ELASTIC SCATTERING OF ELECTRONS AND HADRONS, INCLUDING SPIN

1. DIRAC'S EQUATION

DIRAC SOUGHT A RELATIVISTIC WAVE EQUATION WHICH WOULD BE ONLY A 1ST-ORDER DIFFERENTIAL EQUATION, AS OPPOSED TO THE KLEIN-GORDON EQUATION WHICH IS 2ND ORDER. HE FOUND THIS COULD NOT BE DONE WITH ORDINARY WAVE FUNCTIONS, BUT RATHER 4-COMPONENT (SPINOR) WAVE FUNCTIONS WERE REQUIRED. THESE ARE A GENERALISATION OF THE NON-RELATIVISTIC 2-COMPONENT PAULI SPINORS.

1.a. RELATIVITY + 1ST ORDER DIFFERENTIAL EQUATION $\Rightarrow$ SPIN.

THE DIRAC EQUATION IN FREE SPACE IS

$$i\,\gamma_\mu \partial^\mu \psi = m\psi$$

ψ IS A 4-COMPONENT OBJECT $\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$

$\partial_\mu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$ AS BEFORE

AND γ_μ IS A SET OF FOUR 4x4 MATRICES.

IN TERMS OF THE 2x2 PAULI SPIN MATRICES,

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}.$$

WE WILL RESTRICT OUR DISCUSSION OF DIRAC'S EQUATION TO THE NATURE OF PLANE WAVE SOLUTIONS NEEDED FOR SCATTERING PROBLEMS. WE SEEK SOLUTIONS $\psi = u\,e^{-ipx}$, WHERE THE SPINOR, u, DOES NOT DEPEND ON $x = (t, \vec{x})$

SUBSTITUTION INTO DIRAC'S EQ. GIVES

$$\gamma_\mu p^\mu u = m u$$

PH 529 LECTURE 6

THE PRODUCT OF A 4 VECTOR WITH THE MATRIX 4-VECTOR γ_μ OCCURS SO OFTEN, FEYNMAN INVENTED A SPECIAL NOTATION;

$$\not{a} \equiv a_\mu \gamma^\mu$$

THUS OUR SPINOR EQUATION IS $\not{p}\, u = m u$

WE WISH TO SHOW THAT THIS REQUIRES $m^2 = p^2 = E^2 - \vec{p}^{\,2}$.

THIS CAN BE DONE UTILIZING SOME ALGEBRA OF THE γ MATRICES:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\,\delta_{\mu\nu} \qquad \delta_{\mu\nu} \equiv \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

THEN $\not{a}\not{b} = a_\mu \gamma^\mu b_\nu \gamma^\nu = -b_\nu \gamma^\nu a_\mu \gamma^\mu + 2\delta^{\mu\nu} a_\mu b_\nu$

OR $\not{a}\not{b} = -\not{b}\not{a} + 2ab$

IN PARTICULAR $\not{p}\not{p} = p^2$

BUT $\not{p}\not{p}\, u = \not{p}(m u) = m^2 u$ USING DIRAC'S EQUATION

SO $p^2 u = m^2 u$

OR $p^2 = m^2$, CONSISTENT WITH RELATIVITY.

WE NOW EXAMINE THE NATURE OF THE SPINOR u. FIRST CONSIDER THE CASE OF A PARTICLE AT REST: $p_\mu = (E, 0)$

THEN $\not{p}\, u = m u$ REDUCES TO $E\gamma_0 u = m u$

THE SIMPLEST SPINORS POSSIBLE ARE $u_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $u_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$, $u_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ AND $u_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

RECALL THAT $\gamma_0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

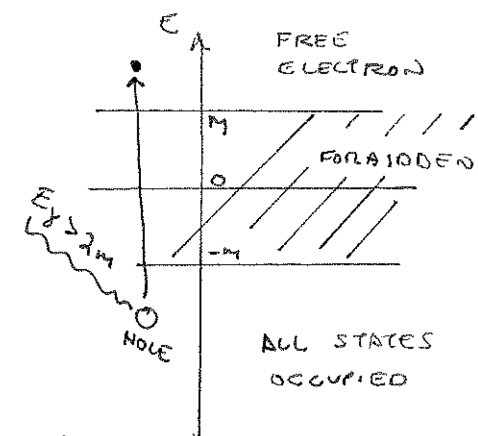
HENCE FOR SPINORS u_1 AND u_2 WE HAVE $E = m$

