Princeton University

Ph501

Electrodynamics

Problem Set 8

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http://kirkmcd.princeton.edu/examples/

1. Wire with a Linearly Rising Current

A neutral wire along the z-axis carries current I that varies with time t according to,

$$I(t) = \begin{cases} 0 & (t \le 0), \\ \alpha t & (t > 0), \qquad \alpha \text{ is a constant.} \end{cases} \tag{1}$$

Deduce the time-dependence of the electric and magnetic fields, **E** and **B**, observed at a point $(r, \theta = 0, z = 0)$ in a cylindrical coordinate system about the wire. Use your expressions to discuss the fields in the two limiting cases that $ct \gg r$ and $ct = r + \epsilon$, where c is the speed of light and $\epsilon \ll r$.

The related, but more intricate case of a solenoid with a linearly rising current is considered in http://kirkmcd.princeton.edu/examples/solenoid.pdf

2. **Harmonic Multipole Expansion**

A common alternative to the multipole expansion of electromagnetic radiation given in Lecture 16 of the Notes assumes from the beginning that the motion of the charges is oscillatory with angular frequency ω. However, we still use the essence of the Hertz method wherein the current density is related to the time derivative of a polarization:[1]

$$\mathbf{J} = \dot{\mathbf{p}}. \tag{2}$$

The radiation fields will be deduced from the retarded vector potential,

$$\mathbf{A} = \frac{1}{c}\int \frac{[\mathbf{J}]}{r}\, d\text{Vol} = \frac{1}{c}\int \frac{[\dot{\mathbf{p}}]}{r}\, d\text{Vol}, \tag{3}$$

which is a solution of the (Lorenz gauge) wave equation,

$$\boldsymbol{\nabla}^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}. \tag{4}$$

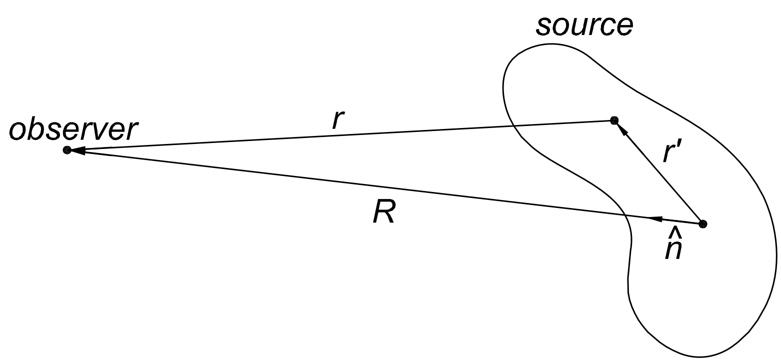
Suppose that the Hertzian electric dipole vector $\mathbf{p}$ has oscillatory time dependence,

$$\mathbf{p}(\mathbf{x}, t) = \mathbf{p}_\omega(\mathbf{x})e^{-i\omega t}. \tag{5}$$

Using the expansion,

$$r = R - \mathbf{r}' \cdot \hat{\mathbf{n}} + ... \tag{6}$$

of the distance r from source to observer,



to show that

$$\mathbf{A} = -i\omega\frac{e^{i(kR-\omega t)}}{cR}\int \mathbf{p}_\omega(\mathbf{r}')\left(1 + \mathbf{r}' \cdot \hat{\mathbf{n}}\left(\frac{1}{R} - ik\right) + ...\right)\, d\text{Vol}', \tag{7}$$

where no assumption is made that $R \gg$ source size or that $R \gg \lambda = 2\pi/k = 2\pi c/\omega$.

Consider now only the leading term in this expansion, which corresponds to electric dipole radiation. Introducing the total electric dipole moment,

$$\mathbf{P} \equiv \int \mathbf{p}_\omega(\mathbf{r}')\, d\text{Vol}', \tag{8}$$

[1]Some consideration of the related topics of Hertz vectors and scalars is given in the Appendix of http://kirkmcd.princeton.edu/examples/smallloop.pdf

use,

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \qquad \text{and} \qquad \boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{9}$$

to show that for an observer in vacuum the electric dipole radiation fields are,,

$$\mathbf{B} = k^2 \frac{e^{i(kR-\omega t)}}{R}\left(1 + \frac{i}{kR}\right)\hat{\mathbf{n}} \times \mathbf{P}, \tag{10}$$

$$\mathbf{E} = k^2 \frac{e^{i(kR-\omega t)}}{R}\left\{\hat{\mathbf{n}} \times (\mathbf{P} \times \hat{\mathbf{n}}) + [3(\hat{\mathbf{n}} \cdot \mathbf{P})\hat{\mathbf{n}} - \mathbf{P}]\left(\frac{1}{k^2R^2} - \frac{i}{kR}\right)\right\}. \tag{11}$$

Alternatively, deduce the electric field from both the scalar and vector potentials via,

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \tag{12}$$

in both the Lorenz and Coulomb gauges.

For large R,

$$\mathbf{B}_{\text{far}} \approx k^2 \frac{e^{i(kR-\omega t)}}{R}\hat{\mathbf{n}} \times \mathbf{P}, \qquad \mathbf{E}_{\text{far}} \approx \mathbf{B}_{\text{far}} \times \hat{\mathbf{n}}, \tag{13}$$

while for small R,

$$\mathbf{B}_{\text{near}} \approx \frac{ik}{R^2}(\hat{\mathbf{n}} \times \mathbf{P})e^{-i\omega t}, \qquad \mathbf{E}_{\text{near}} \approx \frac{3(\hat{\mathbf{n}} \cdot \mathbf{P})\hat{\mathbf{n}} - \mathbf{P}}{R^3}e^{-i\omega t}, \tag{14}$$

Thus, $B_{\text{near}} \ll E_{\text{near}}$, and the electric field $\mathbf{E}_{\text{near}}$ has the shape of the static dipole field of moment $\mathbf{P}$, modulated at frequency ω.

Calculate the Poynting vector of the fields of a Hertzian oscillating electric dipole (10)-(11) at all points in space. Show that the time-averaged Poynting vector has the same form in the near zone as it does in the far zone, which confirms that (classical) radiation exists both close to and far from the source.

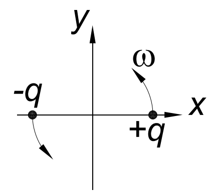
Extend your discussion to the case of an oscillating, point magnetic dipole by noting that if $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ are solutions to Maxwell's equations in free space (*i.e.*, where the charge density ρ and current density $\mathbf{J}$ are zero), then the **dual** fields,

$$\mathbf{E}'(\mathbf{r}, t) = -\mathbf{B}(\mathbf{r}, t), \qquad \mathbf{B}'(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t), \tag{15}$$

are also solutions.

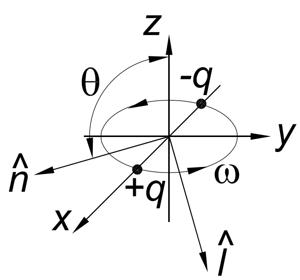
3. **Rotating Electric Dipole**

An electric dipole of moment p_0 lies in the x-y plane and rotates about the x axis with angular velocity ω.



Calculate the radiation fields and the radiated power according to an observer at angle θ to the z axis in the x-z plane.

Define $\hat{\mathbf{n}}$ towards the observer, so that $\hat{\mathbf{n}} \cdot \hat{\mathbf{z}} = \cos\theta$, and let $\hat{\mathbf{l}} = \hat{\mathbf{y}} \times \hat{\mathbf{n}}$.



Show that,

$$\mathbf{B}_{\rm rad} = p_0 k^2 \frac{e^{i(kr-\omega t)}}{r}(\cos\theta\ \hat{\mathbf{y}} - i\ \hat{\mathbf{l}}), \qquad \mathbf{E}_{\rm rad} = p_0 k^2 \frac{e^{i(kr-\omega t)}}{r}(\cos\theta\ \hat{\mathbf{l}} + i\ \hat{\mathbf{y}}), \tag{16}$$

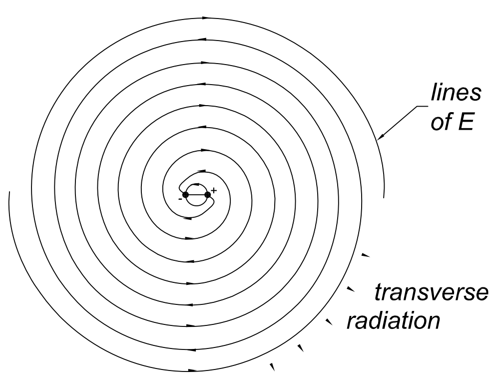
where r is the distance from the center of the dipole to the observer.

Note that for an observer in the x-y plane ($\hat{\mathbf{n}} = \hat{\mathbf{x}}$), the radiation is linearly polarized, while for an observer along the z axis it is circularly polarized.

Show that the (time-averaged) radiated power is given by,

$$\frac{d\langle P\rangle}{d\Omega} = \frac{c}{8\pi} p_0^2 k^4 (1 + \cos^2\theta), \qquad \langle P\rangle = \frac{2cp_0^2k^4}{3} = \frac{2p_0^2\omega^4}{3c^3}\,. \tag{17}$$

This example gives another simple picture of how radiation fields are generated. The field lines emanating from the dipole become twisted into spirals as the dipole rotates. At large distances, the field lines are transverse...



4. **Magnetars**

   The x-ray pulsar SGR1806-20 has recently been observed to have a period T of 7.5 s and a relatively large "spindown" rate $\left|\dot{T}\right| = 8 \times 10^{-11}$. See, C. Kouveliotou *et al.*, *An X-ray pulsar with a superstrong magnetic field in the soft γ-ray repeater SGR1806-20*, Nature **393**, 235-237 (1998).[2]

   Calculate the maximum magnetic field at the surface of this pulsar, assuming it to be a standard neutron star of mass $1.4M_\odot = 2.8 \times 10^{30}$ kg and radius 10 km, that the mass density is uniform, that the spindown is due to electromagnetic radiation, and that the angular velocity vector is perpendicular to the magnetic dipole moment of the pulsar.

   Compare the surface magnetic field strength to the so-called QED critical field strength $m^2c^3/e\hbar = 4.4 \times 10^{13}$ gauss, at which electron-positron pair creation processes become highly probable.

[2] http://kirkmcd.princeton.edu/examples/EM/kouveliotou_nature_393_235_98.pdf

5. **Radiation of Angular Momentum**

Recall that we identified a field momentum density,

$$\mathbf{P}_{\text{field}} = \frac{\mathbf{S}}{c^2} = \frac{U}{c}\hat{\mathbf{k}}, \tag{18}$$

and angular momentum density,

$$\mathcal{L}_{\text{field}} = \mathbf{r} \times \mathbf{P}_{\text{field}}. \tag{19}$$

Show that for oscillatory sources, the time-average angular momentum radiated into unit solid angle per second is (the real part of),

$$\frac{d\langle\mathbf{L}\rangle}{dt\ d\Omega} = \frac{1}{8\pi} r^3 [\mathbf{E}(\hat{\mathbf{n}} \cdot \mathbf{B}^\star) - \mathbf{B}^\star(\hat{\mathbf{n}} \cdot \mathbf{E})]. \tag{20}$$

Thus, the radiated angular momentum is zero for purely transverse fields.

In eq. (11) of Prob. 2 above, we found that for electric dipole radiation there is a term in $\mathbf{E}$ with $\mathbf{E} \cdot \hat{\mathbf{n}} \propto 1/r^2$. Show that for radiation by an oscillating electric dipole $\mathbf{p}$,

$$\frac{d\langle\mathbf{L}\rangle}{dt\ d\Omega} = \frac{ik^3}{4\pi}(\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^\star). \tag{21}$$

If the dipole moment $\mathbf{p}$ is real, eq. (21) tells us that no angular momentum is radiated. However, when $\mathbf{p}$ is real, the radiation is linearly polarized and we expect it to carry no angular momentum.

Rather, we need circular (or elliptical) polarization to have radiated angular momentum.

The radiation fields (16) of Prob. 2 are elliptically polarized. Show that in this case the radiated angular momentum distribution is,

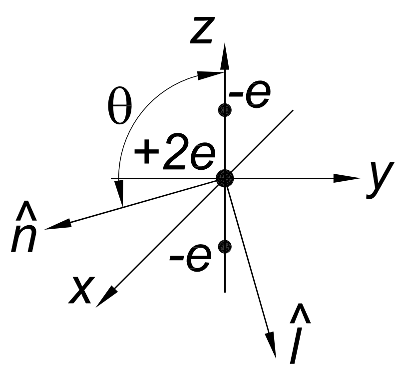
$$\frac{d\langle\mathbf{L}\rangle}{dt\ d\Omega} = -\frac{k^3}{4\pi} p_0^2 \sin\theta\, \hat{\mathbf{l}}, \qquad \text{and} \qquad \frac{d\langle\mathbf{L}\rangle}{dt} = \frac{\langle P\rangle}{\omega}\,\hat{\mathbf{z}}\,. \tag{22}$$

[These relations carry over into the quantum realm where a single (left-hand) circularly polarized photon has $U = \hbar\omega$, $\mathbf{p} = \hbar\mathbf{k}$, and $L = \hbar$.]

For a another view of electromagnetic waves that carry angular momentum, see http://kirkmcd.princeton.edu/examples/oblate_wave.pdf

6. **Oscillating Electric Quadrupole**

An oscillating linear quadrupole consists of charge $2e$ at the origin, and two charges $-e$ each at $z = \pm a \cos \omega t$.



Show that for an observer in the x-z plane at distance r from the origin,

$$\mathbf{E}_{\rm rad} = -4k^3 a^2 e \frac{\sin(2kr - 2\omega t)}{r} \sin\theta \cos\theta \, \hat{\mathbf{l}}, \tag{23}$$

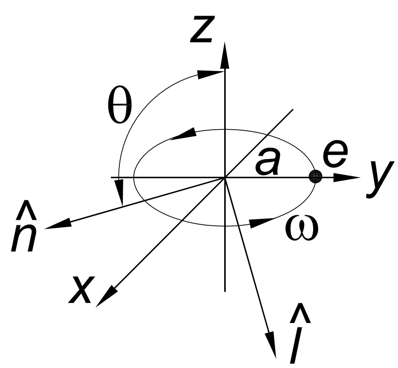
where $\hat{\mathbf{l}} = \hat{\mathbf{y}} \times \hat{\mathbf{n}}$. This radiation is linearly polarized.

Show also that the time-averaged total power is,

$$\langle P \rangle = \frac{16}{15} c k^6 a^4 e^2. \tag{24}$$

7. **Charge with Uniform Circular Motion at Low Velocity**

A single charge e rotates with angular velocity ω in a circle of radius a, centered in the x-y plane. The velocity $v = \omega a$ is much less that c, the speed of light in vacuum.



The time-varying electric-dipole moment of this charge distribution with respect to the origin has magnitude $p = ae$, so from Larmor's formula (prob. 2) we know that the (time-averaged) power in electric dipole radiation is,

$$\langle P_{E1} \rangle = \frac{2a^2e^2\omega^4}{3c^3} . \tag{25}$$

This charge distribution also has a magnetic dipole moment and an electric-quadrupole moment (plus higher moments as well!). Calculate the total radiation fields due to the E1, M1 and E2 moments, as well as the angular distribution of the radiated power and the total radiated power from these three moments. In this pedagogic problem you may ignore the interference between the various moments.

Show, for example, that the part of the radiation due only to the electric-quadrupole moment obeys,

$$\frac{d\,\langle P_{E2} \rangle}{d\Omega} = \frac{a^4e^2\omega^6}{2\pi c^5}(1 - \cos^4\theta), \qquad \langle P_{E2} \rangle = \frac{8a^4e^2\omega^6}{5c^5} . \tag{26}$$

Thus,

$$\frac{\langle P_{E2} \rangle}{\langle P_{E1} \rangle} = \frac{12a^2\omega^2}{5c^2} \propto \frac{v^2}{c^2} . \tag{27}$$

8. **Radiation by a Classical Atom**

a) Consider a classical atom consisting of charge $+e$ fixed at the origin, and charge $-e$ in a circular orbit of radius a. As in Prob. 5, this atom emits electric-dipole radiation $\Rightarrow$ loss of energy $\Rightarrow$ the electron falls into the nucleus!

Calculate the time to fall to the origin supposing the electron's motion is nearly circular at all times (*i.e.*, it spirals into the origin with only a small change in radius per turn). *You may ignore relativistic corrections.*

Show that,

$$t_{\rm fall} = \frac{a^3}{4 r_0^2 c}, \tag{28}$$

where $r_0 = e^2/mc^2$ is the classical electron radius. Evaluate $t_{\rm fall}$ for $a = 5 \times 10^{-9}$ cm = the Bohr radius.

b) The energy loss of part a) can be written as,

$$\frac{dU}{dt} = P_{\rm dipole} \propto \frac{e^2 a \omega^4}{c} = \frac{e^2}{a}\left(\frac{a\omega}{c}\right)^3 \omega \propto U \left(\frac{v}{c}\right)^3 \omega \propto \frac{U}{T}\left(\frac{v}{c}\right)^3, \tag{29}$$

or,

$$\frac{\text{dipole energy loss per revolution}}{\text{energy}} \propto \left(\frac{v}{c}\right)^3. \tag{30}$$

For quadrupole radiation, Prob. 5 shows that,

$$\frac{\text{quadrupole energy loss per revolution}}{\text{energy}} \propto \left(\frac{v}{c}\right)^5. \tag{31}$$

Consider the Earth-Sun system. The motion of the Earth around the Sun causes a quadrupole moment, so gravitational radiation is emitted (although, of course, there is no dipole gravitational radiation since the dipole moment of any system of masses about its center of mass is zero). Estimate the time for the Earth to fall into the Sun due to gravitational-radiation energy loss.

What is the analog of the factor ea^2 that appears in the electrical-quadrupole moment (Prob. 5) for masses m_1 and m_2 that are in circular motion about each other, separated by distance a?

Also note that in Gaussian units the electrical coupling constant k in the force law $F = k e_1 e_2 / r^2$ has been set to 1, but for gravity $k = G$, Newton's constant.

The general-relativity expression for quadrupole radiation in the present example is,[3]

$$P_{\rm G2} = \frac{32}{5}\frac{G}{c^5}\frac{m_1^2 m_2^2}{(m_1 + m_2)^2} a^4 \omega^6. \tag{32}$$

The extra factor of 4 compared to E2 radiation arises because the source term in the gravitational wave equation has a factor of 16π, rather than 4π as for E&M.

[3] P.C. Peters and J. Mathews, Phys. Rev. **131**, 435 (1963), http://kirkmcd.princeton.edu/examples/GR/peters_pr_131_435_63.pdf

9. **Why Doesn't a Steady Current Loop Radiate?**

A steady current in a circular loop presumably involves a large number of electrons in uniform circular motion. Each electron undergoes accelerated motion, and according to Prob. 5 emits radiation. Yet, the current density $\mathbf{J}$ is independent of time in the limit of a continuous current distribution, and therefore does not radiate. How can we reconcile these two views?

The answer must be that the radiation is canceled by destructive interference between the radiation fields of the large number N of electrons that make up the steady current.

Prob. 5 showed that a single electron in uniform circular motion emits electric dipole radiation, whose power is proportional to $(v/c)^4$. But the electric-dipole moment vanishes for two electrons in uniform circular motion at opposite ends of a common diameter; quadrupole radiation is the highest multipole in this case, with power proportional to $(v/c)^6$. It is suggestive that in case of 3 electrons 120° apart in uniform circular motion the (time-dependent) quadrupole moment vanishes, and the highest multipole radiation is octupole. For N electrons evenly spaced around a ring, the highest multipole that radiates in the Nth, and the power of this radiation is proportional to $(v/c)^{2N+2}$. Then, for steady motion with $v/c \ll 1$, the radiated power of a ring of N electrons is very small.

Verify this argument with a detailed calculation.

