = \phi(r, \theta)$ in cylindrical coordinates (r, θ, z).

   The source of the magnetic scalar potential is the imagined magnetic charges associated with the magnetization. Since $\mathbf{M} = M\hat{\mathbf{x}}$, the volume charge density $\rho_M = -\boldsymbol{\nabla}\cdot\mathbf{M} = 0$. However, at the surface of the cylinder at $r = a$, there is a density given by

$$\sigma_M = \mathbf{M} \cdot \hat{\mathbf{r}} = M \cos\theta. \tag{103}$$

   The potential is continuous at the boundary $r = a$, and Gauss' law tells us that

$$4\pi\sigma_M = 4\pi M \cos\theta = H_r(r = a^+) - H_r(r = a^-) = -\frac{\partial\phi(r = a^+)}{\partial r} + \frac{\partial\phi(r = a^-)}{\partial r}. \tag{104}$$

   The potential can be expanded as a harmonic series, but only the term in $\cos\theta$ will contribute in view of (104). Thus,

$$\phi = \begin{cases} -Hr\cos\theta, & r \le a, \\ -H\frac{a^2}{r}\cos\theta, & r \ge a, \end{cases} \tag{105}$$

   satisfies continuity of the potential at $r = a$. Then, (104) also tells us that $H = -2\pi M$. Inside the cylinder we have

$$\begin{aligned} \phi(r < a) &= 2\pi M x, & (106) \\ \mathbf{H}(r < a) &= -2\pi M\hat{\mathbf{x}} = -2\pi\mathbf{M}, & (107) \\ \mathbf{B}(r < a) &= \mathbf{H} + 4\pi\mathbf{M} = 2\pi\mathbf{M}. & (108) \end{aligned}$$

   Outside the cylinder there is no magnetization, and

$$\begin{aligned} \phi(r > a) &= 2\pi M a^2 \frac{\cos\theta}{r}, & (109) \\ \mathbf{H}(r > a) = \mathbf{B}(r > a) &= \frac{2\pi M a^2}{r^2}(\cos\theta\ \hat{\mathbf{r}} + \sin\theta\ \hat{\boldsymbol{\theta}}). & (110) \end{aligned}$$

   b) In case of a moving cylinder, the analysis of part a) holds in the rest frame of the cylinder. When the cylinder has velocity $\mathbf{v} = v\hat{\mathbf{z}}$ in the lab frame, there appears to be an electric field in the lab frame related by

$$\mathbf{E} = -\gamma\frac{\mathbf{v}}{c} \times \mathbf{B}' \approx -\frac{\mathbf{v}}{c} \times \mathbf{B}, \tag{111}$$

   where we ignore terms of order v^2/c^2, so the magnetic field $\mathbf{B}$ in the lab frame is the same as the field $\mathbf{B}'$ given by (108) and (110) in the rest frame. Regarding the sign in (111), we note that a charge which is at rest in the lab frame is moving with velocity $-\mathbf{v}$ in the rest frame of the magnetized cylinder, and so in the latter frame experiences a Lorentz force $-\mathbf{v}/c \times \mathbf{B}$.

Thus,

$$\mathbf{E}(r < a) = -2\pi M \frac{v}{c}\hat{\mathbf{y}} = -2\pi M \frac{v}{c}(\sin\theta \; \hat{\mathbf{r}} + \cos\theta \; \hat{\boldsymbol{\theta}}), \qquad (112)$$

$$\mathbf{E}(r > a) = \frac{2\pi M v a^2}{cr^2}(\sin\theta \; \hat{\mathbf{r}} - \cos\theta \; \hat{\boldsymbol{\theta}}). \qquad (113)$$

There is an electric charge density on the surface of the cylinder given by

$$\sigma = \frac{1}{4\pi}\left[E_r(r = a^+) - E_r(r = a^-)\right] = \frac{Mv}{c}\sin\theta \; . \qquad (114)$$

This can be thought of as arising from a polarization **P** related to the moving magnetization by

$$\mathbf{P} = \frac{\mathbf{v}}{c} \times \mathbf{M}. \qquad (115)$$

See sec. 87 of Becker for a discussion of how **M** and **P** form a relativistic tensor.