BUT FOR u_3 AND u_4 $E = -m$

u_3 AND u_4 ARE THE FAMOUS NEGATIVE ENERGY SOLUTIONS, WHICH DIRAC INTERPRETED AS ANTI-PARTICLES.

WE READILY INTERPRET u_1 AND u_2 AS THE SPINORS FOR THE SPIN UP AND SPIN DOWN STATES OF A SPIN 1/2 PARTICLE. DIRAC WAS ABLE TO GIVE A MEANINGFUL AND PROPHETIC INTERPRETATION OF THE 'NEGATIVE ENERGY' STATES u_3 AND u_4 BY MEANS OF THE HOLE THEORY.

THE NEGATIVE ENERGY STATES SEEM UNPLEASANT BECAUSE OF PARADOXES INVOLVING CONSERVATION OF ENERGY. (SUCH A STATE COULD RADIATE LIGHT, DROPPING TO STILL LOWER ENERGY, THEN RADIATING MORE LIGHT ....) DIRAC ARGUED THAT THE ORDINARY STATE OF AFFAIRS (THE VACUUM) CONSISTS OF A 'SEA' OF NEGATIVE ENERGY PARTICLES, OCCUPYING ALL POSSIBLE NEGATIVE ENERGY STATES. THIS CONVENIENTLY PROHIBITS ANY OF THE PARADOXICAL TRANSITIONS. OF COURSE WE HAVE TO IGNORE THE INFINITE ENERGY & CHARGE OF THIS SEA. (RENORMALISATION)



TRANSITIONS INVOLVING THE NEGATIVE ENERGY SEA ARE POSSIBLE IF ENOUGH ENERGY IS ADDED TO THE SYSTEM. FOR EXAMPLE, A 'VIRTUAL' PHOTON OF ENERGY $E > 2m$, AND MOMENTUM 0, COULD PROMOTE A NEGATIVE ENERGY ELECTRON ACROSS THE 'GAP' INTO THE REALM OF FREE ELECTRONS. THIS WOULD LEAVE A HOLE IN THE NEGATIVE ENERGY SEA. DIRAC INTERPRETED THE HOLE AS A POSITIVELY CHARGED PARTICLE. THE HOLE COULD THEN MOVE AROUND IN THE SEA, OBEYING LAWS APPROPRIATE FOR A POSITIVE ENERGY STATE. TO A LABORATORY OBSERVER, THE 'POSITRON' WILL APPEAR VERY REAL!

OF COURSE, IF AN ELECTRON COLLIDES WITH A HOLE, BOTH MAY DISAPPEAR FROM ORDINARY VIEW, LEAVING ONLY A FLASH OF LIGHT, OF ENERGY EQUAL THAT OF THE ELECTRON + POSITRON.

U_3 IS THE SPINOR OF A SPIN UP, NEGATIVE ENERGY ELECTRON. THE PHYSICALLY MEANINGFUL STATE IS THE HOLE, OR ABSENCE OF A SPIN UP, NEGATIVE E. HENCE u_3 IS INTERPRETED AS A SPIN DOWN, POSITIVE ENERGY, + CHARGE!