Since we do not have on record a time-dependent multipole expansion to arbitrary order, return to the basic expression for the vector potential of the radiation fields,

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int \frac{[\mathbf{J}]}{r}\, d\text{Vol}' \approx \frac{1}{cR}\int [\mathbf{J}]\, d\text{Vol}' = \frac{1}{cR}\int \mathbf{J}(\mathbf{r}', t' = t - r/c)\, d\text{Vol}', \tag{33}$$

where R is the (large) distance from the observer to the center of the ring of radius a. For uniform circular motion of N electrons with angular frequency ω, the current density $\mathbf{J}$ is a periodic function with period $T = 2\pi/\omega$, so a Fourier analysis can be made where,

$$\mathbf{J}(\mathbf{r}', t') = \sum_{m=-\infty}^{\infty} \mathbf{J}_m(\mathbf{r}') e^{-im\omega t'}, \tag{34}$$

with,

$$\mathbf{J}_m(\mathbf{r}') = \frac{1}{T}\int_0^T \mathbf{J}(\mathbf{r}', t') e^{im\omega t'}\, dt'. \tag{35}$$

Then,

$$\mathbf{A}(\mathbf{r},t) = \sum_m \mathbf{A}_m(\mathbf{r}) e^{-im\omega t}, \tag{36}$$

etc. The radiated power follows from the Poynting vector,

$$\frac{dP}{d\Omega} = \frac{c}{4\pi} R^2 \left|\mathbf{B}\right|^2 = \frac{c}{4\pi} R^2 \left|\boldsymbol{\nabla}\times\mathbf{A}\right|^2. \tag{37}$$

However, as discussed on p. 181, Lecture 15 of the Notes, one must be careful in going from a Fourier analysis of an amplitude, such as $\mathbf{B}$, to a Fourier analysis of an intensity that depends on the square of the amplitude. Transcribing the argument there to the present case, a Fourier analysis of the average power radiated during one period T can be given as,

$$\begin{aligned} \frac{d\langle P\rangle}{d\Omega} &= \frac{1}{T}\int_0^T \frac{dP}{d\Omega}\,dt = \frac{cR^2}{4\pi T}\int_0^T |\mathbf{B}|^2\,dt = \frac{cR^2}{4\pi T}\int_0^T \mathbf{B}^\star \sum_m \mathbf{B}_m e^{-im\omega t}\,dt \\ &= \frac{cR^2}{4\pi}\sum_m \mathbf{B}_m \frac{1}{T}\int_0^T \mathbf{B}^\star e^{-im\omega t}\,dt = \frac{cR^2}{4\pi}\sum_{m=-\infty}^{\infty} \mathbf{B}_m \mathbf{B}_m^\star \\ &= \frac{cR^2}{2\pi}\sum_{m=0}^{\infty} |\mathbf{B}_m|^2 \equiv \sum_{m=0}^{\infty} \frac{dP_m}{d\Omega}. \end{aligned} \tag{38}$$

That is, the Fourier components of the time-averaged radiated power can be written as,

$$\frac{dP_m}{d\Omega} = \frac{cR^2}{2\pi}|\mathbf{B}_m|^2 = \frac{cR^2}{2\pi}|\boldsymbol{\nabla}\times\mathbf{A}_m|^2 = \frac{cR^2}{2\pi}|imk\hat{\mathbf{n}}\times\mathbf{A}_m|^2, \tag{39}$$

where $k = \omega/c$ and $\hat{\mathbf{n}}$ points from the center of the ring to the observer.

Evaluate the Fourier components of the vector potential and of the radiated power first for a single electron, with geometry as in Prob. 5, and then for N electrons evenly spaced around the ring. It will come as no surprise that a 3-dimensional problem with charges distributed on a ring leads to Bessel functions, and we must be aware of the integral representation,

$$J_m(z) = \frac{i^m}{2\pi}\int_0^{2\pi} e^{im\phi - iz\cos\phi}\,d\phi. \tag{40}$$

Use the asymptotic expansion for large index and small argument,

$$J_m(mx) \approx \frac{(ex/2)^m}{\sqrt{2\pi m}} \qquad (m \gg 1,\ x \ll 1), \tag{41}$$

to verify the suppression of the radiation for large N.

This problem was first posed (and solved via series expansions without explicit mention of Bessel functions) by J.J. Thomson, Phil. Mag. **45**, 673 (1903).[4] He knew that atoms (in what we now call their ground state) don't radiate, and used this calculation to support his model that the electric charge in an atom must be smoothly distributed. This was a classical precursor to the view of a continuous probability distribution for the electron's position in an atom.

Thomson's work was followed shortly by an extensive treatise by G.A. Schott, *Electromagnetic Radiation* (Cambridge U.P., 1912),[5] that included analyses in term of Bessel functions correct for any value of v/c.

These pioneering works were largely forgotten during the following era of nonrelativistic quantum mechanics, and were reinvented around 1945 when interest emerged in relativistic particle accelerators. See Arzimovitch and Pomeranchuk,[6] and Schwinger.[7]

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[4] http://kirkmcd.princeton.edu/examples/EM/thomson_pm_45_673_03.pdf
[5] http://kirkmcd.princeton.edu/examples/EM/schott_radiation_12.pdf
[6] http://kirkmcd.princeton.edu/examples/EM/arzimovitch_jpussr_9_267_45.pdf
[7] http://kirkmcd.princeton.edu/accel/schwinger.pdf

10. **Spherical Cavity Radiation**

Thus far we have only considered waves arising from the **retarded** potentials, and have ignored solutions via the **advanced** potentials. "Advanced" spherical waves converge on the source rather than propagate away – so we usually ignore them.

Inside a cavity, an outward going wave can bounce off the walls and become an inward going wave. Thus, a general description of cavity radiation should include both kinds of waves.

Reconsider your derivation in Prob. 1 above, this time emphasizing the advanced waves, for which $t' = t + r/c$ is the "advanced" time. It suffices to consider only the fields due to oscillation of an electric dipole moment.

If one superimposes outgoing waves due to oscillating dipole $\mathbf{p} = \mathbf{p}_0 e^{-i\omega t}$ at the origin with incoming waves associated with dipole $-\mathbf{p}$, then we can have standing waves – and zero total dipole moment.

Suppose all this occurs inside a spherical cavity with perfectly conducting walls at radius a. Show that the condition for standing waves associated with the virtual electric dipole is,

$$\cot ka = \frac{1}{ka} - ka \qquad \Rightarrow \qquad \omega_{\rm min} = 2.74\frac{c}{a}\,, \tag{42}$$

where $k = \omega/c$. [A quick estimate would be $\lambda_{\rm max} = 2a \Rightarrow \omega_{\rm min} = \pi c/a$.]

By a simple transformation, use your result to find the condition for standing magnetic dipoles waves inside a spherical cavity,

$$\tan ka = ka \qquad \Rightarrow \qquad \omega_{\rm min} = 4.49\frac{c}{a}\,. \tag{43}$$

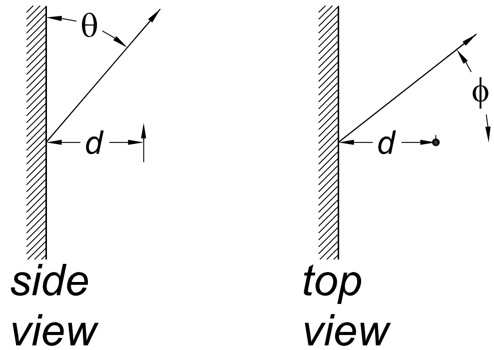
11. **Maximum Energy of a Betatron**

A betatron is a circular device of radius R designed to accelerate electrons (charge e, mass m) via a changing magnetic flux $\dot{\Phi} = \pi R^2 \dot{B}_{\rm ave}$ through the circle.

Deduce the relation between the magnetic field B at radius R and the magnetic field $B_{\rm ave}$ averaged over the area of the circle needed for a betatron to function. Also deduce the maximum energy $\mathcal{E}$ to which an electron could be accelerated by a betatron in terms of B, $\dot{B}_{\rm ave}$ and R.

Hints: The electrons in this problem are relativistic, so it is useful to introduce the factor $\gamma = \mathcal{E}/mc^2$ where c is the speed of light. Recall that Newton's second law has the same form for nonrelativistic and relativistic electrons except that in the latter case the effective mass is γm. Recall also that for circular motion the rest frame acceleration is γ^2 times that in the lab frame.

12. a) An oscillating electric dipole of angular frequency ω is located at distance $d \ll \lambda$ away from a perfectly conducting plane. The dipole is oriented parallel to the plane, as shown below.



Show that the power radiated in the direction (θ, ϕ) is,

$$\frac{dP}{d\Omega} = 4A \sin^2\theta \sin^2\Delta, \tag{44}$$

where,

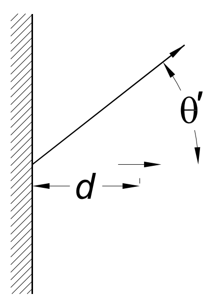
$$\Delta = \frac{2\pi\lambda}{d} \sin\theta \cos\phi, \tag{45}$$

and the power radiated by the dipole alone is,

$$\frac{dP}{d\Omega} = A \sin^2\theta. \tag{46}$$

Sketch the shape of the radiation pattern for $d = \lambda/2$ and $d = \lambda/4$.

b) Suppose instead that the dipole was oriented perpendicular to the conducting plane.



Show that the radiated power in this case is,

$$\frac{dP}{d\Omega} = 4A \sin^2\theta' \cos^2\Delta, \tag{47}$$

where,

$$\Delta = \frac{2\pi\lambda}{d} \cos\theta'. \tag{48}$$

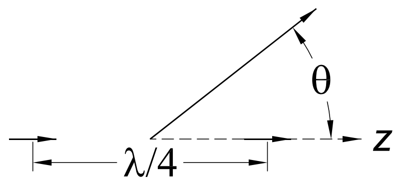
In parts a) and b), the polar angles θ and θ' are measured with respect to the axes of the dipoles.

c) Repeat parts a) and b) for a magnetic dipole oscillator in the two orientations.

13. **Force on an Antenna Array**

In an array of antennas, their relative phases can be adjusted as well as their relative spacing, which leads to additional freedom to shape the radiation pattern.

Consider two short, center-fed linear antennas of length $L \ll \lambda$, peak current I_0 and angular frequency ω, as discussed on p. 191, Lecture 16 of the Notes. The axes of the antennas are collinear, their centers are $\lambda/4$ apart, and the currents have a 90° phase difference.



Show that the angular distribution of the radiated power is,

$$\frac{dU}{dt\, d\Omega} = \frac{\omega^2}{16\pi c^3} I_0^2 L^2 \sin^2\theta \left[1 + \sin\left(\frac{\pi}{2}\cos\theta\right)\right]. \qquad (49)$$

Unlike the radiation patterns of previous examples, this is not symmetric about the plane $z = 0$. Therefore, this antenna array emits nonzero momentum $\mathbf{P}_{\rm rad}$. As a consequence, there is a net reaction force $\mathbf{F} = -d\mathbf{P}_{\rm rad}/dt$. Show that,

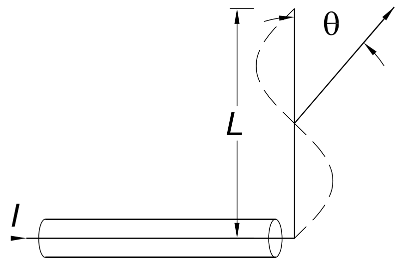
$$\mathbf{F} = -\frac{1}{c}\frac{dU}{dt}\left(1 - \frac{\pi^2}{12}\right)\hat{\mathbf{z}}. \qquad (50)$$

A variant on this problem is at
http://kirkmcd.princeton.edu/examples/endfire.pdf

14. a) Consider a full-wave “end-fire” antenna whose current distribution (along the z axis) is,

$$I(z) = I_0 \sin \frac{2\pi z}{L} e^{-i\omega t}, \qquad (-L/2 < z < L/2), \tag{51}$$

where $L = \lambda = 2\pi c/\omega$.



Use the result of p. 182, Lecture 15 of the Notes to calculate the radiated power “exactly”. Note that the real part of the integral vanishes, so you must evaluate the imaginary part. Show that,

$$\frac{dP}{d\Omega} = \frac{I_0^2}{2\pi c} \frac{\sin^2(\pi \cos\theta)}{\sin^2\theta}. \tag{52}$$

Sketch the radiation pattern.

Use tricks like,

$$\frac{1}{1-u^2} = \frac{1}{1+u} + \frac{1}{1-u} \tag{53}$$

to show that the total radiated power is,

$$P = \int \frac{dP}{d\Omega} \tag{54}$$

(*c.f.*, Abramowitz and Stegun, pp. 231, 244.)
http://kirkmcd.princeton.edu/examples/EM/abramowitz_and_stegun.pdf

b) Calculate the lowest-order nonvanishing multipole radiation. You may need the fact that,

$$\int z^2 \cos z \, dz = (z^2 - 2) \sin z + 2z \cos z. \tag{55}$$

Show that to this order,

$$P = \frac{8\pi^2}{15} \frac{I_0^2}{2c} = 5.26 \frac{I_0^2}{2c}. \tag{56}$$

which gives a sense of the accuracy of the multipole expansion.

15. **Scattering Off a Conducting Sphere**

Calculate the scattering cross section for plane electromagnetic waves of angular frequency ω incident on a perfectly conducting sphere of radius a when the wavelength obeys $\lambda \gg a$ ($ka \ll 1$).

Note that both electric and magnetic dipole moments are induced. Inside the sphere, $\mathbf{B}_{\rm total}$ must vanish. Surface currents are generated such that $\mathbf{B}_{\rm induced} = -\mathbf{B}_{\rm incident}$, and because of the long wavelength, these fields are essentially uniform over the sphere. (*c.f.*, Set 4, Prob. 8a).

Show that,

$$\mathbf{E}_{\rm scat} = a^3 k^2 \frac{e^{i(kr-\omega t)}}{r} \left[(\mathbf{E}_0 \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}} - \frac{1}{2}(\mathbf{B}_0 \times \hat{\mathbf{n}} \right], \qquad (57)$$

where $\hat{\mathbf{n}}$ is along the vector $\mathbf{r}$ that points from the center of the sphere to the distant observer.

Suppose that the incident wave propagates in the $+z$ direction, and the electric field is linearly polarized along direction $\hat{\mathbf{l}}$, so $\mathbf{E}_0 = E_0\hat{\mathbf{l}}$ and $\hat{\mathbf{l}} \cdot \hat{\mathbf{z}} = 0$. Show that in this case the scattering cross section can be written as,

$$\frac{d\sigma}{d\Omega} = a^6 k^4 \left[\left(1 - \frac{\hat{\mathbf{n}} \cdot \hat{\mathbf{z}}}{2} \right)^2 - \frac{3}{4} (\hat{\mathbf{l}} \cdot \hat{\mathbf{n}})^2 \right]. \qquad (58)$$

Consider an observer in the x-z plane to distinguish between the cases of electric polarization parallel and perpendicular to the scattering plane to show that,

$$\frac{d\sigma_\parallel}{d\Omega} = a^6 k^4 \left(\frac{1}{2} - \cos\theta \right)^2, \qquad \frac{d\sigma_\perp}{d\Omega} = a^6 k^4 \left(1 - \frac{\cos\theta}{2} \right)^2. \qquad (59)$$

Then, for an unpolarized incident wave, show that,

$$\frac{d\sigma}{d\Omega} = a^6 k^4 \left[\frac{5}{8}(1 + \cos^2\theta) - \cos\theta \right], \qquad (60)$$

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega = \frac{10\pi}{3} a^6 k^4. \qquad (61)$$

Sketch the angular distribution (60). Note that,

$$\frac{d\sigma(180^\circ)}{d\Omega} \bigg/ \frac{d\sigma(0^\circ)}{d\Omega} = 9, \qquad (62)$$

so the sphere reflects much more backwards than it radiates forwards.

Is there any angle θ for which the scattered radiation is linearly polarized for unpolarized incident waves?

Solutions

1. The suggested approach is to calculate the retarded potentials and then take derivatives to find the fields. The retarded scalar and vector potentials ϕ and $\mathbf{A}$ are given by,

$$\phi(\mathbf{x},t) = \int \frac{\rho(\mathbf{x}', t - R/c)}{R}\, d^3\mathbf{x}', \qquad \text{and} \qquad \mathbf{A}(\mathbf{x},t) = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{x}', t - R/c)}{R}\, d^3\mathbf{x}', \quad (63)$$

where ρ and $\mathbf{J}$ are the charge and current densities, respectively, and $R = |\mathbf{x} - \mathbf{x}'|$.

In the present case, we assume the wire remains neutral when the current flows (compare Prob. 3, Set 4). Then the scalar potential vanishes. For the vector potential, we see that only the component A_z will be nonzero. Also, $\mathbf{J}\, d^3\mathbf{x}'$ can be rewritten as $I\, dz$ for current in a wire along the z-axis. For an observer at $(r, 0, 0)$ and a current element at $(0, 0, z)$, we have $R = \sqrt{r^2 + z^2}$. Further, the condition that I is nonzero only for time $t > 0$ implies that it contributes to the fields only for z such that $(ct)^2 > R^2 = r^2 + z^2$. That is, we need to evaluate the integral only for,

$$|z| < z_0 \equiv \sqrt{(ct)^2 - r^2}, \quad (64)$$

which must be positive to have physical significance. Altogether,

$$\begin{aligned} A_z(r,0,0,t) &= \frac{\alpha}{c}\int_{-z_0}^{z_0} \left(\frac{t}{\sqrt{r^2+z^2}} - \frac{1}{c}\right) dz = \frac{\alpha}{c}\left(t \ln\frac{ct + z_0}{ct - z_0} - \frac{2z_0}{c}\right) \\ &= \frac{2\alpha}{c}\left(t \ln\frac{z_0 + ct}{r} - \frac{z_0}{c}\right). \end{aligned} \quad (65)$$

[The two forms tend to arise depending on whether or not one notices that the integrand is even in z.]

The magnetic field is obtained via $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. Since only A_z is nonzero, the only nonzero component of $\mathbf{B}$ is,

$$B_\phi = -\frac{\partial A_z}{\partial r} = \frac{2\alpha z_0}{c^2 r}\,. \quad (66)$$

[Some chance of algebraic error in this step!]

The only nonzero component of the electric field is,

$$E_z = -\frac{1}{c}\frac{\partial A_z}{\partial t} = -\frac{2\alpha}{c^2}\ln\frac{z_0 + ct}{r}. \quad (67)$$

For long times, $ct \gg r$, $\Rightarrow z_0 \approx ct$, and the fields become,

$$B_\phi \approx \frac{2\alpha t}{cr} = \frac{2I(t)}{cr} = B_0(t), \qquad E_z \approx -\frac{2\alpha}{c^2}\ln\frac{2ct}{r} = -B_0\frac{r}{ct}\ln\frac{2ct}{r} \ll B_0, \quad (68)$$

where $B_0(t) = 2I(t)/cr$ is the instantaneous magnetic field corresponding to current I(t). That is, we recover the magnetostatic limit at large times.

For short times, $ct = r + \epsilon$ with $\epsilon \ll r$, after the fields first become nonzero we have,

$$z_0 = \sqrt{2r\epsilon + \epsilon^2} \approx \sqrt{2r\epsilon}, \quad (69)$$

so,

$$B_\phi \approx \frac{2\alpha}{c^2}\sqrt{\frac{2\epsilon}{r}}, \qquad \text{and} \qquad E_z \approx -\frac{2\alpha}{c^2}\ln\frac{r+\epsilon+\sqrt{2r\epsilon}}{r} \approx -\frac{2\alpha}{c^2}\sqrt{\frac{2\epsilon}{r}} = -B_\phi. \quad (70)$$

In this regime, the fields have the character of radiation, with **E** and **B** of equal magnitude, mutually orthogonal, and both orthogonal to the line of sight to the closest point on the wire. (Because of the cylindrical geometry the radiation fields do not have $1/r$ dependence – which holds instead for static fields.)

In sum, the fields build up from zero only after time $ct = r$. The initial fields propagate outwards at the speed of light and have the character of cylindrical waves. But at a fixed r, the electric field dies out with time, and the magnetic field approaches the instantaneous magnetostatic field due to the current in the wire.

Of possible amusement is a direct calculation of the vector potential for the case of a constant current I_0.

First, from Ampère's law we know that $B_\phi = 2I_0/cr = -\partial A_z/\partial r$, so we have that,

$$A_z = -\frac{2I_0}{c}\ln r + \text{const.} \quad (71)$$

Whereas, if we use the integral form for the vector potential we have,

$$\begin{aligned} A_z(r,0,0) &= \frac{1}{c}\int_{-\infty}^{\infty}\frac{I_0\, dz}{\sqrt{r^2+z^2}} = \frac{2I_0}{c}\int_0^{\infty}\frac{dz}{\sqrt{r^2+z^2}} \\ &= -\frac{2I_0}{c}\ln r + \lim_{z\to\infty}\ln(z+\sqrt{z^2+r^2}). \end{aligned} \quad (72)$$

Only by ignoring the last term, which does not depend on r for a long wire, do we recover the "elementary" result.