9. The magnetic induction $\mathbf{B}$ is related to the magnetic field $\mathbf{H}$ by $\mathbf{B} = \mu\mathbf{H}$, where μ is the permeability. In the gap, $\mu = 1$. The normal component of the magnetic induction is continuous across the boundaries of the gap, since $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. Thus,

$$H_{\rm gap} = B_{\rm gap} = B_{\rm iron} = \mu H_{\rm iron}. \tag{116}$$

For a large permeability $\mu_{\rm iron}$, the magnetic field $H_{\rm iron}$ is negligible.

The magnetic field H at the center of the toroid is related by Ampére's law as

$$\oint H\ dl = H_{\rm gap} w + H_{\rm iron}(2\pi b - w) = \frac{4\pi NI}{c}\,. \tag{117}$$

With the neglect of the small quantity $H_{\rm iron}$, we find

$$H_{\rm gap} = B_{\rm gap} = B_{\rm iron} \approx \frac{4\pi NI}{cw}\,. \tag{118}$$

The magnetic energy is

$$U = \frac{1}{8\pi}\int \mathbf{B} \cdot \mathbf{H}\ d\text{Vol} \approx \frac{\pi a^2 w}{8\pi}\left(\frac{4\pi NI}{cw}\right)^2 = \frac{2\pi^2 a^2 N^2 I^2}{c^2 w}\,. \tag{119}$$

The force tending to close the gap is

$$F = -\frac{dU}{dw} = \frac{2\pi^2 a^2 N^2 I^2}{c^2 w^2}\,. \tag{120}$$

The pressure can also be calculated via the Maxwell stress tensor as

$$P_{\rm gap} = \frac{B_{\rm gap}^2}{8\pi}\,. \tag{121}$$

If $B_{\rm gap} = 15,000$ Gauss, then

$$P_{\rm gap} = 9 \times 10^6\ \text{dyne/cm}^2 = 9\ \text{atmospheres}. \tag{122}$$

10. The current density associated with a uniform current I in a wire of radius a whose axis is the z axis is

$$\mathbf{J} = \frac{I}{\pi a^2}\hat{\mathbf{z}}. \tag{123}$$

Ohm's law gives the electric field inside the wire as

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = \frac{I}{\pi a^2 \sigma}\hat{\mathbf{z}} = IR\hat{\mathbf{z}}, \tag{124}$$

where σ is the conductivity, and $R = 1/\pi a^2 \sigma$ is the resistance per unit length of the wire.

The electric potential inside the wire is therefore,

$$\phi(r < a) = -IRz, \tag{125}$$

where we define $\phi(0, 0, 0) = 0$.

For the region $a < r < b$, we suppose the potential satisfies separation of variables:

$$\phi(a < r < b) = f(r)g(z). \tag{126}$$

Continuity of the potential at $r = a$ is satisfied by the form

$$\phi(a < r < b) = -f(r)IRz. \tag{127}$$

Substituting (127) into Laplace's equation, $\nabla^2\phi = 0$, we find that

$$\frac{1}{r}\frac{d}{dr}r\frac{df}{dr} = 0, \tag{128}$$

so f has the general solution

$$f = A + B\ln r. \tag{129}$$

The boundary conditions on the potential at $r = a$ and b now require that $f(a) = 1$ and $f(b) = 0$. Hence, $f = \ln(r/b)/\ln(a/b)$, and

$$\phi(a < r < b) = -IRz\frac{\ln(r/b)}{\ln(a/b)} = IRz\frac{\ln(r/b)}{\ln(b/a)}. \tag{130}$$

The surface charge density σ_q at the surface of the wire is

$$\begin{aligned}\sigma_q &= \frac{1}{4\pi}\left[E_r(r = a^+) - E_r(r = a^-)\right] = \frac{1}{4\pi}\left[-\frac{\partial\phi(r = a^+)}{\partial r} + \frac{\partial\phi(r = a^-)}{\partial r}\right] \\ &= -\frac{IRz}{4\pi a\ln(b/a)}.\end{aligned} \tag{131}$$

The electric field is

$$\mathbf{E} = -\boldsymbol{\nabla}\phi = \begin{cases} IR\hat{\mathbf{z}}, & r < a, \\ -IRz\hat{\mathbf{r}}/r\ln(b/a) + IR\ln(r/b)\hat{\mathbf{z}}/\ln(a/b), & a < r < b, \\ 0, & b < r. \end{cases} \tag{132}$$

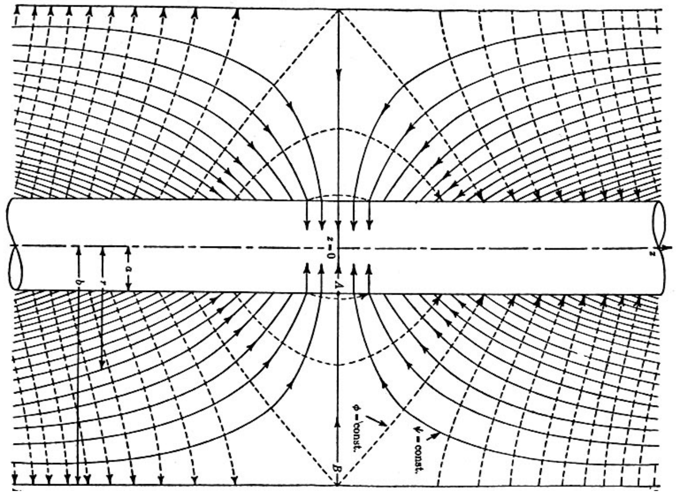

Figure 1: The solid curves show lines of Poynting flux $\mathbf{S}$, and the dashed lines are the electric field $\mathbf{E}$ in the region between the wire and the outer conductor. Because the tangential component of $\mathbf{E}$ is continuous at the boundary $r = a$, and $\mathbf{E} = IR\hat{\mathbf{z}}$ for $r < a$, the field lines for $r > a$ are bent towards positive z. For $|z| < b$ the field lines leave positive surface charges at $r = a$ and end on negative surface charges also at $r = a$; in loop circuits ($b \gtrsim L$) this is the general behavior. From *Electrodynamics* by A. Sommerfeld (Academic Press, 1952), p. 129.

The magnetic field follows from Ampère's law:

$$B_\theta(r) = \begin{cases} 2Ir/a^2c, & r \le a, \\ 2I/cr, & a \le r, \end{cases} \tag{133}$$

The Poynting vector is then,

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \begin{cases} -I^2Rr\hat{\mathbf{r}}/2\pi a^2, & r < a, \\ -I^2R\ln(b/r)\hat{\mathbf{r}}/2\pi r\ln(a/b) - I^2Rz\hat{\mathbf{z}}/2\pi r^2\ln(b/a), & a < r < b, \\ 0, & b < r. \end{cases} \tag{134}$$

The Poynting vector is radially inwards at the surface of the wire, and the energy flux per unit length there is $2\pi aS(r = a) = I^2R$. That is, the Poynting flux energy the wire through its surface provides the I^2R power loss to Joule heating.

The Poynting flux crossing a plane at constant z is

$$\int S_z \, d\text{Area} = -\frac{I^2Rz}{2\pi\ln(b/a)}\int_a^b \frac{2\pi r\ dr}{r^2} = -I^2Rz. \tag{135}$$

Since the flux is zero at $z = 0$, we interpret (135) as indicating that the total Poynting flux crossing a plane at constant z equals the power dissipated by the wire between 0 and z. This flux exists in the region $a < r < b$, *i.e.*, in the air (or vacuum) between the conductors, rather than in the conductors themselves.

For the alternative calculation of the surface charge density, we note that the capacitance per unit length between the inner and outer conductors is

$$C = \frac{1}{2\ln(b/a)}\,, \tag{136}$$

so the charge per unit length needed to support the potential $\phi(a,z) = -IRz$ is

$$Q(z) = C\phi(z) = -\frac{IRz}{2\ln(b/a)}\,, \tag{137}$$

and the corresponding surface charge density is

$$\sigma = \frac{Q}{2\pi a} = -\frac{IRz}{4\pi a\ln(b/a)}\,, \tag{138}$$

as previously found in eq. (131).