2. The expansion,

$$r = R - \mathbf{r}' \cdot \hat{\mathbf{n}} + ... \tag{73}$$

implies that the retarded time derivative of the polarization vector is,

$$\begin{aligned} [\dot{\mathbf{p}}] &= \dot{\mathbf{p}}(\mathbf{r}', t' = t - r/c) \approx -i\omega \mathbf{p}_\omega(\mathbf{r}')\, e^{-i\omega(t - R/c + \mathbf{r}'\cdot\hat{\mathbf{n}}/c)} = -i\omega\, e^{i(kR-\omega t)} \mathbf{p}_\omega(\mathbf{r}')\, e^{-ik\mathbf{r}'\cdot\hat{\mathbf{n}}} \\ &\approx -i\omega\, e^{i(kR-\omega t)} \mathbf{p}_\omega(\mathbf{r}')(1 - ik\mathbf{r}' \cdot \hat{\mathbf{n}}), \end{aligned} \tag{74}$$

where $k = \omega/c$. Likewise,

$$\frac{1}{r} \approx \frac{1}{R}\left(1 + \frac{\mathbf{r}' \cdot \hat{\mathbf{n}}}{R}\right). \tag{75}$$

Then, the retarded vector potential can be written (in the Lorenz gauge) as,

$$\mathbf{A}^{(L)} = \frac{1}{c}\int \frac{[\dot{\mathbf{p}}]}{r}\, d\text{Vol}' \approx -i\omega\, \frac{e^{i(kR-\omega t)}}{cR} \int \mathbf{p}_\omega(\mathbf{r}') \left[1 + \mathbf{r}' \cdot \hat{\mathbf{n}}\left(\frac{1}{R} - ik\right) + ...\right] d\text{Vol}', \tag{76}$$

The electric-dipole ($E1$) approximation is to keep only the first term of eq. (76),

$$\mathbf{A}_{E1}^{(L)} = -i\omega\, \frac{e^{i(kR-\omega t)}}{cR} \int \mathbf{p}_\omega(\mathbf{r}')\, d\text{Vol}' \equiv -ik\, \frac{e^{i(kR-\omega t)}}{R}\mathbf{P} \qquad \text{(Lorenz gauge)}. \tag{77}$$

We obtain the magnetic field by taking the curl of eq. (77). The curl operation with respect to the observer acts only on the distance R. In particular,

$$\boldsymbol{\nabla} R = \frac{\mathbf{R}}{R} = \hat{\mathbf{n}}. \tag{78}$$

Hence,

$$\begin{aligned} \mathbf{B}_{E1} &= \boldsymbol{\nabla} \times \mathbf{A}_{E1}^{(L)} = -ik\boldsymbol{\nabla}\frac{e^{i(kR-\omega t)}}{R} \times \mathbf{P} = -ik\frac{e^{i(kR-\omega t)}}{R}\left(ik\hat{\mathbf{n}} - \frac{\hat{\mathbf{n}}}{R}\right) \times \mathbf{P} \\ &= k^2\frac{e^{i(kR-\omega t)}}{R}\left(1 + \frac{i}{kR}\right)\hat{\mathbf{n}} \times \mathbf{P}. \end{aligned} \tag{79}$$

The 4$^{\text{th}}$ Maxwell equation in vacuum tells us that,

$$\boldsymbol{\nabla} \times \mathbf{B}_{E1} = \frac{1}{c}\frac{\partial \mathbf{E}_{E1}}{\partial t} = -ik\mathbf{E}_{E1}. \tag{80}$$

Hence,

$$\begin{aligned} \mathbf{E}_{E1} &= \frac{i}{k}\boldsymbol{\nabla} \times \mathbf{B}_{E1} = \boldsymbol{\nabla} \times \left[e^{i(kR-\omega t)}\left(\frac{ik}{R^2} - \frac{1}{R^3}\right)\mathbf{R} \times \mathbf{P}\right] \\ &= \boldsymbol{\nabla} e^{i(kR-\omega t)}\left(\frac{ik}{R^2} - \frac{1}{R^3}\right) \times (\mathbf{R} \times \mathbf{P}) + e^{i(kR-\omega t)}\left(\frac{ik}{R^2} - \frac{1}{R^3}\right)\boldsymbol{\nabla} \times (\mathbf{R} \times \mathbf{P}) \\ &= e^{i(kR-\omega t)}\left[-\frac{k^2}{R} - 3\left(\frac{ik}{R^2} - \frac{1}{R^3}\right)\right]\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{P}) - 2\mathbf{P}e^{i(kR-\omega t)}\left(\frac{ik}{R^2} - \frac{1}{R^3}\right) \\ &= k^2\frac{e^{i(kR-\omega t)}}{R}\hat{\mathbf{n}} \times (\mathbf{P} \times \hat{\mathbf{n}}) + e^{i(kR-\omega t)}\left(\frac{ik}{R^2} - \frac{1}{R^3}\right)[\mathbf{P} - 3(\mathbf{P} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}]. \end{aligned} \tag{81}$$

We could also deduce the electric field from the general relation,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\boldsymbol{\nabla} V + ik\mathbf{A}. \tag{82}$$

For this, we need to know the scalar potential $V^{(L)}$, which we can deduce from the Lorenz gauge condition:

$$\boldsymbol{\nabla}\cdot\mathbf{A}^{(L)} + \frac{1}{c}\frac{\partial V^{(L)}}{\partial t} = 0. \tag{83}$$

For an oscillatory source this becomes,

$$V^{(L)} = -\frac{i}{k}\boldsymbol{\nabla}\cdot\mathbf{A}^{(L)}. \tag{84}$$

In the electric-dipole approximation (77) this yields,[8]

$$V_{E1}^{(L)} = e^{i(kR-\omega t)}\left(\frac{1}{R^2} - \frac{ik}{R}\right)(\mathbf{P}\cdot\hat{\mathbf{n}}) \qquad \text{(Lorenz gauge)}. \tag{85}$$

For small R the scalar potential is that of a time-varying dipole,

$$V_{E1,\mathrm{near}}^{(L)} \approx \frac{\mathbf{P}\cdot\hat{\mathbf{n}}}{R^2}e^{-i\omega t}. \tag{86}$$

The electric field is given by,

$$\begin{aligned}
\mathbf{E}_{E1} &= -\boldsymbol{\nabla} V_{E1}^{(L)} + ik\mathbf{A}_{E1}^{(L)} \\
&= (\mathbf{P}\cdot\mathbf{R})\boldsymbol{\nabla} e^{i(kR-\omega t)}\left(\frac{ik}{R^2} - \frac{1}{R^3}\right) + e^{i(kR-\omega t)}\left(\frac{ik}{R^2} - \frac{1}{R^3}\right)\boldsymbol{\nabla}(\mathbf{P}\cdot\mathbf{R}) + k^2\frac{e^{i(kR-\omega t)}}{R}\mathbf{P} \\
&= e^{i(kR-\omega t)}\left[-\frac{k^2}{R} - 3\left(\frac{ik}{R^2} - \frac{1}{R^3}\right)\right](\mathbf{P}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + e^{i(kR-\omega t)}\left(\frac{ik}{R^2} - \frac{1}{R^3}\right)\mathbf{P} \\
&\quad + k^2\frac{e^{i(kR-\omega t)}}{R}\mathbf{P} \\
&= k^2\frac{e^{i(kR-\omega t)}}{R}\hat{\mathbf{n}}\times(\mathbf{P}\times\hat{\mathbf{n}}) + e^{i(kR-\omega t)}\left(\frac{ik}{R^2} - \frac{1}{R^3}\right)[\mathbf{P} - 3(\mathbf{P}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}],
\end{aligned} \tag{87}$$

as before. The angular distribution in the far field (for which the radial dependence is $1/R$) is $\hat{\mathbf{n}}\times(\mathbf{P}\times\hat{\mathbf{n}}) = \mathbf{P} - (\mathbf{P}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}$. The isotropic term $\mathbf{P}$ is due to the vector potential, while the variable term $-(\mathbf{P}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}$ is due to the scalar potential and is purely radial. Spherical waves associated with a scalar potential must be radial (longitudinal), but the transverse character of electromagnetic waves in the far field does not imply the absence of a contribution of the scalar potential; the latter is needed (in the Lorenz gauge) to cancel to radial component of the waves from the vector potential.

[8] Equation (85) is not simply the electrostatic-dipole potential times a spherical wave because the retarded positions at time t of the two charges of a point dipole correspond to two different retarded times t'. For a calculation of the retarded scalar potential via $V^{(L)} = \int_{\mathrm{Vol}}[\rho]/r$, see sec. 11.1.2 of *Introduction to Electrodynamics* by D.J. Griffiths.

We could also work in the Coulomb gauge, meaning that we set $\boldsymbol{\nabla} \cdot \mathbf{A}^{(C)} = 0$. Recall (Lecture 15, p. 174) that the "wave" equations for the potentials in the Coulomb gauge are,

$$\boldsymbol{\nabla}^2 V^{(C)} = -4\pi\rho, \tag{88}$$

$$\boldsymbol{\nabla}^2 \mathbf{A}^{(C)} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}^{(C)}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \boldsymbol{\nabla} V^{(C)}}{\partial t}. \tag{89}$$

Equation (88) is the familiar Poisson equation of electrostatics, so the scalar potential is just the "instantaneous" electric-dipole potential,

$$V_{E1}^{(C)} = \frac{\mathbf{P}\cdot\hat{\mathbf{n}}}{R^2} e^{-i\omega t} \qquad \text{(Coulomb gauge)}. \tag{90}$$

One way to deduce the Coulomb-gauge vector potential is via eq. (82),

$$\begin{aligned}\mathbf{A}_{\mathrm{E1}}^{(C)} &= -\frac{i}{k}\mathbf{E}_{\mathrm{E1}} - \frac{i}{k}\boldsymbol{\nabla} V_{\mathrm{E1}}^{(C)} \\ &= -ik\frac{e^{i(kR-\omega t)}}{R}\hat{\mathbf{n}}\times(\mathbf{P}\times\hat{\mathbf{n}}) + \left[\frac{e^{i(kR-\omega t)}}{R^2} + \frac{i(e^{i(kR-\omega t)} - e^{-i\omega t})}{kR^3}\right][\mathbf{P} - 3(\mathbf{P}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}] \\ &\equiv \mathbf{A}_{\mathrm{far}}^{(C)} + \mathbf{A}_{\mathrm{near}}^{(C)} \qquad \text{(Coulomb gauge)}.\end{aligned} \tag{91}$$

We learn that the far-zone, Coulomb gauge vector potential (*i.e.*, the part of the vector potential that varies as $1/R$) is purely transverse, and can be written as

$$\mathbf{A}_{\mathrm{far}}^{(C)} = -ik\frac{e^{i(kR-\omega t)}}{R}\hat{\mathbf{n}}\times(\mathbf{P}\times\hat{\mathbf{n}}) \qquad \text{(Coulomb gauge)}. \tag{92}$$

Because the radiation part of the Coulomb-gauge vector potential is transverse, the Coulomb gauge is sometimes called the "transverse" gauge.

The Coulomb-gauge scalar potential is negligible in the far zone, and we can say that the radiation fields are entirely due to the far-zone, Coulomb-gauge vector potential. That is,

$$\mathbf{E}_{\mathrm{far}} = ik\mathbf{A}_{\mathrm{far}}^{(C)} = k^2\frac{e^{i(kR-\omega t)}}{R}\hat{\mathbf{n}}\times(\mathbf{P}\times\hat{\mathbf{n}}), \tag{93}$$

$$\mathbf{B}_{\mathrm{far}} = \boldsymbol{\nabla}\times\mathbf{A}_{\mathrm{far}}^{(C)} = ik\times\mathbf{A}_{\mathrm{far}}^{(C)} = k^2\frac{e^{i(kR-\omega t)}}{R}\hat{\mathbf{n}}\times\mathbf{P}. \tag{94}$$

It is possible to choose gauges for the electromagnetic potentials such that some of their components appear to propagate at any velocity v, as discussed by J.D. Jackson, Am. J. Phys. **70***, 917 (2002) and by K.-H. Yang, Am. J. Phys.* **73***, 742 (2005).*[9] *The*

[9] http://kirkmcd.princeton.edu/examples/EM/jackson_ajp_70_917_02.pdf
http://kirkmcd.princeton.edu/examples/EM/yang_ajp_73_742_05.pdf

potentials $\mathbf{A}^{(v)}$ *and* $V^{(v)}$ *in the so-called velocity gauge with the parameter* v *obey the gauge condition,*

$$\boldsymbol{\nabla}\cdot\mathbf{A}^{(v)} + \frac{c}{v^2}\frac{\partial V^{(v)}}{\partial t} = 0. \tag{95}$$

The scalar potential $V^{(v)}$ *is obtained by replacing the speed of light* c *in the Lorenz-gauge scalar potential by* v*. Equivalently, we replace the wave number* $k = \omega/c$ *by* $k' = \omega/v$*. Thus, from eq. (85) we find,*

$$V_{E1}^{(v)} = -e^{i(k'R-\omega t)}\left(\frac{ik'}{R^2} - \frac{1}{R^3}\right)(\mathbf{P}\cdot\mathbf{R}) \qquad \textit{(velocity gauge)}. \tag{96}$$

Then, as in eq. (87) we obtain

$$\begin{aligned} -\boldsymbol{\nabla}V_{E1}^{(v)} &= e^{i(k'R-\omega t)}\left[-\frac{k'^2}{R} - 3\left(\frac{ik'}{R^2} - \frac{1}{R^3}\right)\right](\mathbf{P}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + e^{i(kR-\omega t)}\left(\frac{ik'}{R^2} - \frac{1}{R^3}\right)\mathbf{P} \\ &= -k'^2\frac{e^{i(k'R-\omega t)}}{R}(\mathbf{P}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + e^{i(k'R-\omega t)}\left(\frac{ik'}{R^2} - \frac{1}{R^3}\right)[\mathbf{P} - 3(\mathbf{P}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}]. \end{aligned} \tag{97}$$

The vector potential in the v*-gauge can be obtained from eq. (82) as,*

$$\begin{aligned} \mathbf{A}_{E1}^{(v)} &= -\frac{i}{k}\mathbf{E}_{E1} - \frac{i}{k}\boldsymbol{\nabla}V_{E1}^{(v)} \\ &= -ik\frac{e^{i(kR-\omega t)}}{R}\hat{\mathbf{n}}\times(\mathbf{P}\times\hat{\mathbf{n}}) + e^{i(kR-\omega t)}\left(\frac{1}{R^2} + \frac{i}{kR^3}\right)[\mathbf{P} - 3(\mathbf{P}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}] \\ &\quad -i\frac{k'^2}{k}\frac{e^{i(k'R-\omega t)}}{R}(\mathbf{P}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} - e^{i(k'R-\omega t)}\left(\frac{k'}{kR^2} + \frac{i}{kR^3}\right)[\mathbf{P} - 3(\mathbf{P}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}]. \end{aligned} \tag{98}$$

This vector potential includes terms that propagate with velocity v *both in the near and far zones. When* $v = c$*, then* $k' = k$ *and the velocity-gauge vector potential (98) reduces to the Lorenz-gauge potential (77); and when* $v \to \infty$*, then* $k' = 0$ *and the velocity-gauge vector potential reduces to the Coulomb-gauge potential (91).*[10]

Turning to the question of energy flow, we calculate the Poynting vector,

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}, \tag{99}$$

where we use the real parts of the fields (79) and (81),

$$\mathbf{E} = k^2 P\hat{\mathbf{n}}\times(\hat{\mathbf{P}}\times\hat{\mathbf{n}})\frac{\cos(kR-\omega t)}{R} + P[3\hat{\mathbf{P}}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} - \hat{\mathbf{P}}]\left[\frac{\cos(kR-\omega t)}{R^3} + \frac{k\sin(kR-\omega t)}{R^2}\right], \tag{100}$$

[10]Thanks to J.D Jackson and K.-H. Yang for discussions of the Coulomb gauge and the velocity gauge. See also, J.D. Jackson and L.B. Okun, *Historical roots of gauge invariance*, Rev. Mod. Phys. **73**, 663 (2001), http://kirkmcd.princeton.edu/examples/EM/jackson_rmp_73_663_01.pdf, and http://kirkmcd.princeton.edu/examples/gauge.pdf

$$\mathbf{B} = k^2 P(\hat{\mathbf{n}} \times \hat{\mathbf{P}}) \left[\frac{\cos(kR - \omega t)}{R} - \frac{\sin(kR - \omega t)}{kR^2} \right]. \tag{101}$$

The Poynting vector contains six terms, some of which do not point along the radial vector $\hat{\mathbf{n}}$:

$$\begin{aligned} \mathbf{S} &= \frac{c}{4\pi} \Bigg\{ k^4 P^2 [\hat{\mathbf{n}} \times (\hat{\mathbf{P}} \times \hat{\mathbf{n}})] \times (\hat{\mathbf{n}} \times \hat{\mathbf{P}}) \left[\frac{\cos^2(kR - \omega t)}{R^2} - \frac{\cos(kR - \omega t)\sin(kR - \omega t)}{kR^3} \right] \\ &\quad + k^2 P^2 [3(\hat{\mathbf{P}} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} - \hat{\mathbf{P}}] \times (\hat{\mathbf{n}} \times \hat{\mathbf{P}}) \left[\frac{\cos^2(kR - \omega t) - \sin^2(kR - \omega t)}{R^4} \right. \\ &\qquad \left. + \cos(kR - \omega t)\sin(kR - \omega t) \left(\frac{k}{R^3} - \frac{1}{kR^5} \right) \right] \Bigg\} \\ &= \frac{c}{4\pi} \Bigg\{ k^4 P^2 \sin^2\theta \hat{\mathbf{n}} \left[\frac{\cos^2(kR - \omega t)}{R^2} - \frac{\cos(kR - \omega t)\sin(kR - \omega t)}{kR^3} \right] \\ &\quad + k^2 P^2 [4\cos\theta \hat{\mathbf{P}} + (3\cos^2\theta - 1)\hat{\mathbf{n}}] \left[\frac{\cos^2(kR - \omega t) - \sin^2(kR - \omega t)}{R^4} \right. \\ &\qquad \left. + \cos(kR - \omega t)\sin(kR - \omega t) \left(\frac{k}{R^3} - \frac{1}{kR^5} \right) \right] \Bigg\}, \end{aligned} \tag{102}$$

where θ is the angle between vectors $\hat{\mathbf{n}}$ and $\mathbf{P}$. As well as the expected radial flow of energy, there is a flow in the direction of the dipole moment $\mathbf{P}$. Since the product $\cos(kR - \omega t)\sin(kR - \omega t)$ can be both positive and negative, part of the energy flow is inwards at times, rather than outwards as expected for pure radiation.

However, we obtain a simple result if we consider only the time-average Poynting vector, $\langle \mathbf{S} \rangle$. Noting that $\langle \cos^2(kR - \omega t) \rangle = \left\langle \sin^2(kR - \omega t) \right\rangle = 1/2$ and $\langle \cos(kR - \omega t)\sin(kR - \omega t) \rangle = (1/2) \langle \sin 2(kR - \omega t) \rangle = 0$, eq (102) leads to

$$\langle \mathbf{S} \rangle = \frac{ck^4 P^2 \sin^2\theta}{8\pi R^2} \hat{\mathbf{n}}. \tag{103}$$

The time-average Poynting vector is purely radially outwards, and falls off as $1/R^2$ at all radii, as expected for a flow of energy that originates in the oscillating point dipole (which must be driven by an external power source). The time-average angular distribution $d\langle P \rangle / d\Omega$ of the radiated power is related to the Poynting vector by

$$\frac{d\langle P \rangle}{d\Omega} = R^2 \hat{\mathbf{n}} \cdot \langle \mathbf{S} \rangle = \frac{ck^4 P^2 \sin^2\theta}{8\pi} = \frac{P^2 \omega^4 \sin^2\theta}{8\pi c^3}, \tag{104}$$

which is the expression often quoted for dipole radiation in the far zone. Here we see that this expression holds in the near zone as well.

We conclude that radiation, as measured by the time-average Poynting vector, exists in the near zone as well as in the far zone.

Our considerations of an oscillating electric dipole can be extended to include an oscillating magnetic dipole by noting that if $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ are solutions to Maxwell's equations in free space (*i.e.*, where the charge density ρ and current density $\mathbf{J}$ are zero), then the **dual** fields,

$$\mathbf{E}'(\mathbf{r},t) = -\mathbf{B}(\mathbf{r},t), \qquad \mathbf{B}'(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t), \tag{105}$$

are solutions also. The Poynting vector is the same for the dual fields as for the original fields,

$$\mathbf{S}' = \frac{c}{4\pi}\mathbf{E}' \times \mathbf{B}' = -\frac{c}{4\pi}\mathbf{B} \times \mathbf{E} = \mathbf{S}. \tag{106}$$

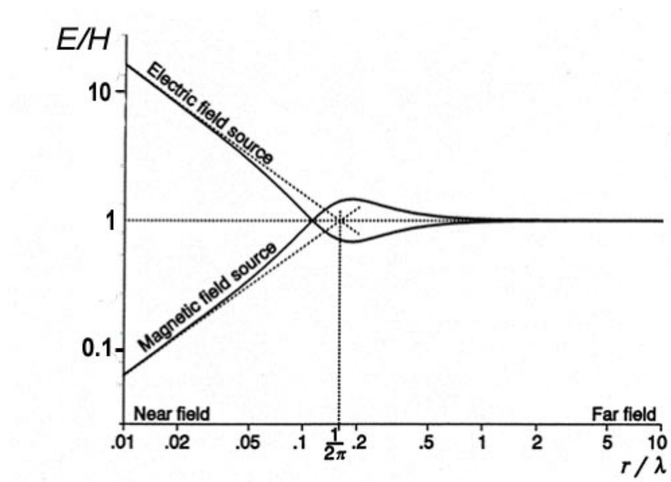
Taking the dual of fields (10)-(11), we find the fields,

$$\mathbf{E}' = \mathbf{E}_{\rm M1} = -k^2 \frac{e^{i(kR-\omega t)}}{R}\left(1 + \frac{i}{kR}\right)\hat{\mathbf{n}} \times \mathbf{M}, \tag{107}$$

$$\mathbf{B}' = \mathbf{B}_{\rm M1} = k^2 \frac{e^{i(kR-\omega t)}}{R}\left\{\hat{\mathbf{n}} \times (\mathbf{M} \times \hat{\mathbf{n}}) + [3(\hat{\mathbf{n}} \cdot \mathbf{M})\hat{\mathbf{n}} - \mathbf{M}]\left(\frac{1}{k^2R^2} - \frac{i}{kR}\right)\right\}. \tag{108}$$

which are also solutions to Maxwell's equations. These are the fields of an oscillating point magnetic dipole, whose peak magnetic moment is $\mathbf{M}$. In the near zone, the magnetic field (108) looks like that of a (magnetic) dipole.