This argument helps us understand how the charge distribution and electric field in the central region of the wire is insensitive to the physical details of the ends of the wire. The capacitance per unit length might be different from the expression (136) for a few wire diameters in z from the ends of the wire, but it is quite accurate over most of the length of the wire. Hence, we are less surprised that the potential (130) was obtained without ever specifying the boundary conditions at the ends of the wire. Those boundary conditions only affect the potential very near the ends of the wire, and the potential over most of the wire must have the form (130) in any case.

The potential (130) can be thought of as a kind of zero-frequency mode of the cavity between the inner and outer conductors. This cavity more has a "natural" behavior at the ends, found by inserting $z_{\rm end}$ into eq. (130). We readily see that this radial potential distribution would hold if the ends of the cable are terminated "naturally" in plates of uniform conductivity, so $E_r \propto j_r \propto 1/r$, and $\phi(r) \propto \ln r$.

If the coaxial cable transmits energy from a source (battery) at one end to a load (resistor) at the other, there is a net momentum $\int d\mathrm{Vol}\, \epsilon\mu \mathbf{S}/c^2$ *stored in the fields, which is very small due to the factor* $1/c^2$*. If the cable is an isolated system, then it also has an equal and opposite mechanical momentum. See,*
http://physics.princeton.edu/~mcdonald/examples/hidden.pdf

11. a) If long coil 1 carries steady current I_1, then the magnetic field inside that coil is axial with magnitude

$$B_1 = \frac{4\pi N_1 I_1}{cl}, \tag{139}$$

by an application of Ampére's law, ignoring end effects. Outside the coil, the magnetic field is zero. The flux linked by coil 1 is therefore,

$$\phi_1 = N_1 \pi r_1^2 B_1 = \frac{4\pi^2 N_1^2 r_1^2 I_1}{cl} = cL_1 I_1, \tag{140}$$

so the self inductance of coil 1 is

$$L_1 = \frac{4\pi^2 N_1^2 r_1^2}{c^2 l}. \tag{141}$$

Similarly, the self inductance of coil 2 is

$$L_2 = \frac{4\pi^2 N_2^2 r_2^2}{c^2 l}. \tag{142}$$

The mutual inductance can be calculated via the flux linked in coil 2 when coil 1 carries current I_1. Since the magnetic field due to current I_1 is zero outside coil 1, which is inside coil 2, we have

$$\phi_{12} = N_2 \pi r_1^2 B_1 = \frac{4\pi^2 N_1 N_2 r_1^2 I_1}{cl} = cMI_1, \tag{143}$$

so the mutual inductance is

$$M = \frac{4\pi^2 N_1 N_2 r_1^2}{c^2 l}. \tag{144}$$

Since $r_2 > r_1$, we have $L_1 L_2 > M^2$.

In solving the coupled circuit equations in the presence of an oscillatory driving voltage at frequency ω, we use complex notation, and divide out the common factor $e^{i\omega t}$. Then the symbols I_1 and I_2 are complex numbers such that the real current is $Re\ I_1 e^{i\omega t}$, *etc.*

The coupled equations are

$$V_0 = I_1 R_1 + \dot{I}_1 L_1 + \dot{I}_2 M = I_1 R_1 + i\omega I_1 L_1 + i\omega I_2 M, \tag{145}$$

$$0 = I_2 R_2 + \dot{I}_2 L_2 + \dot{I}_1 M = I_2 R_2 + i\omega I_2 L_2 + i\omega I_1 M. \tag{146}$$

These are readily solved as

$$I_1 = \frac{(R_2 + i\omega L_2)V_0}{R_1^2 - \omega^2(L_1 L_2 - M^2) + i\omega(R_1 L_2 + R_2 L_1)}, \tag{147}$$

$$I_2 = -\frac{i\omega M V_0}{R_1^2 - \omega^2(L_1 L_2 - M^2) + i\omega(R_1 L_2 + R_2 L_1)}, \tag{148}$$

The time-average power dissipated in coil 2 is then,

$$\langle P_2 \rangle = \frac{|I_2^2|\, R_2}{2} = \frac{\omega^2 M^2 R_2 V_0^2}{2\left\{\left[R_1^2 - \omega^2(L_1 L_2 - M^2)\right]^2 + \omega^2(R_1 L_2 + R_2 L_1)^2\right\}}. \tag{149}$$

b) To calculate the Poynting vector $\mathbf{S}$, we need the electric and magnetic fields. The (complex) magnetic field is

$$B_z(r) = \begin{cases} 4\pi(N_1 I_1 + N_2 I_2)/cl, & r < r_1, \\ 4\pi N_2 I_2/cl, & r_1 < r < r_2, \\ 0, & r < r_2. \end{cases} \tag{150}$$

The electric field is azimuthal, as follows from Faraday's law:

$$\begin{aligned} E_\theta &= -\frac{1}{2\pi rc}\frac{d}{dt}\int_0^r B_z\ 2\pi r\ dr = -\frac{i\omega}{rc}\int_0^r B_z\ r\ dr \\ &= \begin{cases} -2\pi i\omega r(N_1 I_1 + N_2 I_2)/c^2 l, & r < r_1, \\ -2\pi i\omega(r_1^2 N_1 I_1 + r^2 N_2 I_2)/c^2 lr, & r_1 < r < r_2, \\ -2\pi i\omega(r_1^2 N_1 I_1 + r_2^2 N_2 I_2)/c^2 lr, & r_2 < r, \end{cases} \end{aligned} \tag{151}$$

The Poynting vector is radial, and positive if both E_θ and B_z are positive. Its time-average value is $\langle S_r \rangle = (c/8\pi) Re E_\theta^\star B_z$. For $r < r_1$, $E_\theta^\star B_z$ is pure imaginary, so $\langle S_r \rangle = 0$ here. Since $B_z = 0$ for $r > r_2$, $\langle S_r \rangle = 0$ here also. The remaining region gives

$$\begin{aligned} \langle S_r(r_1 < r < r_2) \rangle &= \frac{c}{8\pi} Re \frac{2\pi i\omega}{c^2 lr}(r_1^2 N_1 I_1^\star + r^2 N_2 I_2^\star)\frac{4\pi N_2 I_2}{cl} = -\frac{\pi r_1^2 \omega N_1 N_2}{c^2 l^2 r} Im(I_1^\star I_2) \\ &= \frac{\pi r_1^2 \omega N_1 N_2}{c^2 l^2 r} \frac{\omega M R_2 V_0^2}{\left[R_1^2 - \omega^2(L_1 L_2 - M^2)\right]^2 + \omega^2(R_1 L_2 + R_2 L_1)^2} \\ &= \frac{\omega^2 M^2 R_2 V_0^2}{4\pi lr\left\{\left[R_1^2 - \omega^2(L_1 L_2 - M^2)\right]^2 + \omega^2(R_1 L_2 + R_2 L_1)^2\right\}} \\ &= \frac{1}{2\pi rl}\langle P_2 \rangle . \end{aligned} \tag{152}$$

Since $2\pi rl \langle S_r \rangle$ is the power transported by the electromagnetic field across the cylinder of radius r and length l, we interpret the power consumed in the outer coil as flowing from the inner, driven coil.

c) If, instead, coil 2 is driven, then the solutions to the coupled equations are obtained from (147-148) by swapping indices 1 and 2. Likewise, the power consumed in coil 1 is obtained from (149) by the same swap of indices. The expressions (150-151) for the electric and magnetic fields in terms of the currents remain the same, as does the first line of (152) for the Poynting vector. However, in the rest of (152) we must swap indices 1 and 2, and note the sign change that occurs in $I_1^\star$. Thus, we find

$$\langle S_r(r_1 < r < r_2) \rangle = -\frac{1}{2\pi rl}\langle P_1 \rangle . \tag{153}$$

Again, power flows from the driven coil to the load coil.

12. a) The field angular momentum is given by

$$\mathbf{L}_{\mathrm{EM}} = \int \mathbf{r}\times\mathbf{P}_{\mathrm{field}}\, d\mathrm{Vol} = \frac{1}{4\pi c}\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})\, d\mathrm{Vol} = \frac{1}{4\pi c}\int[(\mathbf{r}\cdot\mathbf{B})\mathbf{E} - (\mathbf{r}\cdot\mathbf{E})\mathbf{B}]\, d\mathrm{Vol}. \tag{154}$$