While the fields of eqs. (10)-(11) are not identical to those of eqs. (107)-(108), the Poynting vectors are the same in the two cases. Hence, the time-average Poynting vector, and also the angular distribution of the time-averaged radiated power are the same in the two cases. The radiation of a point electric dipole is the same as that of a point magnetic dipole (assuming that $\mathbf{M} = \mathbf{P}$), both in the near and in the far zones. Measurements of only the intensity of the radiation could not distinguish the two cases.



However, if measurements were made of both the electric and magnetic fields, then the near zone fields of an oscillating electric dipole, eqs. (10)-(11), would be found to be quite different from those of a magnetic dipole, eqs. (107)-(108). This is illustrated in the figure on the previous page, which plots the ratio $E/H = E/B$ of the magnitudes of the electric and magnetic fields as a function of the distance r from the center of the dipoles.

To distinguish between the cases of electric and magnetic dipole radiation, it suffices to measure only the polarization (*i.e.*, the direction, but not the magnitude) of either the electric of the magnetic field vectors.

3. The rotating dipole $\mathbf{p}$ can be thought of as two oscillating linear dipoles oriented 90° apart in space, and phased 90° apart in time. This is conveniently summarized in complex vector notation:

$$\mathbf{p} = p_0(\hat{\mathbf{x}} + i\hat{\mathbf{y}})\, e^{-i\omega t}, \tag{109}$$

for a rotation from the $+\hat{\mathbf{x}}$ axis towards the $+\hat{\mathbf{y}}$ axis. Thus,

$$[\ddot{\mathbf{p}}] = \ddot{\mathbf{p}}(t' = t - r/c) = -\omega^2 p_0(\hat{\mathbf{x}} + i\hat{\mathbf{y}})\, e^{i(kr-\omega t)}. \tag{110}$$

The radiation fields of this oscillating dipole are given by,

$$\begin{aligned} \mathbf{B}_{\rm rad} &= \frac{[\ddot{\mathbf{p}}] \times \hat{\mathbf{n}}}{c^2 r} = -\frac{k^2 p_0\, e^{-i(kr-\omega t)}}{r}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) \times \hat{\mathbf{n}} \\ &= \frac{k^2 p_0\, e^{-i(kr-\omega t)}}{r}(\cos\theta\ \hat{\mathbf{y}} - i\ \hat{\mathbf{l}}), \end{aligned} \tag{111}$$

$$\mathbf{E}_{\rm rad} = \mathbf{B}_{\rm rad} \times \hat{\mathbf{n}} = \frac{k^2 p_0\, e^{-i(kr-\omega t)}}{r}(\cos\theta\ \hat{\mathbf{l}} + i\ \hat{\mathbf{y}}). \tag{112}$$

The time-averaged angular distribution of the radiated power is given by,

$$\frac{d\langle P\rangle}{d\Omega} = \frac{c}{8\pi} r^2 |B_{\rm rad}|^2 = \frac{c}{8\pi} k^4 p_0^2 (1 + \cos^2\theta), \tag{113}$$

since $\hat{\mathbf{l}}$ and $\hat{\mathbf{y}}$ are orthogonal. The total radiated power is therefore,

$$\langle P\rangle = \int \frac{d\langle P\rangle}{d\Omega}\, d\Omega = \frac{2c}{3} k^4 p_0^2 = \frac{2}{3c^3}\omega^4 p_0^2. \tag{114}$$

The total power also follows from the Larmor formula,

$$\langle P\rangle = \frac{1}{2}\frac{2\,|\ddot{\mathbf{p}}|^2}{3c^3} = \frac{2}{3c^3}\omega^4 p_0^2, \tag{115}$$

since $|\ddot{\mathbf{p}}| = \sqrt{2}\omega^2 p_0$ in the present example.

4. According to the Larmor formula, the rate of magnetic dipole radiation is,

$$\frac{dU}{dt} = \frac{2}{3}\frac{\ddot{\mathbf{m}}^2}{c^3} = \frac{2}{3}\frac{m^2\omega^4}{c^3}, \tag{116}$$

where $\omega = 2\pi/T$ is the angular velocity, taken to be perpendicular to the magnetic dipole moment $\mathbf{m}$.

The radiated power (116) is derived from a decrease in the rotational kinetic energy, $U = I\omega^2/2$, of the pulsar:

$$\frac{dU}{dt} = -I\omega\dot{\omega} = \frac{2}{5}MR^2\omega\left|\dot{\omega}\right|, \tag{117}$$

where the moment of inertia I is taken to be that of a sphere of uniform mass density. Combining eqs. (116) and (117), we have,

$$m^2 = \frac{3}{5}\frac{MR^2\left|\dot{\omega}\right|c^3}{\omega^3}. \tag{118}$$

Substituting $\omega = 2\pi/T$, and $|\dot{\omega}| = 2\pi\left|\dot{T}\right|/T^2$, we find,

$$m^2 = \frac{3}{20\pi^2}MR^2T\left|\dot{T}\right|c^3. \tag{119}$$

The static magnetic field $\mathbf{B}$ due to dipole $\mathbf{m}$ is,

$$\mathbf{B} = \frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3}, \tag{120}$$

so the peak field at radius R is,

$$\mathbf{B} = \frac{2\mathbf{m}}{R^3}. \tag{121}$$

Inserting this in eq. (119), the peak surface magnetic field is related by,

$$B^2 = \frac{3}{5\pi^2}\frac{MT\left|\dot{T}\right|c^3}{R^4} = \frac{3}{5\pi^2}\frac{(2.8\times10^{33})(7.5)(8\times10^{-11})(3\times10^{10})^3}{(10^6)^4} = 2.8\times10^{30}\ \text{gauss}^2. \tag{122}$$

Thus, $B_{\text{peak}} = 1.7\times10^{15}$ G $= 38B_{\text{crit}}$, where $B_{\text{crit}} = 4.4\times10^{13}$ G.

When electrons and photons of kinetic energies greater than 1 MeV exist in a magnetic field with $B > B_{\text{crit}}$, they rapidly lose this energy via electron-positron pair creation.

Kouveliotou *et al.* report that $B_{\text{peak}} = 8\times10^{14}$ G without discussing details of their calculation.

5. The time-average field momentum density is given in terms of the Poynting vector as (the real part of),

$$\langle \mathbf{P} \rangle_{\rm field} = \frac{\langle \mathbf{S} \rangle}{c^2} = \frac{c}{8\pi} \mathbf{E} \times \mathbf{B}^\star. \tag{123}$$

Hence, the time-averaged angular momentum density is,

$$\langle \mathcal{L} \rangle_{\rm field} = \mathbf{r} \times \mathbf{P}_{\rm field} = \frac{1}{8\pi c} \mathbf{r} \times (\mathbf{E} \times \mathbf{B}^\star) = \frac{1}{8\pi c} r[\mathbf{E}(\hat{\mathbf{n}} \cdot \mathbf{B}^\star) - \mathbf{B}^\star(\hat{\mathbf{n}} \cdot \mathbf{E})]. \tag{124}$$

writing $\mathbf{r} = r\hat{\mathbf{n}}$.

The time-average rate of radiation of angular momentum into solid angle $d\Omega$ is therefore,

$$\frac{d\langle \mathbf{L} \rangle}{dt\ d\Omega} = cr^2 \langle \mathcal{L} \rangle_{\rm field} = \frac{1}{8\pi} r^3 [\mathbf{E}(\hat{\mathbf{n}} \cdot \mathbf{B}^\star) - \mathbf{B}^\star(\hat{\mathbf{n}} \cdot \mathbf{E})], \tag{125}$$

since the angular momentum density $\mathcal{L}$ is moving with velocity c.

The radiation fields of an oscillating electric dipole moment $\mathbf{p}$ including both the $1/r$ and $1/r^2$ terms of eqs. (79) and (87) are,

$$\mathbf{E} = k\frac{e^{i(kr-\omega t)}}{r} \left[\left(k + \frac{i}{r} \right) \mathbf{p} - \left(k + \frac{3i}{r} \right) (\hat{\mathbf{n}} \cdot \mathbf{p}) \hat{\mathbf{n}} \right], \tag{126}$$

$$\mathbf{B} = k^2 \frac{e^{i(kr-\omega t)}}{r} \left(1 + \frac{i}{kr} \right) (\hat{\mathbf{n}} \times \mathbf{p}). \tag{127}$$

Since $\hat{\mathbf{n}} \cdot \mathbf{B}^\star = 0$ for this case, only the second term in eq. (125) contributes to the radiated angular momentum. We therefore find,

$$\frac{d\langle \mathbf{L} \rangle}{dt\ d\Omega} = -\frac{k^3 r}{8\pi} \left(1 - \frac{i}{kr} \right) (\hat{\mathbf{n}} \times \mathbf{p}^\star) \left(-\frac{2i}{r} \right) (\hat{\mathbf{n}} \cdot \mathbf{p}) = \frac{ik^3}{4\pi} (\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^\star), \tag{128}$$

ignoring terms in the final expression that have positive powers of r in the denominator, as these grow small at large distances.

For the example of a rotating dipole moment (Prob. 2),

$$\mathbf{p} = p_0(\hat{\mathbf{x}} + i\hat{\mathbf{y}})\, e^{-i\omega t}, \tag{129}$$

we have,

$$\frac{d\langle \mathbf{L} \rangle}{dt\ d\Omega} = Re \frac{ik^3 p_0^2}{4\pi} [\hat{\mathbf{n}} \cdot (\hat{\mathbf{x}} + i\hat{\mathbf{y}})][\hat{\mathbf{n}} \times (\hat{\mathbf{x}} - i\hat{\mathbf{y}})] = Re \frac{ik^3 p_0^2}{4\pi} \sin\theta [\cos\theta\ \hat{\mathbf{y}} + i\ \hat{\mathbf{l}}]$$
$$= -\frac{k^3}{4\pi} p_0^2 \sin\theta\ \hat{\mathbf{l}}. \tag{130}$$

To find $d\langle \mathbf{L} \rangle / dt$ we integrate eq. (129) over solid angle. When vector $\hat{\mathbf{n}}$ is in the x-z plane, vector $\hat{\mathbf{l}}$ can be expressed as,

$$\hat{\mathbf{l}} = \cos\theta\ \hat{\mathbf{x}} - \sin\theta\ \hat{\mathbf{z}}. \tag{131}$$

As we integrate over all directions of $\hat{\mathbf{n}}$, the contributions to $d\langle \mathbf{L}\rangle/dt$ in the x-y plane sum to zero, and only its z component survives. Hence,

$$\begin{aligned}\frac{d\langle \mathbf{L}\rangle}{dt} &= \hat{\mathbf{z}}\int \frac{d\langle L_z\rangle}{dt\, d\Omega}\, d\Omega = 2\pi\frac{k^3}{4\pi}p_0^2\,\hat{\mathbf{z}}\int_{-1}^{1}\sin^2\theta\; d\cos\theta = \frac{2k^3}{3}p_0^2\,\hat{\mathbf{z}} \\ &= \frac{2ck^3}{3\omega}p_0^2\,\hat{\mathbf{z}} = \frac{\langle P\rangle}{\omega}\hat{\mathbf{z}}, \qquad (132)\end{aligned}$$

recalling eq. (115) for the radiated power $\langle P\rangle$. Of course, the motion described by eq. (129) has its angular momentum along the $+z$ axis.

6. The magnetic field radiated by a time-dependent, axially symmetric quadrupole is given by,

$$\mathbf{B} = \frac{[\dddot{\mathbf{Q}}] \times \hat{\mathbf{n}}}{6c^3 r}, \tag{133}$$

where the unit vector $\hat{\mathbf{n}}$ has rectangular components,

$$\hat{\mathbf{n}} = (\sin\theta, 0, \cos\theta), \tag{134}$$

and the quadrupole vector $\mathbf{Q}$ is related to the quadrupole tensor Q_{ij} by,

$$Q_i = Q_{ij} n_j. \tag{135}$$

The charge distribution is symmetric about the z axis, so the quadrupole-moment tensor Q_{ij} may be expressed entirely in terms of,

$$Q_{zz} = \int \rho(3z^2 - r^2)\, d\mathrm{Vol} = -4a^2 e \cos^2 \omega t = -2a^2 e(1 + \cos 2\omega t). \tag{136}$$

Thus,

$$Q_{ij} = \begin{pmatrix} -Q_{zz}/2 & 0 & 0 \\ 0 & -Q_{zz}/2 & 0 \\ 0 & 0 & Q_{zz} \end{pmatrix}, \tag{137}$$

and the quadrupole vector can be written as,

$$\begin{aligned} \mathbf{Q} &= \left(-\frac{Q_{zz}\sin\theta}{2}, 0, Q_{zz}\cos\theta\right) = -\frac{Q_{zz}}{2}\hat{\mathbf{n}} + \frac{3Q_{zz}\cos\theta}{2}\hat{\mathbf{z}} \\ &= a^2 e(1 + \cos 2\omega t)(\hat{\mathbf{n}} - 3\hat{\mathbf{z}}\cos\theta). \end{aligned} \tag{138}$$

Then,

$$[\dddot{\mathbf{Q}}] = 8\omega^3 a^2 e \sin 2\omega t'(\hat{\mathbf{n}} - 3\hat{\mathbf{z}}\cos\theta), \tag{139}$$

where the retarded time is $t' = t - r/c$. Hence,

$$\mathbf{B} = \frac{[\dddot{\mathbf{Q}}] \times \hat{\mathbf{n}}}{6c^3 r} = -\frac{4k^3 a^2 e}{r}\hat{\mathbf{y}} \sin(2kr - 2\omega t)\ \sin\theta\cos\theta, \tag{140}$$

since $\hat{\mathbf{z}} \times \hat{\mathbf{n}} = \hat{\mathbf{y}}\sin\theta$. The radiated electric field is given by,

$$\mathbf{E} = \mathbf{B} \times \hat{\mathbf{n}} = -\frac{4k^3 a^2 e}{r}\hat{\mathbf{l}} \sin(2kr - 2\omega t)\ \sin\theta\cos\theta, \tag{141}$$

using $\hat{\mathbf{y}} \times \hat{\mathbf{n}} = \hat{\mathbf{l}}$.

As we are not using complex notation, we revert to the basic definitions to find that the time-averaged angular distribution of radiated power is,

$$\frac{d\langle P\rangle}{d\Omega} = r^2 \langle \mathbf{S}\rangle \cdot \hat{\mathbf{n}} = \frac{cr^2}{4\pi}\langle \mathbf{E} \times \mathbf{B} \cdot \hat{\mathbf{n}}\rangle = \frac{2ck^6 a^4 e^2}{\pi}\ \sin^2\theta\cos^2\theta, \tag{142}$$

since $\hat{\mathbf{l}} \times \hat{\mathbf{y}} = \hat{\mathbf{n}}$. This integrates to give,

$$\langle P\rangle = 2\pi \int_{-1}^{1} \frac{d\langle P\rangle}{d\Omega}\ d\cos\theta = \frac{16ck^6 a^4 e^2}{15}. \tag{143}$$

7. Since the charge is assumed to rotate with constant angular velocity, the magnetic moment it generates is constant in time, and there is no magnetic-dipole radiation. Hence, we consider only electric-quadrupole radiation in addition to the electric-dipole radiation. The radiated fields are therefore,

$$\mathbf{B} = \frac{[\ddot{\mathbf{p}}] \times \hat{\mathbf{n}}}{c^2 r} + \frac{[\dddot{\mathbf{Q}}] \times \hat{\mathbf{n}}}{6c^3 r}, \qquad \mathbf{E} = \mathbf{B} \times \hat{\mathbf{n}}. \tag{144}$$

The electric dipole radiation fields are given by eqs. (111) and 112) when we write $p_0 = ae$.

The charge distribution is not azimuthally symmetric about any fixed axis, so we must evaluate the full quadrupole tensor,

$$Q_{ij} = e(3r_i r_j - r^2 \delta_{ij}). \tag{145}$$

to find the components of the quadrupole vector $\mathbf{Q}$. The position vector of the charge has components,

$$r_i = (a \cos \omega t, a \sin \omega t, 0), \tag{146}$$

so the nonzero components of Q_{ij} are,

$$Q_{xx} = e(3x^2 - r^2) = a^2 e(3\cos^2 \omega t - 1) = \frac{a^2 e}{2}(1 + 3\cos 2\omega t), \tag{147}$$

$$Q_{yy} = e(3y^2 - r^2) = a^2 e(3\sin^2 \omega t - 1) = \frac{a^2 e}{2}(1 - 3\cos 2\omega t), \tag{148}$$

$$Q_{zz} = -er^2 = -a^2 e, \tag{149}$$

$$Q_{xy} = Q_{yx} = 3exy = 3a^2 e \sin \omega t \cos \omega t = \frac{3a^2 e}{2} \sin 2\omega t. \tag{150}$$

Only the time-dependent part of Q_{ij} contributes to the radiation, so we write,

$$Q_{ij}(\text{time dependent}) = \frac{3a^2 e}{2} \begin{pmatrix} \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & -\cos 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{151}$$

The unit vector $\hat{\mathbf{n}}$ towards the observer has components given in eq. (134), so the time-dependent part of the quadrupole vector $\mathbf{Q}$ has components,

$$Q_i = Q_{ij} n_j = \frac{3a^2 e}{2}(\cos 2\omega t \ \sin\theta, \sin 2\omega t \ \sin\theta, 0). \tag{152}$$

Thus,

$$[\dddot{Q}_i] = \dddot{Q}_i(t' = t - r/c) = -12a^2 e\omega^3 \sin\theta(\sin(2kr - 2\omega t), \cos(2kr - 2\omega t), 0). \tag{153}$$

It is preferable to express this vector in terms of the orthonormal triad $\hat{\mathbf{n}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{l}} = \hat{\mathbf{y}} \times \hat{\mathbf{n}}$, by noting that,

$$\hat{\mathbf{x}} = \hat{\mathbf{n}} \sin\theta - \hat{\mathbf{l}} \cos\theta. \tag{154}$$

Hence,

$$[\dddot{\mathbf{Q}}] = -12a^2 e\omega^3 \sin\theta(\hat{\mathbf{n}} \sin\theta \sin(2kr - 2\omega t) - \hat{\mathbf{l}} \cos\theta \sin(2kr - 2\omega t) + \hat{\mathbf{y}} \cos(2kr - 2\omega t)). \tag{155}$$

The fields due to electric-quadrupole radiation are therefore,

$$\mathbf{B}_{E2} = \frac{[\dddot{\mathbf{Q}}] \times \hat{\mathbf{n}}}{6c^3 r} = -\frac{2a^2 e k^3}{r} \sin\theta(\hat{\mathbf{l}} \cos(2kr - 2\omega t) + \hat{\mathbf{y}} \cos\theta \sin(2kr - 2\omega t)), \tag{156}$$

$$\mathbf{E}_{E2} = \mathbf{B}_{E2} \times \hat{\mathbf{n}} = \frac{2a^2 e k^3}{r} \sin\theta(\hat{\mathbf{y}} \cos(2kr - 2\omega t) - \hat{\mathbf{l}} \cos\theta \sin(2kr - 2\omega t)). \tag{157}$$

The angular distribution of the radiated power can be calculated from the combined electric-dipole and electric-quadrupole fields, and will include a term $\propto k^4$ due only to dipole radiation as found in Prob. 2 above, a term $\propto k^6$ due only to quadrupole radiation, and a complicated cross term $\propto k^5$ due to both dipole and quadrupole fields. Here, we only display the term due to the quadrupole fields by themselves:

$$\frac{d\langle P_{E2}\rangle}{d\Omega} = \frac{cr^2}{4\pi} \langle \mathbf{E}_{E2} \times \mathbf{B}_{E2} \cdot \hat{\mathbf{n}} \rangle = \frac{ca^4 e^2 k^6}{2\pi}(1 - \cos^4\theta), \tag{158}$$

which integrates to give,[11]

$$\langle P_{E2}\rangle = 2\pi \int_{-1}^{1} \frac{d\langle P\rangle}{d\Omega}\, d\cos\theta = \frac{8ca^4 e^2 k^6}{5}\,. \tag{162}$$

[11]It is stated in eq. (71.5) of http://kirkmcd.princeton.edu/examples/EM/landau_ctf_75.pdf that, noting Landau's $D_{\alpha\beta}$ of his eq. (41.3) is our Q_{ij},

$$P_{E2} = \frac{\dddot{Q}_{ij}^2}{180c^5}. \tag{159}$$

From the time-dependent part of the quadrupole tensor, eq. (151), we have,

$$\dddot{Q}_{ij} = 12e\omega^3 a^2 \begin{pmatrix} \sin 2\omega t & -\cos 2\omega t & 0 \\ -\cos 2\omega t & -\sin 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \dddot{Q}_{ij}^2 = \sum_{i,j} \dddot{Q}_{ij}^2 = 288e^2\omega^6 a^4, \tag{160}$$

such that eq. (159) implies,

$$P_{E2} = \frac{2e^2\omega^6 a^4}{5c^5} = \frac{8ca^4 e^2 k^6}{5}\,, \tag{161}$$

as in eq. (162).