From the symmetry of the problem, we infer that the angular momentum will be along the z axis, and that the electric and magnetic field are independent of azimuth φ in spherical coordinates (r,θ,φ). Thus, we desire

$$\mathbf{L}_{\mathrm{EM},z} = \frac{1}{2c}\int_0^\infty r^3\, dr \int_{-1}^{1} d\cos\theta (B_r E_z - E_r B_z). \tag{155}$$

The magnetic field due to magnetic dipole $m\hat{\mathbf{z}}$ is

$$\mathbf{B} = \frac{3\cos\theta\hat{\mathbf{r}} - \hat{\mathbf{z}}}{r^3}m = \frac{2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}}{r^3}m. \tag{156}$$

The components we need are

$$B_r = \frac{2mP_1(\cos\theta)}{r^3}, \qquad \text{and} \qquad B_z = \cos\theta B_r - \sin\theta B_\theta = \frac{2mP_2(\cos\theta)}{r^3}. \tag{157}$$

The electric field can be gotten from the electric potential ϕ of a charged ring, p. 59 of the Notes with $\cos\theta_0 = 0$:

$$\phi = \begin{cases} \frac{Q}{a}\sum_n \left(\frac{r}{a}\right)^n P_n(0)P_n(\cos\theta), & r < a, \\ \frac{Q}{r}\sum_n \left(\frac{a}{r}\right)^n P_n(0)P_n(\cos\theta), & r > a. \end{cases} \tag{158}$$

Since $P_n(0) = 0$ for odd n, only even n terms contribute to the potential. The electric field components are

$$E_r = -\frac{\partial\phi}{\partial r} = \begin{cases} -\frac{Q}{ar}\sum_n n\left(\frac{r}{a}\right)^n P_n(0)P_n(\cos\theta), & r < a, \\ \frac{Q}{r^2}\sum_n (n+1)\left(\frac{a}{r}\right)^n P_n(0)P_n(\cos\theta), & r > a. \end{cases} \tag{159}$$

$$E_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \begin{cases} -\frac{Q}{ar}\sum_n \left(\frac{r}{a}\right)^n P_n(0)P_n^1(\cos\theta), & r < a, \\ -\frac{Q}{r^2}\sum_n \left(\frac{a}{r}\right)^n P_n(0)P_n^1(\cos\theta), & r > a. \end{cases} \tag{160}$$

$$E_\varphi = 0, \tag{161}$$

using the fact that

$$\frac{dP_n(\cos\theta)}{d\theta} = P_n^1(\cos\theta), \tag{162}$$

where $P_n^m(\cos\theta)$ is an associated Legendre polynomial. See eq. (3.39) of Jackson.

We also need $E_z = \cos\theta E_r - \sin\theta E_\theta$, for which it is useful to note two recurrence relations (Gradshetyn and Ryzhik, 8.731.2 and 8.735.2):

$$\cos\theta P_n(\cos\theta) = \frac{(n+1)P_{n+1}(\cos\theta) + nP_{n-1}(\cos\theta)}{2n+1}, \tag{163}$$

$$\begin{aligned} \sin\theta P_n^1(\cos\theta) &= n\cos\theta P_n(\cos\theta) - nP_{n-1}(\cos\theta) \\ &= \frac{n(n+1)}{2n+1}[P_{n+1}(\cos\theta) - P_{n-1}(\cos\theta)]. \end{aligned} \tag{164}$$

Then

$$E_z = \begin{cases} -\frac{Q}{ar}\sum_n n\left(\frac{r}{a}\right)^n P_n(0)P_{n-1}(\cos\theta), & r < a, \\ \frac{Q}{r^2}\sum_n (n+1)\left(\frac{a}{r}\right)^n P_n(0)P_{n+1}(\cos\theta), & r > a. \end{cases} \tag{165}$$