8. a) The dominant energy loss is from electric-dipole radiation, which obeys eq. (25),

$$\frac{dU}{dt} = -\langle P_{E1}\rangle = -\frac{2a^2e^2\omega^4}{3c^3}\,. \tag{163}$$

For an electron of charge $-e$ and mass m in an orbit of radius a about a fixed nucleus of charge $+e$, $F = ma$ tells us that

$$\frac{e^2}{a^2} = m\frac{v^2}{a} = m\omega^2 a, \tag{164}$$

so that,

$$\omega^2 = \frac{e^2}{ma^3}\,, \tag{165}$$

and the total energy (kinetic plus potential) is,

$$U = -\frac{e^2}{a} + \frac{1}{2}mv^2 = -\frac{e^2}{2a}\,. \tag{166}$$

Using eqs. (165) and (166) in (163), we have,

$$\frac{dU}{dt} = \frac{e^2}{2a^2}\dot{a} = -\frac{2e^6}{3a^4m^2c^3}\,, \tag{167}$$

or

$$a^2\dot{a} = \frac{1}{3}\frac{da^3}{dt} = -\frac{4e^4}{3m^2c^3} = -\frac{4}{3}r_0^2c, \tag{168}$$

where $r_0 = e^2/mc^2$ is the classical electron radius. Hence,

$$a^3 = a_0^3 - 4r_0^2ct. \tag{169}$$

The time to fall to the origin is,

$$t_{\rm fall} = \frac{a_0^3}{4r_0^2c}\,. \tag{170}$$

With $r_0 = 2.8 \times 10^{-13}$ cm and $a_0 = 5.3 \times 10^{-9}$ cm, $t_{\rm fall} = 1.6 \times 10^{-11}$ s.

This is of the order of magnitude of the lifetime of an excited hydrogen atom, but the ground state appears to have infinite lifetime in Nature.

This classical puzzle is pursued further in Prob. 9 below.

b) The analog of the quadrupole factor ea^2 in Prob. 7 above for masses m_1 and m_2 in circular orbits with distance a between them is $m_1r_1^2 + m_2r_2^2$, where r_1 and r_2 are measured from the center of mass. That is,

$$m_1r_1 = m_2r_2, \qquad \text{and} \qquad r_1 + r_2 = a, \tag{171}$$

so that,

$$r_1 = \frac{m_2}{m_1+m_2}a, \qquad r_2 = \frac{m_1}{m_1+m_2}a, \tag{172}$$

and the quadrupole factor is,

$$m_1 r_1^2 + m_2 r_2^2 = \frac{m_1 m_2}{m_1 + m_2} a^2. \tag{173}$$

We are then led by eq. (26) to say that the power in gravitational quadrupole radiation is,

$$P_{G2} = \frac{8}{5}\frac{G}{c^5}\left(\frac{m_1 m_2}{m_1 + m_2}\right)^2 a^4 \omega^6. \tag{174}$$

We insert a single factor of Newton's constant G in this expression, since it has dimensions of mass^2, and Gm^2 is the gravitational analog of the square of the electric charge in eq. (26).

We note that a general-relativity calculation[12] yields a result a factor of 4 larger than eq. (174):

$$P_{G2} = \frac{32}{5}\frac{G}{c^5}\left(\frac{m_1 m_2}{m_1 + m_2}\right)^2 a^4 \omega^6. \tag{175}$$

To find $t_{\rm fall}$ due to gravitational radiation, we follow the argument of part a):

$$\frac{G m_1 m_2}{a^2} = m_1 \frac{v_1^2}{r_1} = m_1 \omega^2 r_1 = m_2 \frac{v_2^2}{r_2}\,, \tag{176}$$

so that,

$$\omega^2 = \frac{G(m_1 + m_2)}{a^3}\,, \tag{177}$$

and also the total energy (kinetic plus potential) is,

$$U = -\frac{G m_1 m_2}{a} + \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = -\frac{G m_1 m_2}{2a}\,. \tag{178}$$

Using eqs. (177) and (178) in (175), we have,

$$\frac{dU}{dt} = \frac{G m_1 m_2}{2a^2}\dot{a} = -\frac{32 G^4 m_1^2 m_2^2 (m_1 + m_2)}{5 a^5 c^5}\,, \tag{179}$$

or,

$$a^3 \dot{a} = \frac{1}{4}\frac{da^4}{dt} = -\frac{64 G^3 m_1 m_2 (m_1 + m_2)}{5 c^5}\,. \tag{180}$$

Hence,

$$a^4 = a_0^4 - \frac{256 G^3 m_1 m_2 (m_1 + m_2)}{5 c^5} t. \tag{181}$$

The time to fall to the origin is,

$$t_{\rm fall} = \frac{5 a_0^4 c^5}{256 G^3 m_1 m_2 (m_1 + m_2)}\,. \tag{182}$$

For the Earth-Sun system, $a_0 = 1.5 \times 10^{13}$ cm, $m_1 = 6 \times 10^{27}$ gm, $m_2 = 2 \times 10^{33}$ cm, and $G = 6.7 \times 10^{-10}$ cm^2/(g-s^2), so that $t_{\rm fall} \approx 1.5 \times 10^{36}$ s $\approx 5 \times 10^{28}$ years!

[12] http://kirkmcd.princeton.edu/examples/GR/peters_pr_131_435_63.pdf

9. The solution given here follows the succinct treatment by Landau, *Classical Theory of Fields*, §74, http://kirkmcd.princeton.edu/examples/EM/landau_ctf_71.pdf

For charges in steady motion at angular frequency ω in a ring of radius a, the current density $\mathbf{J}$ is periodic with period $2\pi/\omega$, so the Fourier analysis (34) at the retarded time t' can be evaluated via the usual approximation that $r \approx R - \mathbf{r}' \cdot \hat{\mathbf{n}}$, where R is the distance from the center of the ring to the observer, $\mathbf{r}'$ points from the center of the ring to the electron, and $\hat{\mathbf{n}}$ is the unit vector pointing from the center of the ring to the observer. Then,

$$\begin{aligned} [\mathbf{J}] &= \mathbf{J}(\mathbf{r}', t' = t - r/c) = \sum_m \mathbf{J}_m(\mathbf{r}')\, e^{-im\omega(t-R/c+\mathbf{r}'\cdot\hat{\mathbf{n}}/c} \\ &= \sum_m e^{im(kR-\omega t)} \mathbf{J}_m(\mathbf{r}')\, e^{-im\omega\mathbf{r}'\cdot\hat{\mathbf{n}}/c}, \end{aligned} \tag{183}$$

where $k = \omega/c$.

We first consider a single electron, whose azimuth varies as $\phi = \omega t + \phi_0$, and whose velocity is, of course, $v = a\omega$. The current density of a point electron of charge e can be written using Dirac delta functions in a cylindrical coordinate system (ρ, ϕ, z) (with volume element $\rho d\rho\ d\phi\ dz$) as,

$$\mathbf{J} = \rho_{\rm charge} v\, \hat{\boldsymbol{\phi}} = ev\, \delta(\rho - a)\, \delta(z)\, \delta(\rho(\phi - \omega t - \phi_0))\, \hat{\boldsymbol{\phi}}. \tag{184}$$

The Fourier components $\mathbf{J}_m$ are given by,

$$\mathbf{J}_m = \frac{1}{T} \int_0^T \mathbf{J}(\mathbf{r}, t)\, e^{imt}\, dt = ev\, \delta(\rho - a)\, \delta(z)]\, \frac{e^{im(\phi-\phi_0)}}{\rho\omega T}\, \hat{\boldsymbol{\phi}}. \tag{185}$$

Also,

$$\mathbf{r}' = \rho(\cos\phi\, \hat{\mathbf{x}} + \sin\phi\, \hat{\mathbf{y}}), \quad \hat{\mathbf{n}} = \sin\theta\, \hat{\mathbf{x}} + \cos\theta\, \hat{\mathbf{z}}, \quad \text{and} \quad \hat{\boldsymbol{\phi}} = -\sin\phi\, \hat{\mathbf{x}} + \cos\phi\, \hat{\mathbf{y}}. \tag{186}$$

Using eqs. (185) and (186) in (183) and noting that $\omega T = 2\pi$, we find,

$$[\mathbf{J}] = \frac{ev}{2\pi\rho} \sum_m e^{im(kR-\omega t)}\, e^{im(\phi-\phi_0-\omega\rho\sin\theta\cos\phi/c)}\, \delta(\rho - a)\, \delta(z)\, \hat{\boldsymbol{\phi}}. \tag{187}$$

Inserting this in eq. (33), we have,

$$\begin{aligned} \mathbf{A} &\approx \frac{1}{cR} \int [\mathbf{J}] \rho\ d\rho\ d\phi\ dz = \frac{ev}{2\pi cR} \sum_m e^{im(kR-\omega t-\phi_0)} \int_0^{2\pi} e^{im(\phi-\omega a\sin\theta\cos\phi/c)}\, \hat{\boldsymbol{\phi}}\ d\phi \\ &= \sum_m \mathbf{A}_m\, e^{-im\omega t}, \end{aligned} \tag{188}$$

so that the Fourier components of the vector potential are,

$$\mathbf{A}_m = \frac{ev}{2\pi cR}\, e^{im(kR-\phi_0)} \int_0^{2\pi} e^{im(\phi - v\sin\theta\cos\phi/c)} (-\sin\phi\ \hat{\mathbf{x}} + \cos\phi\ \hat{\mathbf{y}})\ d\phi. \tag{189}$$

The integrals yield Bessel functions with the aid of the integral representation (40). The $\hat{\mathbf{y}}$ part of eq. (189) can be found by taking the derivative of this relation with respect to z:

$$J'_m(z) = -\frac{i^{m+1}}{2\pi}\int_0^{2\pi} e^{im\phi - iz\cos\phi}\cos\phi\, d\phi, \tag{190}$$

For the $\hat{\mathbf{x}}$ part of eq. (189) we play the trick,

$$\begin{aligned} 0 &= \int_0^{2\pi} e^{i(m\phi - z\cos\phi)} d(m\phi - z\cos\phi) \\ &= m\int_0^{2\pi} e^{im\phi - iz\cos\phi}\, d\phi + z\int_0^{2\pi} e^{im\phi - iz\cos\phi}\sin\phi\, d\phi, \end{aligned} \tag{191}$$

so that,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{im\phi - iz\cos\phi}\sin\phi\, d\phi = -\frac{m}{z}\frac{1}{2\pi}\int_0^{2\pi} e^{im\phi - iz\cos\phi}\, d\phi = -\frac{m}{i^m z}J_m(z). \tag{192}$$

Using eqs. (190) and (192) with $z = mv\sin\theta/c$ in (189) we have,

$$\mathbf{A}_m = \frac{ev}{cR}e^{im(kR-\phi_0)}\left(\frac{1}{i^m v\sin\theta/c}J_m(mv\sin\theta/c)\ \hat{\mathbf{x}} - \frac{1}{i^{m+1}}J'_m(mv\sin\theta/c)\ \hat{\mathbf{y}}\right). \tag{193}$$

We skip the calculation of the electric and magnetic fields from the vector potential, and proceed immediately to the angular distribution of the radiated power according to eq. (39),

$$\begin{aligned} \frac{dP_m}{d\Omega} &= \frac{cR^2}{2\pi}\left|imk\,\hat{\mathbf{n}}\times\mathbf{A}_m\right|^2 = \frac{ck^2m^2R^2}{2\pi}\left|\hat{\mathbf{n}}\times\mathbf{A}_m\right|^2 \\ &= \frac{ck^2m^2R^2}{2\pi}\left(\cos^2\theta\left|A_{m,x}\right|^2 + \left|A_{m,y}\right|^2\right) \\ &= \frac{ce^2k^2m^2}{2\pi}\left(\cot^2\theta J_m^2(mv\sin\theta/c) + \frac{v^2}{c^2}J'^2_m(mv\sin\theta/c)\right). \end{aligned} \tag{194}$$

The present interest in this result is for $v/c \ll 1$, but in fact it holds for any value of v/c. As such, it can be used for a detailed discussion of the radiation from a relativistic electron that moves in a circle, which emits so-called synchrotron radiation. This topic is discussed further in Lecture 20 of the Notes.

We now turn to the case of N electrons uniformly spaced around the ring. The initial azimuth of the $n^{\rm th}$ electron can be written as,

$$\phi_n = \frac{2\pi n}{N}\,. \tag{195}$$

The $m^{\rm th}$ Fourier component of the total vector potential is simply the sum of components (193) inserting ϕ_n in place of ϕ_0:

$$\begin{aligned} \mathbf{A}_m &= \sum_{n=1}^{N}\frac{ev}{cR}e^{im(kR-\phi_n)}\left(\frac{1}{i^m v\sin\theta/c}J_m(mv\sin\theta/c)\,\hat{\mathbf{x}} - \frac{1}{i^{m+1}}J'_m(mv\sin\theta/c)\,\hat{\mathbf{y}}\right) \\ &= \frac{ev\,e^{imkR}}{cR}\left(\frac{1}{i^m v\sin\theta/c}J_m(mv\sin\theta/c)\,\hat{\mathbf{x}} - \frac{1}{i^{m+1}}J'_m(mv\sin\theta/c)\,\hat{\mathbf{y}}\right)\sum_{n=1}^{N}e^{-i2\pi mn/N}. \end{aligned} \tag{196}$$

This sum vanishes unless m is a multiple of N, in which case the sum is just N. The lowest nonvanishing Fourier component has order N, and the radiation is at frequency $N\omega$. We recognize this as $N^{\rm th}$-order multipole radiation, whose radiated power follows from eq. (194) as,

$$\frac{dP_N}{d\Omega} = \frac{ce^2k^2N^2}{2\pi}\left(\cot^2\theta J_N^2(Nv\sin\theta/c) + \frac{v^2}{c^2}J_N'^2(Nv\sin\theta/c)\right). \qquad (197)$$

For large N, but $v/c \ll 1$, we can use the asymptotic expansion (41), and its derivative,

$$J'_m(mx) \approx \frac{(ex/2)^m}{\sqrt{2\pi m}\ x} \qquad (m \gg 1, x \ll 1), \qquad (198)$$

to write eq. (197) as,

$$\frac{dP_N}{d\Omega} \approx \frac{ce^2k^2N}{4\pi^2\sin^2\theta}\left(\frac{e}{2}\frac{v}{c}\sin\theta\right)^{2N}(1+\cos^2\theta) \ll N\frac{dP_{E1}}{d\Omega} \qquad (N \gg 1, v/c \ll 1). \qquad (199)$$

In eqs. (198) and (199) the symbol e inside the parentheses is not the charge but rather the base of natural logarithms, 2.718...

For currents in, say, a loop of copper wire, $v \approx 1$ cm/s, so $v/c \approx 10^{-10}$, while $N \approx 10^{23}$. The radiated power predicted by eq. (199) is extraordinarily small!

Note, however, that this nearly complete destructive interference depends on the electrons being uniformly distributed around the ring. Suppose instead that they were distributed with random azimuths ϕ_n. Then the square of the magnetic field at order m has the form ,

$$|\mathbf{B}_m|^2 \propto \left|\sum_{n=1}^{N} e^{-im\phi_n}\right|^2 = N + \sum_{l\neq n} e^{-im(\phi_l-\phi_n)} = N. \qquad (200)$$

Thus, for random azimuths the power radiated by N electrons (at any order) is just N times that radiated by one electron.

If the charge carriers in a wire were localized to distances much smaller than their separation, radiation of “steady” currents could occur. However, in the quantum view of metallic conduction, such localization does not occur.

The random-phase approximation is relevant for electrons in a so-called storage ring, for which the radiated power is a major loss of energy – or source of desirable photon beams of synchrotron radiation, depending on one’s point of view. We cannot expound here on the interesting topic of the “formation length” for radiation by relativistic electrons, which length sets the scale for interference of multiple electrons. See, for example, http://kirkmcd.princeton.edu/accel/weizsacker.pdf

10. We repeat the derivation of Prob. 1 above, this time emphasizing the advanced fields. The advanced vector potential for the point electric dipole $\mathbf{p} = \mathbf{p}_0\, e^{-i\omega t}$ located at the origin is,

$$\mathbf{A}_{E1,\mathrm{adv}} = \frac{\{\dot{\mathbf{p}}\}}{cr} = \frac{\dot{\mathbf{p}}(t' = t + r/c)}{cr} = -i\omega\, \frac{e^{-i(kr+\omega t)}}{cr}\mathbf{p}_0 = -ik\, \frac{e^{-i(kr+\omega t)}}{r}\mathbf{p}_0, \qquad (201)$$

where $k = \omega/c$.

We obtain the magnetic field by taking the curl of eq. (201),

$$\begin{aligned} \mathbf{B}_{E1,\mathrm{adv}} &= \boldsymbol{\nabla} \times \mathbf{A}_{E1,\mathrm{adv}} = -ik\boldsymbol{\nabla}\frac{e^{-i(kr+\omega t)}}{r} \times \mathbf{p}_0 = -ik\, \frac{e^{-i(kr+\omega t)}}{r}\left(-ik\, \hat{\mathbf{n}} - \frac{\hat{\mathbf{n}}}{r}\right) \times \mathbf{p}_0 \\ &= k^2\, \frac{e^{-i(kr+\omega t)}}{r}\left(-1 + \frac{i}{kr}\right) \hat{\mathbf{n}} \times \mathbf{p}_0. \qquad (202) \end{aligned}$$

Then,

$$\begin{aligned} \mathbf{E}_{E1,\mathrm{adv}} &= \frac{i}{k}\boldsymbol{\nabla} \times \mathbf{B}_{E1,\mathrm{adv}} = -\boldsymbol{\nabla} \times \left[e^{-i(kr+\omega t)}\left(\frac{ik}{r^2} + \frac{1}{r^3}\right)\mathbf{r} \times \mathbf{p}_0\right] \\ &= -\boldsymbol{\nabla}e^{-i(kr+\omega t)}\left(\frac{ik}{r^2} + \frac{1}{r^3}\right) \times (\mathbf{r} \times \mathbf{p}_0) - e^{-i(kr+\omega t)}\left(\frac{ik}{r^2} + \frac{1}{r^3}\right)\boldsymbol{\nabla} \times (\mathbf{r} \times \mathbf{p}_0) \\ &= e^{-i(kr+\omega t)}\left[-\frac{k^2}{r} + 3\left(\frac{ik}{r^2} + \frac{1}{r^3}\right)\right] \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{p}_0) + 2\mathbf{p}_0\, e^{-i(kr+\omega t)}\left(\frac{ik}{r^2} + \frac{1}{r^3}\right) \\ &= e^{-i(kr+\omega t)}\left\{\left[-\frac{k^2}{r} + 3\left(\frac{ik}{r^2} + \frac{1}{r^3}\right)\right](\mathbf{p}_0 \cdot \hat{\mathbf{n}})\, \hat{\mathbf{n}} + \left[\frac{k^2}{r} - \frac{ik}{r^2} - \frac{1}{r^3}\right]\mathbf{p}_0\right\}. \qquad (203) \end{aligned}$$

The retarded fields due to a point dipole $-\mathbf{p}$ are, from Prob. 1,

$$\begin{aligned} \mathbf{B}_{E1,\mathrm{ret}} &= -k^2\frac{e^{i(kr-\omega t)}}{r}\left(1 + \frac{i}{kr}\right)\hat{\mathbf{n}} \times \mathbf{p}_0, \qquad (204) \\ \mathbf{E}_{E1,\mathrm{ret}} &= e^{i(kr-\omega t)}\left\{\left[\frac{k^2}{r} + 3\left(\frac{ik}{r^2} - \frac{1}{r^3}\right)\right](\mathbf{p}_0 \cdot \hat{\mathbf{n}})\, \hat{\mathbf{n}} - \left[\frac{k^2}{r} + \frac{ik}{r^2} - \frac{1}{r^3}\right]\mathbf{p}_0\right\}. (205) \end{aligned}$$

We now consider the superposition of the fields (202)-(205) inside a conducting sphere of radius a. The spatial part of the total electric field is then,

$$\begin{aligned} \mathbf{E}_{E1} &= \left[\left(\frac{k^2}{r} - \frac{3}{r^3}\right)(e^{ikr} - e^{-ikr}) + 3\frac{ik}{r^2}(e^{ikr} + e^{-ikr})\right](\mathbf{p}_0 \cdot \hat{\mathbf{n}})\, \hat{\mathbf{n}} \\ &\quad - \left[\left(\frac{k^2}{r} - \frac{1}{r^3}\right)(e^{ikr} - e^{-ikr}) + \frac{ik}{r^2}(e^{ikr} + e^{-ikr})\right]\mathbf{p}_0 \\ &= 2i\left[\left(\frac{k^2}{r} - \frac{3}{r^3}\right)\sin kr + 3\frac{k}{r^2}\cos kr\right](\mathbf{p}_0 \cdot \hat{\mathbf{n}})\, \hat{\mathbf{n}} \\ &\quad -2i\left[\left(\frac{k^2}{r} - \frac{1}{r^3}\right)\sin kr + \frac{k}{r^2}\cos kr\right]\mathbf{p}_0. \qquad (206) \end{aligned}$$

Remarkably, this electric field is finite at the origin, although each of the fields (203) and (205) diverges there. We also recognize that this electric field could be expressed in terms of the so-called spherical Bessel functions,

$$j_0(x) = \frac{\sin x}{x}, \quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \quad j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3\cos x}{x^2}, \quad \ldots \tag{207}$$

An expansion of the spherical cavity field in terms of spherical Bessel functions occurs "naturally" when we use the more standard approach to this problem, seeking solutions to the Helmholtz wave equation via separation of variables in spherical coordinates. See *Electromagnetic Theory* by J.A. Stratton (McGraw-Hill, 1941)[13] for details of this method.