so that

$$\begin{aligned} \mathbf{L}_{\mathrm{EM},z} &= \frac{1}{2c}\int_0^\infty r^3\ dr \int_{-1}^1 d\cos\theta (B_r E_z - E_r B_z) \\ &= \frac{mQ}{c}\begin{cases} \int_0^a r^3\ dr \frac{1}{ar^4}\sum_n n\left(\frac{a}{r}\right)^2 P_n(0)\int_{-1}^1 d\mu\,[P_2(\mu)P_n(\mu) - P_1(\mu)P_{n-1}(\mu)] \\ \int_a^\infty r^3\ dr \frac{1}{r^5}\sum_n (n+1)\left(\frac{a}{r}\right)^2 P_n(0)\int_{-1}^1 d\mu\,[P_1(\mu)P_{n+1}(\mu) - P_2(\mu)P_n(\mu)] \end{cases} \\ &= \frac{mQ}{c}\begin{cases} \int_0^a r^3\ dr \frac{1}{ar^4} 2\left(\frac{a}{r}\right)^2\left(-\frac{1}{2}\right)\left(\frac{2}{5} - \frac{2}{3}\right) \\ \int_a^\infty r^3\ dr \frac{1}{r^5}\left[\frac{2}{3} - 3\left(\frac{a}{r}\right)^2\left(-\frac{1}{2}\right)\frac{2}{5}\right] \end{cases} \\ &= \frac{mQ}{c}\begin{cases} \frac{4}{15a}\int_0^a r\ dr \\ \frac{2}{3}\int_a^\infty \frac{dr}{r^2} + \frac{3a^2}{5}\int_a^\infty \frac{dr}{r^4} \end{cases} \\ &= \begin{cases} \frac{2mQ}{15ac}, & r < a, \\ \frac{13mQ}{15ac}, & r > a. \end{cases} \end{aligned} \tag{166}$$

where we have used the facts that $P_0(0) = 1$, $P_2(0) = -1/2$, and $\int_{-1}^1 P_m(\mu)P_n(\mu)\, d\mu = 2\delta_{mn}/(2n+1)$. Altogether,

$$\mathbf{L}_{\mathrm{EM},z} = \frac{mQ}{ac}\,, \tag{167}$$

b) As the magnetic moment m goes to zero, the electromagnetic angular momentum vanishes. But, the consequent change in the flux through the charged ring results in an azimuthal electric field E_φ around the ring, which causes a torque that increases the mechanical angular momentum:

$$\frac{dL_{\mathrm{mech},z}}{dt} = aQE_\varphi = -\frac{aQ}{2\pi ac}\frac{d\Phi}{dt}\,. \tag{168}$$

This integrates to

$$L_{\mathrm{mech},z,\mathrm{final}} = \frac{Q}{2\pi c}\Phi_{\mathrm{initial}} = \frac{Q}{2\pi c}\int_0^a 2\pi r\ dr\ B_z \tag{169}$$

If we use (157) for B_z, the result diverges. However, this form does not correctly account for the flux inside the small coil at the origin. We avoid this issue by noting that, since $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, the magnetic flux through the loop of radius a is the negative of the flux across the plane $z = 0$ outside the loop. In that plane, $\cos\theta = 0$, and since $P_2(0) = -1/2$, we have

$$L_{\mathrm{mech},z,\mathrm{final}} = -\frac{Q}{c}\int_a^\infty r\ dr\ B_z = \frac{mQ}{c}\int_a^\infty \frac{dr}{r^2} = \frac{mQ}{ac} = L_{\mathrm{EM},z,\mathrm{initial}}. \tag{170}$$

Remark: Equation (169) can be given another interpretation. The magnetic flux can be expressed in terms of the vector potential:

$$\Phi_{\mathrm{initial}} = \int \mathbf{B}\cdot d\mathbf{S} = \oint \mathbf{A}\cdot d\mathbf{l} = 2\pi a A_{\mathrm{initial},\varphi}. \tag{171}$$

Thus,

$$L_{\text{mech},z,\text{final}} = (\mathbf{r} \times \mathbf{P}_{\text{final}})_z = aP_{\text{final},\varphi} = QA_{\text{initial},\varphi} \tag{172}$$

Since $P_{\text{initial},\varphi} = 0 = A_{\text{final},\varphi}$, we can write

$$\left[\mathbf{r} \times \left(\mathbf{P} + \frac{Q\mathbf{A}}{c}\right)\right]_z = \text{ constant.} \tag{173}$$

This is the z component of the canonical angular momentum of a charged particle in an electromagnetic field. Hence, another view of the Feynman disk paradox is that it illustrates the conservation of canonical angular momentum.

13. The key to this problem is conservation of canonical momentum, $\mathbf{P} + e\mathbf{A}/c$, where $\mathbf{A}$ is the vector potential (in Gaussian units).