Because the sum of the magnetic fields (202) and (204) is purely transverse, this cavity mode is called a TM mode.

The boundary conditions at the surface of the sphere are that the radial component of the magnetic field and the transverse component of the electric field must vanish. Since the magnetic fields (202) and (204) are transverse at any radius, we examine the electric field at $r = a$. Of the terms in eq. (206), only those in $\mathbf{p}_0$ have transverse components, so the boundary condition is,

$$0 = \frac{k}{a^2}\cos ka + \sin ka\left(\frac{k^2}{a} - \frac{1}{a^3}\right), \tag{208}$$

or,

$$\cot ka = \frac{1}{ka} - ka, \qquad \Rightarrow \qquad ka = 2.744. \tag{209}$$

In case of a point magnetic dipole $\mathbf{m} = \mathbf{m}_0\, e^{-i\omega t}$ at the origin, the fields have the same form as for an electric dipole, but with $\mathbf{E}$ and $\mathbf{B}$ interchanged. That is, the advanced fields would be,

$$\mathbf{E}_{M1,\mathrm{adv}} = k^2\,\frac{e^{-i(kr+\omega t)}}{r}\left(-1 + \frac{i}{kr}\right)\hat{\mathbf{n}} \times \mathbf{m}_0, \tag{210}$$

$$\mathbf{B}_{M1,\mathrm{adv}} = e^{-i(kr+\omega t)}\left\{\left[-\frac{k^2}{r} + 3\left(\frac{ik}{r^2} + \frac{1}{r^3}\right)\right](\mathbf{m}_0 \cdot \hat{\mathbf{n}})\,\hat{\mathbf{n}} + \left[\frac{k^2}{r} - \frac{ik}{r^2} - \frac{1}{r^3}\right]\mathbf{m}_0\right\}. \tag{211}$$

and the retarded field due to magnetic dipole $-\mathbf{m}$ would be

$$\mathbf{E}_{M1,\mathrm{ret}} = -k^2\,\frac{e^{i(kr-\omega t)}}{r}\left(1 + \frac{i}{kr}\right)\hat{\mathbf{n}} \times \mathbf{m}_0, \tag{212}$$

$$\mathbf{B}_{M1,\mathrm{ret}} = e^{i(kr-\omega t)}\left\{\left[\frac{k^2}{r} + 3\left(\frac{ik}{r^2} - \frac{1}{r^3}\right)\right](\mathbf{m}_0 \cdot \hat{\mathbf{n}})\,\hat{\mathbf{n}} - \left[\frac{k^2}{r} + \frac{ik}{r^2} - \frac{1}{r^3}\right]\mathbf{m}_0\right\}. \tag{213}$$

If the advanced and retarded magnetic dipole fields are superposed inside a spherical cavity of radius a, the condition that the transverse electric field vanish at the conducting surface is,

$$0 = e^{ika}\left[1 + \frac{i}{ka}\right] + e^{-ika}\left[1 - \frac{i}{ka}\right] = 2\left[\cos ka - \frac{\sin ka}{ka}\right], \tag{214}$$

[13] http://kirkmcd.princeton.edu/examples/EM/stratton_electromagnetic_theory.pdf

or,

$$\tan ka = ka, \qquad \Rightarrow \qquad ka = 4.493. \tag{215}$$

The electric field of this mode is purely transverse, so it is called a TE mode.

Clearly, other modes of a spherical cavity can be found by superposing the advanced and retarded fields due to higher multipoles at the origin.

11. This problem is due to D. Iwanenko and I. Pomeranchuk, *On the Maximal Energy Attainable in a Betatron*, Phys. Rev. **65**, 343 (1944).[14]

The electron is held in its circular orbit by the Lorentz force due to the magnetic field **B**. Newton's law, $F = ma$, for this circular motion can be written as,

$$F = \gamma ma = \frac{\gamma mv^2}{R} = e\frac{v}{c}B. \tag{216}$$

For a relativistic electron, $v \approx c$, so we have,

$$\gamma \approx \frac{eRB}{mc^2}. \tag{217}$$

The electron is being accelerated by the electric field that is induced by the changing magnetic flux. Applying the integral form of Faraday's law to the circle of radius R, we have (ignoring the sign),

$$2\pi RE_\phi = \frac{\dot{\Phi}}{c} = \frac{\pi R^2 \dot{B}_{\rm ave}}{c}\,, \tag{218}$$

and hence,

$$E_\phi = \frac{R\dot{B}_{\rm ave}}{2c}\,, \tag{219}$$

The rate of change of the electron's energy $\mathcal{E}$ due to E_ϕ is,

$$\frac{d\mathcal{E}}{dt} = \mathbf{F}\cdot\mathbf{v} \approx ecE_\phi = \frac{eR\dot{B}_{\rm ave}}{2}\,, \tag{220}$$

Since $\mathcal{E} = \gamma mc^2$, we can write,

$$\dot{\gamma} mc^2 = \frac{eR\dot{B}_{\rm ave}}{2}\,, \tag{221}$$

which integrates to,

$$\gamma = \frac{eRB_{\rm ave}}{2mc^2}\,. \tag{222}$$

Comparing with eq. (217), we find the required condition on the magnetic field:

$$B = \frac{B_{\rm ave}}{2}\,. \tag{223}$$

As the electron accelerates it radiates energy at rate given by the Larmor formula in the rest frame of the electron,

$$\frac{d\mathcal{E}^\star}{dt^\star} = -\frac{2e^2\ddot{p}^{\star 2}}{3c^3} = -\frac{2e^2a^{\star 2}}{3c^3}\,. \tag{224}$$

[14] http://kirkmcd.princeton.edu/examples/EM/iwanenko_pr_65_343_44.pdf

Because $\mathcal{E}$ and t are both the time components of 4-vectors, their transforms from the rest frame to the lab frame have the same form, and the rate $d\mathcal{E}/dt$ is invariant. However, acceleration at right angles to velocity transforms according to $a^\star = \gamma^2 a$. Hence, the rate of radiation in the lab frame is,

$$\frac{d\mathcal{E}}{dt} = -\frac{2e^2\gamma^4 a^2}{3c^3} = -\frac{2e^4\gamma^2 B^2}{3m^2c^3}\,, \tag{225}$$

using eq. (216) for the acceleration a.

The maximal energy of the electrons in the betatron obtains when the energy loss (225) cancels the energy gain (220), *i.e.*, when,

$$\frac{eR\dot{B}_{\rm ave}}{2} = \frac{2e^4\gamma_{\rm max}^2 B^2}{3m^2c^3}\,, \tag{226}$$

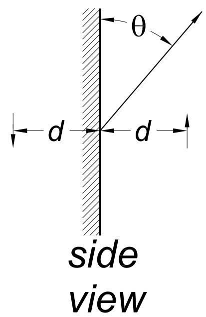
and,

$$\gamma_{\rm max} = \sqrt{\frac{3m^2c^3R\dot{B}_{\rm ave}}{4e^3B^2}} = \sqrt{\frac{3R}{4\alpha c}\frac{\dot{B}_{\rm ave}}{B}\frac{B_{\rm crit}}{B}} \approx \sqrt{\frac{3R}{4\alpha c\tau}\frac{B_{\rm crit}}{B}}\,, \tag{227}$$

where $\alpha = e^2/\hbar c = 1/137$ is the fine structure constant, $B_{\rm crit} = m^2c^3/e\hbar = 4.4\times 10^{13}$ G is the so-called QED critical field strength, and τ is the characteristic cycle time of the betatron such that $\dot{B}_{\rm ave} = B/\tau$. For example, with $R = 1$ m, $\tau = 0.03$ sec (30 Hz), and $B = 10^4$ G, we find that $\gamma_{\rm max} \approx 200$, or $\mathcal{E}_{\rm max} \approx 100$ MeV.

We have ignored the radiation due to the longitudinal acceleration of the electron, since in the limiting case this acceleration ceases.

12. Since the dipole is much less than a wavelength away from the conducting plane, the fields between the dipole and the plane are essentially the instantaneous static fields. Thus, charges arranges themselves on the plane as if there were an image dipole at distance d on the other side of the plane. The radiation from the moving charges on the plan is effectively that due to the oscillating image dipole. A distant observer sees the sum of the radiation fields from the dipole and its image.



The image dipole is inverted with respect to the original, *i.e.*, the two dipoles are 180° out of phase.

Furthermore, there is a difference s in path length between the two dipole and the distant observer at angles (θ, ϕ). We first calculate in a spherical coordinate system with z axis along the first dipole, and x axis pointing from the plane to that dipole. Then,the path difference is,

$$s = 2d\hat{\mathbf{x}} \cdot \hat{\mathbf{n}} = 2d \sin\theta \cos\phi. \tag{228}$$

This path difference results in an additional phase difference δ between the fields from the two dipoles at the observer, in the amount,

$$\delta = 2\pi \frac{s}{\lambda} = \frac{4\pi d}{\lambda} \sin\theta \cos\phi. \tag{229}$$

If we label the electric fields due to the original and image dipoles as $\mathbf{E}_1$ and $\mathbf{E}_2$, respectively, then the total field is,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{E}_1(1 - e^{i\delta}), \tag{230}$$

and, recalling eq. (46), the power radiated is,

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{|\mathbf{E}|^2}{|\mathbf{E}_1|^2} \frac{dP_1}{d\Omega} = \left|1 - e^{i\delta}\right|^2 A \sin^2\theta = 2A \sin^2\theta (1 - \cos\delta) = 4A \sin^2\theta \sin^2 \delta/2 \\ &= 4A \sin^2\theta \sin^2 \Delta, \end{aligned} \tag{231}$$

where,

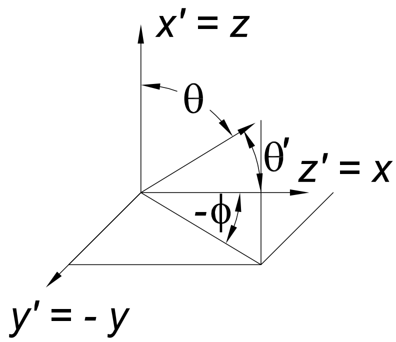
$$\Delta = \frac{\delta}{2} = \frac{2\pi d}{\lambda} \sin\theta \cos\phi. \tag{232}$$

Suppose we had chosen to use a spherical coordinate system (r, θ', ϕ') with the z' axis pointing from the plane to dipole 1, and the x' axis parallel to dipole 1. Then, the phase difference would have the simple form,

$$\Delta = \frac{\delta}{2} = \frac{\pi}{\lambda} 2d\ \hat{\mathbf{z}}' \cdot \hat{\mathbf{n}} = \frac{2\pi d}{\lambda} \cos\theta', \tag{233}$$

but the factor $\sin^2\theta$ would now become,

$$\sin^2\theta = n_x^2 + n_y^2 = n_{z'}^2 + n_{y'}^2 = \cos^2\theta' + \sin^2\theta' \cos^2\phi' = 1 - \sin^2\theta' \sin^2\phi'. \tag{234}$$



If $d = \lambda/4$, then,

$$\frac{dP}{d\Omega} = 4A \sin^2\theta \sin^2\left(\frac{\pi}{2} \sin\theta \cos\phi\right). \tag{235}$$

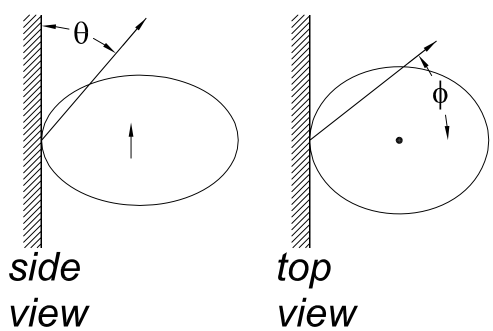
In the "side" view, $\phi = 0$, so the pattern has shape,

$$\sin^2\theta \sin^2\left(\frac{\pi}{2} \sin\theta\right) \qquad \text{(side view)}, \tag{236}$$

while in the "top" view, $\theta = \pi/2$ and the shape is,

$$\sin^2\left(\frac{\pi}{2} \cos\phi\right) \qquad \text{(top view)}. \tag{237}$$

This pattern has a single lobe in the forward hemisphere, as illustrated below:



If instead, $d = \lambda/2$, then,

$$\frac{dP}{d\Omega} = 4A \sin^2\theta \sin^2(\pi \sin\theta \cos\phi). \tag{238}$$

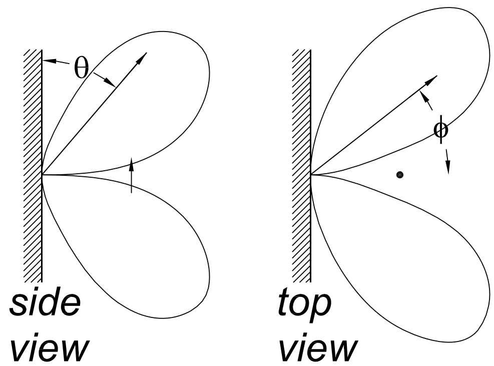
In the "side" view, $\phi = 0$, so the pattern has shape,

$$\sin^2\theta \sin^2(\pi \sin\theta) \qquad \text{(side view)}, \tag{239}$$

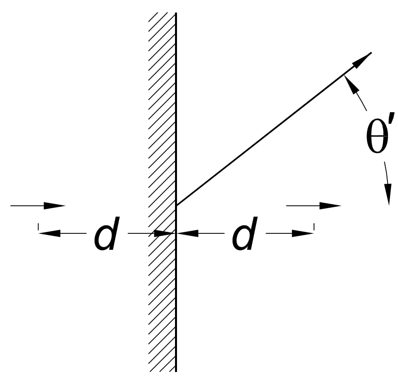
while in the "top" view, $\theta = \pi/2$ and the shape is,

$$\sin^2(\pi \cos\phi) \qquad \text{(top view)}. \tag{240}$$

This pattern does not radiate along the line from the plane to the dipole, as illustrated below:



b) If the electric dipole is aligned with the line from the plane to the dipole, its image has the same orientation.



The only phase difference between the radiation fields of the dipole and its image is that due to the path difference δ, whose value has been given in eqs. (232) and (233). It is simpler to use the angles (θ', ϕ') in this case, since the radiation pattern of a single dipole varies as $\sin^2\theta'$. Then,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{E}_1(1 + e^{i\delta}), \tag{241}$$

and, recalling eq. (46), the power radiated is,

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{|\mathbf{E}|^2}{|\mathbf{E}_1|^2}\frac{dP_1}{d\Omega} = \left|1 + e^{i\delta}\right|^2 A\sin^2\theta' = 2A\sin^2\theta'(1+\cos\delta) = 4A\sin^2\theta'\cos^2\delta/2 \\ &= 4A\sin^2\theta'\cos^2\Delta, \end{aligned} \tag{242}$$

with,

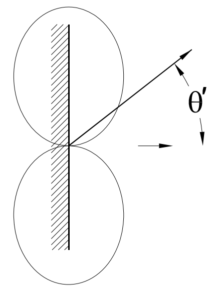
$$\Delta = \frac{\delta}{2} = \frac{2\pi d}{\lambda}\cos\theta'. \tag{243}$$

This radiation pattern is axially symmetric about the line from the plane to the dipole.

If $d = \lambda/4$, then,

$$\frac{dP}{d\Omega} = 4A \sin^2\theta' \cos^2\left(\frac{\pi}{2}\cos\theta'\right). \tag{244}$$

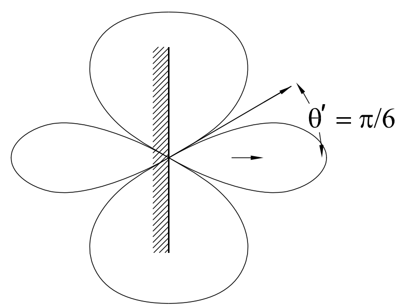
This pattern is a flattened version of the "donut" pattern $\sin^2\theta'$, as illustrated below:



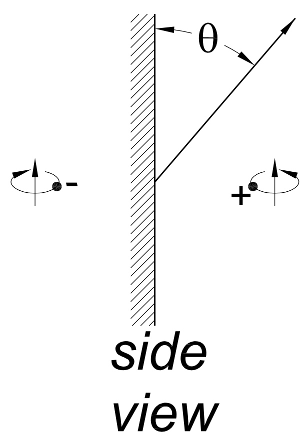
If instead, $d = \lambda/2$, then,

$$\frac{dP}{d\Omega} = 4A \sin^2\theta' \cos^2\left(\pi\cos\theta'\right). \tag{245}$$

This pattern has a forward lobe for $\theta' < \pi/6$ and a "donut" for $\pi/6 < \theta' < \pi/2$, as illustrated below:



c) For a magnetic dipole with axis parallel to the conducting plane, the image dipole has the same orientation, the image consists of the opposite charge rotating in the opposite direction, as shown below:



We use angles (θ, ϕ) and modify the argument of part a) to find,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{E}_1(1 + e^{i\delta}), \tag{246}$$

and, recalling eq. (46), the power radiated is,

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{|\mathbf{E}|^2}{|\mathbf{E}_1|^2}\frac{dP_1}{d\Omega} = \left|1+e^{i\delta}\right|^2 A\sin^2\theta = 2A\sin^2\theta(1+\cos\delta) = 4A\cos^2\theta\sin^2\delta/2 \\ &= 4A\sin^2\theta\cos^2\Delta, \end{aligned} \tag{247}$$

where,

$$\Delta = \frac{\delta}{2} = \frac{2\pi d}{\lambda}\sin\theta\cos\phi. \tag{248}$$

If $d = \lambda/4$, then,

$$\frac{dP}{d\Omega} = 4A\sin^2\theta\cos^2\left(\frac{\pi}{2}\sin\theta\cos\phi\right). \tag{249}$$

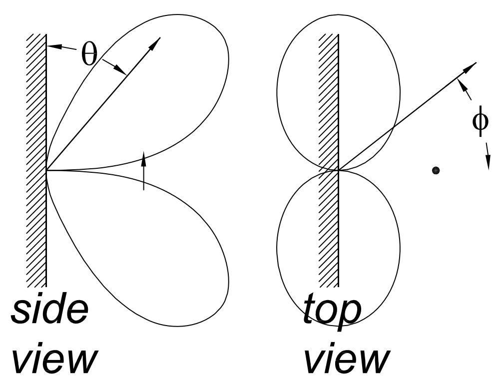
In the "side" view, $\phi = 0$, so the pattern has shape,

$$\sin^2\theta\cos^2\left(\frac{\pi}{2}\sin\theta\right) \qquad \text{(side view)}, \tag{250}$$

while in the "top" view, $\theta = \pi/2$ and the shape is,

$$\cos^2\left(\frac{\pi}{2}\cos\phi\right) \qquad \text{(top view)}. \tag{251}$$

This pattern, shown below, is somewhat similar to that of part a) for $d = \lambda/2$.



If instead, $d = \lambda/2$, then,

$$\frac{dP}{d\Omega} = 4A\sin^2\theta\cos^2(\pi\sin\theta\cos\phi). \tag{252}$$

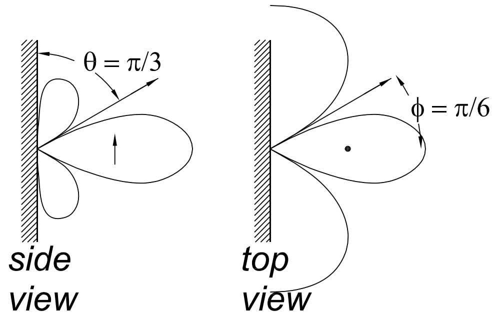
In the "side" view, $\phi = 0$, so the pattern has shape,

$$\sin^2\theta\cos^2(\pi\sin\theta) \qquad \text{(side view)}, \tag{253}$$

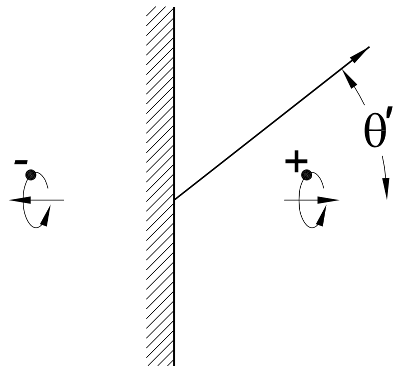
while in the "top" view, $\theta = \pi/2$ and the shape is,

$$\cos^2(\pi\cos\phi) \qquad \text{(top view)}. \tag{254}$$

This pattern, shown below, is somewhat similar to that of part b) for $d = \lambda/2$.