    It turns out to be even more effective to consider the canonical angular momentum, which is $\mathbf{L} = \mathbf{r} \times (\mathbf{P} + e\mathbf{A}/c)$.

    We want $P_\phi = 0$ outside the magnet. This implies $L_z = rP_\phi = 0$ also. Therefore, we need $r(P_\phi + eA_\phi/c) = 0$ inside the magnet.

    A solenoid magnet with field B_z has vector potential $A_\phi = rB_z/2$. To see this, recall that the integral of the vector potential around a loop is equal to the magnetic flux through the loop: $2\pi r A_\phi = \pi r^2 B_z$.

    For a particle with average momentum in the z direction, its trajectory inside the magnet is a helix whose center is at some radius R_G (called R_0 in the statement of the problem) from the magnetic axis. The radius R_B (called R in the statement of the problem) of the helix can be obtained from $F = ma$:

$$\frac{mv_\perp^2}{R_B} = e\frac{v_\perp}{c}B_z, \tag{174}$$

so

$$R_B = \frac{eB_z}{cP_\perp}. \tag{175}$$

The direction of rotation around the helix is in the $-z$ direction (Lenz' law).

Since the canonical angular momentum is a constant of the motion, we can evaluate it at any convenient point on the particle's trajectory. In particular, we consider the point at which the trajectory is closest to the magnetic axis. As shown in Fig. 2, this point obeys $r = R_G - R_B$, and so

$$L_z = (R_G - R_B)\,P_\perp + \frac{eB_z}{2c}\,(R_G - R_B)^2 = \left(R_G^2 - R_B^2\right)\frac{eB_z}{2c}. \tag{176}$$

Note that $R_G^2 - R_B^2$ is the product of the closest and farthest distances between the trajectory and the magnetic axis.

Hence, the canonical angular momentum vanishes for motion in a solenoid field if and only if $R_G = R_B$, *i.e.*, if and only if the particle's trajectory passes through the magnetic axis.

We also see that if the trajectory does not contain the magnetic axis, the canonical angular momentum is positive; while if the trajectory contains the magnetic axis, the canonical angular momentum is negative.

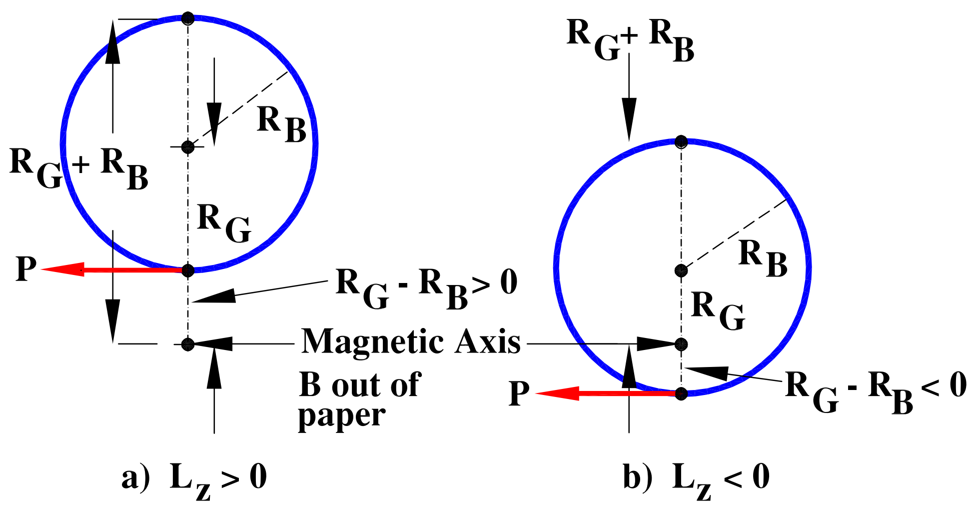

Figure 2: The projection onto a plane perpendicular to the magnetic axis of the helical trajectory a charge particle of transverse momentum P. The magnetic field B_z is out of the paper, so the rotation of the helix is clockwise for a positively charged particle. a) The trajectory does not contain the magnetic axis, and $L_z > 0$. b) The trajectory contains the magnetic axis, and $L_z < 0$.