Finally, we consider the case of a magnetic dipole aligned with the line from the plane to the dipole, in which case its image has the opposite orientation.



As in part b), the only phase difference between the radiation fields of the dipole and its image is that due to the path difference δ, whose value has been given in eqs. (232) and (233). We use the angles (θ', ϕ') in this case, since the radiation pattern of a single dipole varies as $\sin^2\theta'$. Then,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{E}_1(1 - e^{i\delta}), \tag{255}$$

and, recalling eq. (46), the power radiated is,

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{|\mathbf{E}|^2}{|\mathbf{E}_1|^2}\frac{dP_1}{d\Omega} = \left|1 - e^{i\delta}\right|^2 A\sin^2\theta' = 2A\sin^2\theta'(1-\cos\delta) = 4A\sin^2\theta'\sin^2\delta/2 \\ &= 4A\sin^2\theta'\sin^2\Delta, \end{aligned} \tag{256}$$

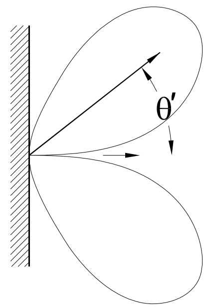
with,

$$\Delta = \frac{\delta}{2} = \frac{2\pi d}{\lambda}\cos\theta'. \tag{257}$$

This radiation pattern is axially symmetric about the line from the plane to the dipole. If $d = \lambda/4$, then,

$$\frac{dP}{d\Omega} = 4A\sin^2\theta'\sin^2\left(\frac{\pi}{2}\cos\theta'\right). \tag{258}$$

This pattern, shown below, is somewhat similar to that of part a) for $d = \lambda/2$.



If instead, $d = \lambda/2$, then,

$$\frac{dP}{d\Omega} = 4A\sin^2\theta' \sin^2(\pi\cos\theta')\,. \tag{259}$$

This pattern is qualitatively similar to that for $d = \lambda/4$, shown just above, but the maximum occurs at a larger value of θ'.

13. From p. 191, Lecture 16 of the Notes we recall that a single, short, center-fed, linear antenna of dipole moment,

$$p(t) = i\frac{I_0 L\, e^{-i\omega t}}{2\omega} \tag{260}$$

radiates time-averaged power (according to the Larmor formula),

$$\frac{dU_1}{dtd\Omega} = \frac{\langle \ddot{p}^2 \rangle \sin^2\theta}{4\pi c^3} = \frac{\omega^2 I_0^2 L^2 \sin^2\theta}{32\pi c^3}. \tag{261}$$

For the record, the current distribution in this short antenna is well approximated by the triangular waveform,

$$I(z,t) = I_0\, e^{-i\omega t}\left(1 - \frac{2\,|z|}{L}\right). \tag{262}$$

The associated charge distribution $\rho(z,t)$ is related by charge conservation, $\boldsymbol{\nabla}\cdot\mathbf{J} = -\dot{\rho}$, which for a 1-d distribution is simply,

$$\dot{\rho} = -\frac{\partial I}{\partial z} = -I_0\, e^{-i\omega t}\left(\mp\frac{2}{L}\right), \tag{263}$$

so that,

$$\rho = \pm\frac{2iI_0\, e^{-i\omega t}}{\omega L}, \tag{264}$$

and the dipole moment is given by,

$$p = \int_{-L/2}^{L/2} \rho z\, dz = i\frac{I_0 L\, e^{-i\omega t}}{2\omega}, \tag{265}$$

as claimed above.

Turning to the case to two antennas, we proceed as in the previous problem and write their combined electric field as,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{E}_1(1 - e^{i\delta}), \tag{266}$$

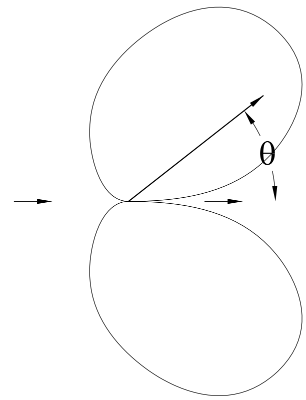
where now the phase difference δ has contributions due to the path difference for radiation from the two antennas (whose separation is $d = \lambda/4$), as well as from their intrinsic phase difference of 90°. That is,

$$\delta = \frac{2\pi}{\lambda}\frac{\lambda}{4}\cos\theta + \frac{\pi}{2} = \frac{\pi}{2}(1 + \cos\theta). \tag{267}$$

From eqs. (261), (266) and (267) we find

$$\begin{aligned} \frac{dU}{dtd\Omega} &= \frac{dU_1}{dtd\Omega}\left|1 - e^{i\delta}\right|^2 = \frac{\omega^2 I_0^2 L^2 \sin^2\theta}{16\pi c^3}(1 - \cos\delta) \\ &= \frac{\omega^2 I_0^2 L^2 \sin^2\theta}{16\pi c^3}\left[1 + \sin\left(\frac{\pi}{2}\cos\theta\right)\right]. \end{aligned} \tag{268}$$

This angular distribution favors the forward hemisphere, as shown in the sketch:



The total radiated power is,

$$\frac{dU}{dt} = \frac{\omega^2 I_0^2 L^2}{6c^3}. \tag{269}$$

Associated with energy U radiated in direction $\hat{\mathbf{n}}$ is momentum $\mathbf{P} = U\hat{\mathbf{n}}/c$. Thus, the angular distribution of radiated momentum is,

$$\frac{d\mathbf{P}}{dt\, d\Omega} = \frac{\omega^2 I_0^2 L^2 \sin^2\theta}{16\pi c^4}\left[1 + \sin\left(\frac{\pi}{2}\cos\theta\right)\right]\hat{\mathbf{n}}. \tag{270}$$

On integrating this over solid angle to find the total momentum radiated, only the z-component is nonzero,

$$\begin{aligned} \frac{dP_z}{dt} &= 2\pi\frac{\omega^2 I_0^2 L^2}{16\pi c^4}\int_{-1}^{1}\sin^2\theta\left[1 + \sin\left(\frac{\pi}{2}\cos\theta\right)\right]\cos\theta\; d\cos\theta \\ &= \frac{2\omega^2 I_0^2 L^2}{\pi^2 c^4}\left(\frac{12}{\pi^2} - 1\right) = \frac{12}{\pi^2 c}\frac{dU}{dt}\left(\frac{12}{\pi^2} - 1\right) \approx \frac{0.26}{c}\frac{dU}{dt}. \end{aligned} \tag{271}$$

The radiation reaction force on the antenna is $F_z = -dP_z/dt$. For a broadcast antenna radiating 10^5 Watts, the reaction force would be only $\approx 10^{-4}$ N.

The radiation-reaction force (271) cannot be deduced as the sum over charges of the radiation-reaction force of Planck, $\mathbf{F}_{\rm rad} = 2e^2\dddot{\mathbf{v}}/3c^3$. Planck's result is obtained by an integration by parts of the integral of the radiated power over a period. This procedure can be carried out if the power is a sum/integral of a square, as holds for an isolated radiating charge. But it cannot be carried out when the power is the square of a sum/integral as holds for (coherent) radiation by an extended charge/current distribution. Rather, the radiation reaction force on a current distribution must be deduced from the rate of radiation of momentum, as done here.

14. According to p. 181, Lecture 16 of the Notes, the power radiated from a known current distribution that oscillates at angular frequency ω is given by,

$$\frac{dP_\omega}{d\Omega} = \frac{1}{8\pi c}\left|\int \mathbf{J}_\omega(\mathbf{r}') \times \mathbf{k}\, e^{-i\mathbf{k}\cdot\mathbf{r}'}\, d\mathrm{Vol}'\right|^2, \tag{272}$$

where $\mathbf{k} = \hat{\mathbf{n}}\,\omega/c$.

We take the z-axis along the antenna, so the radiated power due to the current distribution,

$$I(z) = I_0 \sin\frac{2\pi z}{L}\, e^{-i\omega t} \qquad (-L/2 < z < L/2), \tag{273}$$

is,

$$\begin{aligned}\frac{dP_\omega}{d\Omega} &= \frac{1}{8\pi c}\left|\int_{-L/2}^{L/2} I_0 \sin\left(\frac{2\pi z}{L}\right) k\sin\theta\, e^{-ikz\cos\theta}\, dz\right|^2 \\ &= \frac{k^2 I_0^2}{8\pi c}\sin^2\theta \left|\int\right|^2, \end{aligned}\tag{274}$$

where,

$$\int = \int_{-L/2}^{L/2} \sin\left(\frac{2\pi z}{L}\right)[\cos(kz\cos\theta) - i\sin(kz\cos\theta)]\, dz. \tag{275}$$

For $L = \lambda$, we have $k = 2\pi/L$ and,

$$\int = \int_{-\pi/k}^{\pi/k} \sin kz[\cos(kz\cos\theta) - i\sin(kz\cos\theta)]\, dz. \tag{276}$$

The real part of this integral vanishes, while

$$\mathrm{Im}\int = -2\int_0^{\pi/k} \sin kz \sin(kz\sin\theta)\, dz = -\frac{2}{k}\int_0^{\pi} \sin x \sin(x\sin\theta)\, dx. \tag{277}$$

Using the identity,

$$2\sin A \sin B = \cos(A - B) - \cos(A + B), \tag{278}$$

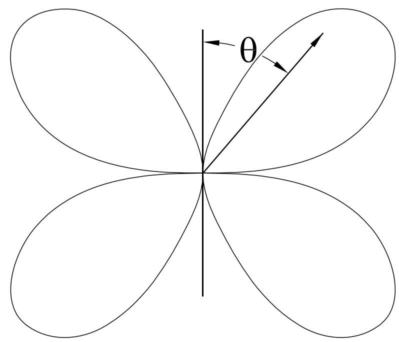
we have,

$$\begin{aligned}\mathrm{Im}\int &= \frac{1}{k}\int_0^{\pi}\{[\cos[x(1+\cos\theta)] - \cos[x(1-\cos\theta)]\}\, dx. \\ &= \frac{1}{k}\left\{\frac{\sin[\pi(1+\cos\theta)]}{1+\cos\theta} - \frac{\sin[\pi(1-\cos\theta)]}{1-\cos\theta}\right\} \\ &= \frac{1}{k}\left\{-\frac{\sin(\pi\cos\theta)}{1+\cos\theta} - \frac{\sin(\pi\cos\theta)}{1-\cos\theta}\right\} \\ &= \frac{2\sin(\pi\cos\theta)}{k\sin^2\theta}. \end{aligned}\tag{279}$$

Inserting this in eq. (274), we find,

$$\frac{dP_\omega}{d\Omega} = \frac{I_0^2}{2\pi c}\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}. \tag{280}$$

The radiation pattern is sketched below:



The total radiated power is, setting $\cos\theta = u$,

$$P = \int \frac{dP_\omega}{d\Omega}, d\Omega = \frac{I_0^2}{c}\int_{-1}^{1}\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}\,d\cos\theta = \frac{I_0^2}{2c}\int_{-1}^{1}\frac{1-\cos(2\pi u)}{1-u^2}\,du$$
$$= \frac{I_0^2}{4c}\int_{-1}^{1}\left(\frac{1-\cos(2\pi u)}{1+u} + \frac{1-\cos(2\pi u)}{1-u}\right)du = \frac{I_0^2}{2c}\int_{-1}^{1}\frac{1-\cos(2\pi u)}{1+u}\,du. \tag{281}$$

To cast this in the form of a known special function, we let $1 + u = v/2\pi$, so that,

$$P = \frac{I_0^2}{2c}\int_0^{4\pi}\frac{1-\cos v}{v}\,dv = \frac{I_0^2}{2c}\mathrm{Cin}(4\pi) = 3.11\frac{I_0^2}{2c}, \tag{282}$$

where Cin is the so-called cosine integral.

b) To calculate the radiation in the multipole approximation, we need to convert the current distribution $I(z)\,e^{-i\omega t}$ to a charge distribution $\rho(z,t)$. This is accomplished via the continuity equation,

$$\frac{\partial I}{\partial z} = -\dot\rho = i\omega\rho. \tag{283}$$

For the current distribution (273) we find,

$$\rho = -\frac{2\pi i}{\omega L}I_0\cos\frac{2\pi z}{L}\,e^{-i\omega t}. \tag{284}$$

The dipole moment of this distribution is,

$$p = \int_{-L/2}^{L/2}\rho z\;dz = 0, \tag{285}$$

so there is no electric-dipole radiation. As the current flows along a line, and not in a loop, there is no magnetic-dipole radiation either.

The charge distribution is symmetric about the z axis, so its tensor quadrupole moment can be characterized in terms of the single quantity,

$$\begin{aligned} Q_{zz} &= \int_{-L/2}^{L/2} 2\rho z^2\, dz = 4\int_0^{L/2} \rho z^2\, dz = -\frac{8\pi i}{\omega L} I_0 \int_0^{L/2} z^2 \cos\frac{2\pi z}{L}\, dz \\ &= -\frac{L^2 i}{\pi^2 \omega} I_0 \int_0^{\pi} x^2 \cos x\, dx = \frac{2L^2 i}{\pi\omega} I_0, \end{aligned} \tag{286}$$

using the integral (55).

The total power radiated by the symmetric quadrupole moment is, according to p. 190, Lecture 16 of the Notes,

$$P_{\rm E2} = \frac{|Q_{zz}|^2 \omega^6}{240 c^5} = \frac{L^4 \omega^4}{30\pi^2 c^4} \frac{I_0^2}{2c}. \tag{287}$$

When $L = \lambda = 2\pi c/\omega$, this becomes,

$$P_{\rm E2} = \frac{8\pi^2}{15} \frac{I_0^2}{2c} = 5.26 \frac{I_0^2}{2c}. \tag{288}$$

In this example, higher multipoles must contribute significantly to the total power, reducing it to the "exact" result (282).

15. *This problem is also discussed at* http://kirkmcd.princeton.edu/examples/small_sphere.pdf

The scattering cross section is given by,

$$\frac{d\sigma}{d\Omega} = \frac{\text{power scattered into } d\Omega}{\text{incident power per unit area}} = r^2 \frac{\langle \mathbf{S}_{\text{scat}}(\theta,\phi)\rangle}{\langle \mathbf{S}_{\text{incident}}\rangle} = r^2 \frac{|\mathbf{E}_{\text{scat}}|^2}{E_0^2}, \tag{289}$$

where in the dipole approximation, the far-zone scattered electric field is,

$$\mathbf{E}_{\text{scat}} = k^2 \frac{e^{i(kr-\omega t)}}{r} \left[(\hat{\mathbf{n}} \times \mathbf{p}_0) \times \hat{\mathbf{n}} - \hat{\mathbf{n}} \times \mathbf{m}_0\right], \tag{290}$$

and $\mathbf{p}_0 e^{i\omega t}$ and $\mathbf{m}_0 e^{-i\omega t}$ are the electric- and magnetic-dipole moments induced in the conducting sphere by the incident wave.

Because the incident wavelength is large compared to the radius of the sphere, the incident fields are essentially uniform over the sphere, and the induced fields near the sphere are the same as the static fields of a conducting sphere in an otherwise uniform electric and magnetic field. Then, from p. 57, Lecture 5 of the Notes, the induced electric-dipole moment is given by

$$\mathbf{p}_0 = a^3 \mathbf{E}_0. \tag{291}$$

For the induced magnetic dipole, we recall p. 98, Lecture 8 of the Notes, remembering that a conducting sphere can be thought of a permeable sphere with zero permeability and a dielectric sphere of infinite dielectric constant. Hence, the magnetic-dipole moment is,

$$\mathbf{m}_0 = -\frac{a^3}{2}\mathbf{B}_0. \tag{292}$$

Then,

$$\mathbf{E}_{\text{scat}} = -k^2 a^3 \frac{e^{i(kr-\omega t)}}{r} \left[\hat{\mathbf{n}} \times (\mathbf{E}_0 \times \hat{\mathbf{n}}) + \frac{1}{2}(\hat{\mathbf{n}} \times \mathbf{B}_0\right], \tag{293}$$

where $\hat{\mathbf{n}}$ is along the vector $\mathbf{r}$ that points from the center of the sphere to the distant observer.

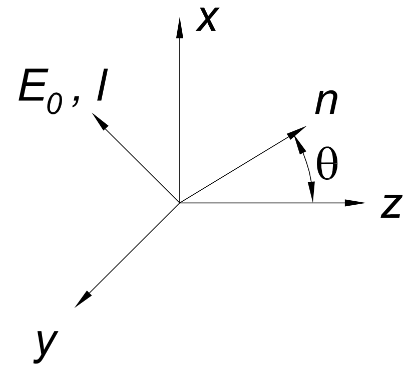
For a wave propagating in the $+z$ direction with electric field linearly polarized along direction $\hat{\mathbf{l}}$, $\mathbf{E}_0 = E\,\hat{\mathbf{l}}$, and the magnetic field obeys $\mathbf{B}_0 = \hat{\mathbf{z}} \times \mathbf{E}_0$. Then,

$$\begin{aligned} \mathbf{E}_{\text{scat}} &= -k^2 a^3 E_0 \frac{e^{i(kr-\omega t)}}{r} \left[\hat{\mathbf{n}} \times (\hat{\mathbf{l}} \times \hat{\mathbf{n}}) + \frac{1}{2}\hat{\mathbf{n}} \times (\hat{\mathbf{z}} \times \hat{\mathbf{l}})\right] \\ &= -k^2 a^3 E_0 \frac{e^{i(kr-\omega t)}}{r} \left[\hat{\mathbf{l}}\left(1 - \frac{(\hat{\mathbf{n}} \cdot \hat{\mathbf{z}})}{2}\right) + \left(\hat{\mathbf{n}} - \frac{\hat{\mathbf{z}}}{2}\right)(\hat{\mathbf{n}} \cdot \hat{\mathbf{l}})\right]. \end{aligned} \tag{294}$$

Inserting this in (289) we find,

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left[\left(1 - \frac{\hat{\mathbf{n}} \cdot \hat{\mathbf{z}}}{2}\right)^2 - \frac{3}{4}(\hat{\mathbf{n}} \cdot \hat{\mathbf{l}})^2\right]. \tag{295}$$

For an observer in the x-z plane, $\hat{\mathbf{n}}\cdot\hat{\mathbf{z}} = \cos\theta$. Then, for electric polarization parallel to the scattering plane $\hat{\mathbf{n}}\cdot\hat{\mathbf{l}} = \sin\theta$, while for polarization perpendicular to the scattering plane $\hat{\mathbf{n}}\cdot\hat{\mathbf{l}} = 0$.



Thus, eq. (295) yields,

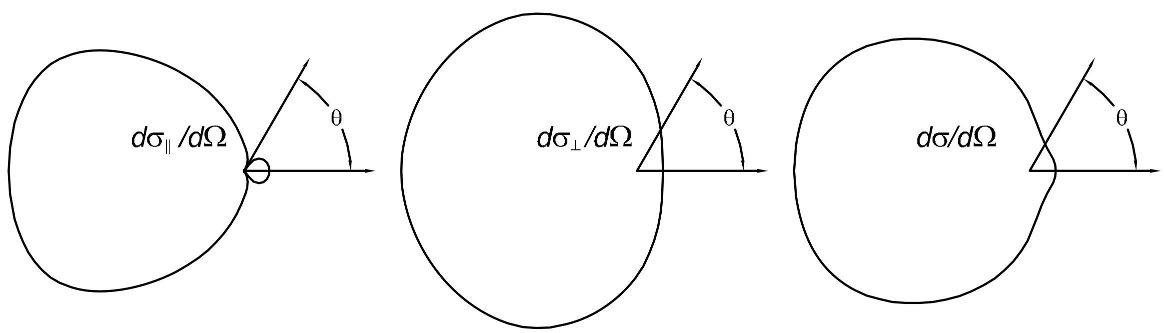
$$\frac{d\sigma_\parallel}{d\Omega} = k^4 a^6 \left(\frac{1}{2} - \cos\theta\right)^2, \qquad \frac{d\sigma_\perp}{d\Omega} = a^6 k^4 \left(1 - \frac{\cos\theta}{2}\right)^2. \tag{296}$$

For an unpolarized incident wave,

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{d\sigma_\parallel}{d\Omega} + \frac{d\sigma_\perp}{d\Omega}\right) = k^4 a^6 \left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta\right], \tag{297}$$

and so,

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega = \frac{10\pi}{8} k^4 a^6 \int_{-1}^{1} (1+\cos^2\theta)\, d\cos\theta = \frac{10\pi a^2}{3} k^4 a^4. \tag{298}$$



From eqs. (296) we see that $d\sigma_\perp/d\Omega$ is always nonzero, but $d\sigma_\parallel/d\Omega = 0$ for $\theta = \pi/3$, so for this angle, the scattered radiation is linearly polarized parallel to the scattering plane for arbitrary incident polarization.

Addendum: The Fields and Poynting Vector Close to the Sphere

Using the results of Prob. 2 above we can also discuss the fields close to the sphere. In particular, from eqs. (11) and (107) the scattered electric field at any position $\mathbf{r}$ outside the sphere is,

$$\mathbf{E}_{\rm scat}(\mathbf{r},t) \;=\; k^2 \frac{e^{i(kr-\omega t)}}{r} \left\{ (\hat{\mathbf{n}}\times\mathbf{p}_0)\times\hat{\mathbf{n}} + [3(\hat{\mathbf{n}}\cdot\mathbf{p}_0)\,\hat{\mathbf{n}} - \mathbf{p}_0]\left(\frac{1}{k^2r^2} - \frac{i}{kr}\right)\right.$$

$$-\left(1+\frac{i}{kr}\right)\hat{\mathbf{n}}\times\mathbf{m}_0\Bigg\}$$

$$= k^2a^3\frac{e^{i(kr-\omega t)}}{r}\Bigg\{(\hat{\mathbf{n}}\times\mathbf{E}_0)\times\hat{\mathbf{n}}+[3(\hat{\mathbf{n}}\cdot\mathbf{E}_0)\,\hat{\mathbf{n}}-\mathbf{E}_0]\left(\frac{1}{k^2r^2}-\frac{i}{kr}\right)$$

$$+\frac{1}{2}\left(1+\frac{i}{kr}\right)\hat{\mathbf{n}}\times\mathbf{B}_0\Bigg\}, \qquad (299)$$

also using eqs. (291) and (292). In this Addendum, we suppose that the electric field of the incident plane wave is along the x-axis, so that $\mathbf{E}_0 = E\,\hat{\mathbf{x}}$ and $\mathbf{B}_0 = E\,\hat{\mathbf{y}}$, while the point of observation is at $\mathbf{r} = (r, \theta, \phi)$. We express the electric-field vector in spherical coordinates, noting that,

$$\hat{\mathbf{n}} = \hat{\mathbf{r}}, \qquad (300)$$

$$\hat{\mathbf{x}} = \sin\theta\cos\phi\;\hat{\mathbf{r}}+\cos\theta\cos\phi\;\hat{\boldsymbol{\theta}}-\sin\phi\;\hat{\boldsymbol{\phi}}, \qquad (301)$$

$$\hat{\mathbf{y}} = \sin\theta\sin\phi\;\hat{\mathbf{r}}+\cos\theta\sin\phi\;\hat{\boldsymbol{\theta}}+\cos\phi\;\hat{\boldsymbol{\phi}}, \qquad (302)$$

$$\hat{\mathbf{z}} = \cos\theta\;\hat{\mathbf{r}}-\sin\theta\;\hat{\boldsymbol{\theta}}. \qquad (303)$$

Thus,

$$\mathbf{E}_{\rm scat}(\mathbf{r},t) = k^2a^3E_0\frac{e^{i(kr-\omega t)}}{r}\Bigg\{\cos\theta\cos\phi\;\hat{\boldsymbol{\theta}}-\sin\phi\;\hat{\boldsymbol{\phi}}$$

$$+(2\sin\theta\cos\phi\;\hat{\mathbf{r}}-\cos\theta\cos\phi\;\hat{\boldsymbol{\theta}}+\sin\phi\;\hat{\boldsymbol{\phi}})\left(\frac{1}{k^2r^2}-\frac{i}{kr}\right)$$

$$-\frac{1}{2}\left(1+\frac{i}{kr}\right)(\cos\phi\;\hat{\boldsymbol{\theta}}-\cos\theta\sin\phi\;\hat{\boldsymbol{\phi}})\Bigg\}$$

$$= k^2a^3E_0\frac{e^{i(kr-\omega t)}}{r}\Bigg\{2\sin\theta\cos\phi\;\left(\frac{1}{k^2r^2}-\frac{i}{kr}\right)\hat{\mathbf{r}}$$

$$+\cos\phi\;\left[\cos\theta\left(1-\frac{1}{k^2r^2}+\frac{i}{kr}\right)-\frac{1}{2}\left(1+\frac{i}{kr}\right)\right]\hat{\boldsymbol{\theta}}$$

$$-\sin\phi\;\left[1-\frac{1}{k^2r^2}+\frac{i}{kr}-\frac{\cos\theta}{2}\left(1+\frac{i}{kr}\right)\right]\hat{\boldsymbol{\phi}}\Bigg\}. \qquad (304)$$

Similarly, using eqs. (8) and (108) the scattered magnetic field can be written as,

$$\mathbf{B}_{\rm scat}(\mathbf{r},t) = k^2\,\frac{e^{i(kr-\omega t)}}{r}\Bigg\{(\hat{\mathbf{n}}\times\mathbf{m}_0)\times\hat{\mathbf{n}}+[3(\hat{\mathbf{n}}\cdot\mathbf{m}_0)\,\hat{\mathbf{n}}-\mathbf{m}_0]\left(\frac{1}{k^2r^2}-\frac{i}{kr}\right)$$

$$+\left(1+\frac{i}{kr}\right)\hat{\mathbf{n}}\times\mathbf{p}_0\Bigg\}$$

$$= -k^2a^3\,\frac{e^{i(kr-\omega t)}}{2r}\Bigg\{(\hat{\mathbf{n}}\times\mathbf{B}_0)\times\hat{\mathbf{n}}+[3(\hat{\mathbf{n}}\cdot\mathbf{B}_0)\,\hat{\mathbf{n}}-\mathbf{B}_0]\left(\frac{1}{k^2r^2}-\frac{i}{kr}\right)$$

$$-2\left(1+\frac{i}{kr}\right)\hat{\mathbf{n}}\times\mathbf{E}_0\Bigg\}$$

$$= -k^2a^3E_0\,\frac{e^{i(kr-\omega t)}}{2r}\Bigg\{\cos\theta\sin\phi\,\hat{\boldsymbol{\theta}}+\cos\phi\,\hat{\boldsymbol{\phi}}$$

$$, \qquad +(2\sin\theta\sin\phi\,\hat{\mathbf{r}}-\cos\theta\sin\phi\,\hat{\boldsymbol{\theta}}-\cos\phi\,\hat{\boldsymbol{\phi}})\left(\frac{1}{k^2r^2}-\frac{i}{kr}\right)$$

$$
\begin{aligned}
& -2\left(1+\frac{i}{kr}\right)(\sin\phi\,\hat{\boldsymbol{\theta}}+\cos\theta\cos\phi\,\hat{\boldsymbol{\phi}})\Big\} \\
= & -k^2a^3E_0\,\frac{e^{i(kr-\omega t)}}{2r}\Big\{2\sin\theta\sin\phi\,\left(\frac{1}{k^2r^2}-\frac{i}{kr}\right)\hat{\mathbf{r}} \\
& +\sin\phi\left[\cos\theta\left(1-\frac{1}{k^2r^2}+\frac{i}{kr}\right)-2\left(1+\frac{i}{kr}\right)\right]\hat{\boldsymbol{\theta}} \\
& +\cos\phi\left[1-\frac{1}{k^2r^2}+\frac{i}{kr}-2\cos\theta\left(1+\frac{i}{kr}\right)\right]\hat{\boldsymbol{\phi}}\Big\}. \qquad (305)
\end{aligned}
$$

On the surface of the sphere, $r = a$, the scattered electromagnetic fields are, to the leading approximation when $ka \ll 1$,

$$
\begin{aligned}
\mathbf{E}_{\rm scat}(r=a) &\approx E_0\,e^{-i\omega t}(2\sin\theta\cos\phi\,\hat{\mathbf{r}}-\cos\theta\cos\phi\,\hat{\boldsymbol{\theta}}+\sin\phi\,\hat{\boldsymbol{\phi}}), \qquad (306) \\
\mathbf{B}_{\rm scat}(r=a) &\approx -\frac{E_0}{2}\,e^{-i\omega t}(2\sin\theta\sin\phi\,\hat{\mathbf{r}}-\cos\theta\sin\phi\,\hat{\boldsymbol{\theta}}-\cos\phi\,\hat{\boldsymbol{\phi}}). \qquad (307)
\end{aligned}
$$

In the same approximation, the incident electromagnetic fields at the surface of the sphere are,

$$
\begin{aligned}
\mathbf{E}_{\rm in}(r=a) &\approx E_0\,e^{-i\omega t}\,\hat{\mathbf{x}} = E_0\,e^{-i\omega t}(\sin\theta\cos\phi\,\hat{\mathbf{r}}+\cos\theta\cos\phi\,\hat{\boldsymbol{\theta}}-\sin\phi\,\hat{\boldsymbol{\phi}}), \qquad (308) \\
\mathbf{B}_{\rm in}(r=a) &\approx E_0\,e^{-i\omega t}\,\hat{\mathbf{y}} = E_0\,e^{-i\omega t}(\sin\theta\sin\phi\,\hat{\mathbf{r}}+\cos\theta\sin\phi\,\hat{\boldsymbol{\theta}}+\cos\phi\,\hat{\boldsymbol{\phi}}). \qquad (309)
\end{aligned}
$$

Thus, the total electric field,

$$
\mathbf{E}_{\rm tot}(r=a) = \mathbf{E}_{\rm in}(r=a)+\mathbf{E}_{\rm scat}(r=a) = 3E_0\,e^{-i\omega t}\sin\theta\cos\phi\,\hat{\mathbf{r}}, \qquad (310)
$$

on the surface of the sphere is purely radial, and the total magnetic field,

$$
\mathbf{B}_{\rm tot}(r=a) = \mathbf{B}_{\rm in}(r=a)+\mathbf{B}_{\rm scat}(r=a) = \frac{3}{2}E_0\,e^{-i\omega t}(\cos\theta\sin\phi\,\hat{\boldsymbol{\theta}}+\cos\phi\,\hat{\boldsymbol{\phi}}), \qquad (311)
$$

is purely tangential, as expected for a perfect conductor.

The total charge density $\sigma_{\rm tot}$ on the surface of the conducting sphere follows from Gauss' law as,

$$
\sigma_{\rm tot} = \frac{\mathbf{E}_{\rm tot}(r=a)\cdot\hat{\mathbf{r}}}{4\pi} = \frac{3E_0}{4\pi}\,e^{-i\omega t}\sin\theta\cos\phi = \frac{3}{2}\,\sigma_{\rm scat}, \qquad (312)
$$

where $\sigma_{\rm scat}$ is the surface charge density corresponding to the scattered field (306). Similarly, the total current density $\mathbf{K}_{\rm tot}$ on the surface of the sphere follows from Ampère's law as,

$$
\mathbf{K}_{\rm tot} = \frac{c}{4\pi}\hat{\mathbf{r}}\times\mathbf{B}_{\rm tot}(r=a) = \frac{3cE_0}{8\pi}\,e^{-i\omega t}(-\cos\phi\,\hat{\boldsymbol{\theta}}+\cos\theta\sin\phi\,\hat{\boldsymbol{\phi}}) = 3\,\mathbf{K}_{\rm scat}, \qquad (313)
$$

where $\mathbf{K}_{\rm scat}$ is the surface charge density corresponding to the scattered field (307).

We can now discuss the energy flow in the vicinity of the conductor sphere from two perspectives. These two views have the same implications for energy flow in the far zone, but differ in their description of the near zone.

First, we can consider the Poynting vector constructed from the total electromagnetic fields,

$$\mathbf{S}_{\rm tot} = \frac{c}{4\pi}\mathbf{E}_{\rm tot} \times \mathbf{B}_{\rm tot}. \tag{314}$$

Because the tangential component of the total electric field vanishes at the surface of the sphere, lines of the total Poynting vector do not begin or end on the sphere, but rather they pass by it tangentially. In this view, the sphere does not absorb or emit energy, but simply redirects (scatters) the flow of energy from the incident wave.

However, this view does not correspond closely to the "microscopic" interpretation that atoms in the sphere are excited by the incident wave and emit radiation as a result, thereby creating the scattered wave. We obtain a second view of the energy flow that better matches the "microscopic" interpretation if we write,

$$\begin{aligned}
\mathbf{S}_{\rm tot} &= \frac{c}{4\pi}\mathbf{E}_{\rm tot} \times \mathbf{B}_{\rm tot} \\
&= \frac{c}{4\pi}(\mathbf{E}_{\rm in} + \mathbf{E}_{\rm scat}) \times (\mathbf{B}_{\rm in} + \mathbf{B}_{\rm scat}) \\
&= \frac{c}{4\pi}\mathbf{E}_{\rm in} \times \mathbf{B}_{\rm in} + \frac{c}{4\pi}(\mathbf{E}_{\rm in} \times \mathbf{B}_{\rm scat} + \mathbf{E}_{\rm scat} \times \mathbf{B}_{\rm in}) + \frac{c}{4\pi}\mathbf{E}_{\rm scat} \times \mathbf{B}_{\rm scat} \\
&= \mathbf{S}_{\rm in} + \mathbf{S}_{\rm interaction} + \mathbf{S}_{\rm scat}.
\end{aligned} \tag{315}$$

Since the scattered fields (306)-(307) at the surface of the sphere include tangential components for both the electric and the magnetic field, the scattered Poynting vector, $\mathbf{S}_{\rm scat}$, has a radial component, whose time average we wish to interpret as the flow of energy radiated by the sphere. The scattered Poynting vector at any r is given by,

$$\begin{aligned}
\langle \mathbf{S}_{\rm scat} \rangle &= \frac{c}{8\pi} Re(\mathbf{E}^\star_{\rm scat} \times \mathbf{B}_{\rm scat}) \\
&= \frac{c}{8\pi} Re\Big[(E^\star_{\theta,\rm scat} B_{\phi,\rm scat} - E^\star_{\phi,\rm scat} B_{\theta,\rm scat})\,\hat{\mathbf{r}} + (E^\star_{\phi,\rm scat} B_{r,\rm scat} - E^\star_{r,\rm scat} B_{\phi,\rm scat})\,\hat{\boldsymbol{\theta}} \\
&\qquad + (E^\star_{r,\rm scat} B_{\theta,\rm scat} - E^\star_{\theta,\rm scat} B_{r,\rm scat})\,\hat{\boldsymbol{\phi}} \Big] \\
&= \frac{c}{8\pi}\frac{k^4 a^6 E_0^2}{r^2} \left\{ \left[\cos^2\phi \left(\frac{1}{2} - \cos\theta\right)^2 + \sin^2\phi \left(1 - \frac{\cos\theta}{2}\right)^2 \right] \hat{\mathbf{r}} \right. \\
&\qquad \left. - \frac{1}{k^4 r^4} \left(\frac{\cos\theta}{2}\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}} \right) \right\}.
\end{aligned} \tag{316}$$

The radial term of eq. (316) in square brackets is identical to the far-zone Poynting vector. However, close to the sphere we find additional terms in $\langle \mathbf{S}_{\rm scat} \rangle$, so that in the near zone $\langle \mathbf{S}_{\rm rad} \rangle \neq \langle \mathbf{S}_{\rm scat} \rangle$. Indeed, at the surface of the sphere we find

$$\begin{aligned}
\langle \mathbf{S}_{\rm scat}(r = a) \rangle &= \frac{c}{8\pi} E_0^2 \left\{ k^4 a^4 \left[\cos^2\phi \left(\frac{1}{2} - \cos\theta\right)^2 + \sin^2\phi \left(1 - \frac{\cos\theta}{2}\right)^2 \right] \hat{\mathbf{r}} \right. \\
&\qquad \left. - \left(\frac{\cos\theta}{2}\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}} \right) \right\} \\
&\approx -\frac{c}{8\pi} E_0^2 \left(\frac{\cos\theta}{2}\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}} \right).
\end{aligned} \tag{317}$$

Of course, the conducting sphere is not an energy source by itself, and the radiated energy is equal to the energy absorbed from the incident wave. For a description of the flow of energy that is absorbed, we look to the time-average of the incident and interaction terms of eq. (315). Lines of the incident Poynting vector,

$$\langle \mathbf{S}_{\rm in} \rangle = \frac{c}{8\pi} E_0^2 \hat{\mathbf{z}} = \frac{c}{8\pi} E_0^2 (\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}), \qquad (318)$$

enter and leave the sphere with equal strength, and are therefore not to be associated with energy transfer to the radiation fields. So, we look to the interaction term,

$$\begin{aligned}
\langle \mathbf{S}_{\rm interaction} \rangle &= \frac{c}{8\pi} Re \Big[(E^\star_{\theta,{\rm scat}} B_{\phi,{\rm in}} + E^\star_{\theta,{\rm in}} B_{\phi,{\rm scat}} - E^\star_{\phi,{\rm scat}} B_{\theta,{\rm in}} - E^\star_{\phi,{\rm in}} B_{\theta,{\rm scat}})\,\hat{\mathbf{r}} \\
&\quad + (E^\star_{\phi,{\rm scat}} B_{r,{\rm in}} + E^\star_{\phi,{\rm in}} B_{r,{\rm scat}} - E^\star_{r,{\rm scat}} B_{\phi,{\rm in}} - E^\star_{r,{\rm in}} B_{\phi,{\rm scat}}),\hat{\boldsymbol{\theta}} \\
&\quad + (E^\star_{r,{\rm scat}} B_{\theta,{\rm in}} + E^\star_{r,{\rm in}} B_{\theta,{\rm scat}} - E^\star_{\theta,{\rm scat}} B_{r,{\rm in}} - E^\star_{\theta,{\rm in}} B_{r,{\rm scat}})\,\hat{\boldsymbol{\phi}} \Big] \\
&= \frac{c}{8\pi} \frac{k^2 a^3 E_0^2}{r} \Bigg\{ \Bigg[-\cos[kr(1-\cos\theta)] \frac{\cos\theta}{2k^2 r^2} \\
&\quad + (1+\cos\theta) \left[\cos^2\phi \left(\frac{1}{2} - \cos\theta \right) + \sin^2\phi \left(\frac{\cos\theta}{2} - 1 \right) \right] \times \\
&\quad \left(\frac{\sin[kr(1-\cos\theta)]}{kr} - \cos[kr(1-\cos\theta)] \right) \Bigg]\,\hat{\mathbf{r}} \\
&\quad + \left[\cos[kr(1-\cos\theta)] \frac{\sin\theta}{k^2 r^2} \left(2 - \frac{9}{2}\cos^2\phi \right) + \ldots \right] \hat{\boldsymbol{\theta}} \\
&\quad + \left[\frac{9}{8} \cos[kr(1-\cos\theta)] \frac{\sin 2\theta \sin 2\phi}{k^2 r^2} + \ldots \right] \hat{\boldsymbol{\phi}} \Bigg\}, \qquad (319)
\end{aligned}$$

where the omitted terms are small close to the sphere. Note that in the far zone the time-average interaction Poynting vector contains terms that vary as $1/r$ times $\cos[kr(1-\cos\theta)]$. These large terms oscillate with radius r with period λ, and might be said to describe a radial "sloshing" of energy in the far zone, rather than a radial flow. It appears in practice that one cannot detect this "sloshing" by means of a small antenna placed in the far zone, so we consider these terms to be unphysical. Nonetheless, it is interesting that they appear in the formalism.

At the surface of the sphere we have, again for $ka \ll 1$,

$$\langle \mathbf{S}_{\rm interaction}(r=a) \rangle = \frac{c}{8\pi} E_0^2 \left[-\frac{\cos\theta}{2}\,\hat{\mathbf{r}} + \sin\theta \left(2 - \frac{9}{2}\cos^2\phi \right) \hat{\boldsymbol{\theta}} + \frac{9}{8} \sin 2\theta \sin 2\phi\,\hat{\boldsymbol{\phi}} \right]. \qquad (320)$$

The total Poynting vector on the surface of the sphere is the sum of eqs. (317), (318) and (320),

$$\langle \mathbf{S}_{\rm tot}(r=a) \rangle = \frac{c}{8\pi} E_0^2 \left(-\frac{9}{2} \sin\theta \cos^2\phi\,\hat{\boldsymbol{\theta}} + \frac{9}{8} \sin 2\theta \sin 2\phi\,\hat{\boldsymbol{\phi}} \right). \qquad (321)$$

The radial component of the total Poynting vector vanishes on the surface of the sphere, as expected for a perfect conductor.

This exercise permits an additional perspective, of possible relevance to thinking about radiation from antennas. Suppose that instead of knowing that a plane wave was incident on the conducting sphere, we were simply given the surface current distribution $\mathbf{K}_{\rm scat}$ of eq. (313). Then, by use of retarded potentials, or the "antenna formula" (p. 182, Lecture 15 of the Notes), we could calculate the radiated power in the far zone, and would arrive at the usual expression (316) (ignoring the terms that fall off as $1/r^6$). However, this procedure would lead to an incomplete understanding of the near zone. In particular, the excitation of the conducting sphere by an external plane wave leads to a total surface current that is three times larger than the current $\mathbf{K}_{\rm scat}$. For a good, but not perfectly conducting sphere, an analysis based on $\mathbf{K}_{\rm scat}$ alone would lead to only 1/9 the actual amount of Joule heating of the sphere. And, if we attempted to assign some kind of impedance or radiation resistance to the sphere via the form $P_{\rm rad} = R_{\rm rad} I_0^2/2$ where I_0 is meant to be a measure of the peak total current, an analysis based only on knowledge the current $\mathbf{K}_{\rm scat}$ would lead to a value of $R_{\rm rad}$ that is 9 times larger than desired.