IT IS CONVENTIONAL TO LABEL ANTI PARTICLE SPINORS BY v RATHER THAN u. WE WOULD LIKE TO USE POSITIVE ENERGIES IN THE WAVE FUNCTION $v e^{-ipx}$, SO IT IS CUSTOMARY TO REWRITE THE DIRAC EQUATION FOR ANTI-PARTICLES AS

$$\not{p} v = -m v \qquad \text{(ANTI PARTICLES)}$$

NOW CONSIDER THE SPINORS FOR A PARTICLE OF MOMENTUM $\bar{p}$, AND $E = \sqrt{\bar{p}^2 + m^2}$. WE CAN WRITE THE 4-COMPONENT DIRAC EQUATION AS 2 2-COMPONENT EQUATIONS, DEFINING

$$u = \begin{pmatrix} \chi \\ \varphi \end{pmatrix} \quad \text{WHERE } \chi, \varphi \text{ ARE 2 COMPONENT SPINORS}$$

$$\not{p} u = m u \Rightarrow \left[E \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \bar{p} \cdot \begin{pmatrix} 0 & \bar{\sigma} \\ -\bar{\sigma} & 0 \end{pmatrix} \right] \begin{pmatrix} \chi \\ \varphi \end{pmatrix} = m \begin{pmatrix} \chi \\ \varphi \end{pmatrix}$$

THEN
$$E\chi - \bar{p}\cdot\bar{\sigma}\,\varphi = m\chi \Rightarrow \chi = \frac{\bar{p}\cdot\bar{\sigma}}{E-m}\varphi$$
$$-E\varphi + \bar{p}\cdot\bar{\sigma}\,\chi = m\varphi \Rightarrow \varphi = \frac{\bar{p}\cdot\bar{\sigma}}{E+m}\chi$$

HENCE
$$u \approx \begin{pmatrix} \chi \\ \frac{\bar{p}\cdot\bar{\sigma}}{E+m}\chi \end{pmatrix}$$
IS THE GENERAL PARTICLE SPINOR

SIMILARLY
$$v \approx \begin{pmatrix} \frac{\bar{p}\cdot\bar{\sigma}}{E+m}\varphi \\ \varphi \end{pmatrix}$$
IS THE GENERAL ANTIPARTICLE SPINOR (RECALL $\not{p} v = -m v$)

FOR NON-ZERO VELOCITIES, ALL 4 COMPONENTS ARE USED IN BOTH PARTICLE AND ANTI-PARTICLE SPINORS. THE DIRAC EQUATION DOES NOT IN GENERAL SPLIT INTO TWO INDEPENDENT EQUATIONS. ANOTHER COMPLEXITY OF RELATIVISTIC QUANTUM MECHANICS.

Ph 529 Lecture 6

As a final topic in our brief introduction to the Dirac equation, we consider how to normalize the spinors.

We follow the usual pattern to look for a probability current obeying $\partial_\mu j^\mu = 0$.

Recall the original form of Dirac's equation

$$i\,\gamma_\mu \partial^\mu \psi = m\psi$$

Let $\psi^+ = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$ be the conjugate of $\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$

Then $$i\,\psi^+ \gamma_0 \frac{\partial\psi}{\partial t} + i\,\psi^+ \bar{\gamma}\cdot\bar{\nabla}\psi = m\,\psi^+\psi$$

The usual trick is to conjugate this equation and subtract

Now $\gamma_0^+ = \gamma_0$, but $\bar{\gamma}^+ = \begin{pmatrix}0 & \bar{\sigma}\\ -\bar{\sigma} & 0\end{pmatrix}^+ = \begin{pmatrix}0 & -\bar{\sigma}\\ \bar{\sigma} & 0\end{pmatrix} = -\bar{\gamma}$

This leads to trouble.

The combination $\gamma_0\bar{\gamma}$ is better behaved $\gamma_0\bar{\gamma} = \begin{pmatrix}0 & \bar{\sigma}\\ \bar{\sigma} & 0\end{pmatrix} = (\gamma_0\bar{\gamma})^+$

So instead we multiply the Dirac equation by $\psi^+\gamma_0$:

$$i\,\psi^+\gamma_0\gamma_0\frac{\partial\psi}{\partial t} + i\,\psi^+\gamma_0\bar{\gamma}\cdot\bar{\nabla}\psi = m\,\psi^+\gamma_0\psi$$

or $$i\,\psi^+\frac{\partial\psi}{\partial t} + i\,\psi^+\gamma_0\bar{\gamma}\cdot\bar{\nabla}\psi = m\,\psi^+\gamma_0\psi \qquad \text{using } (\gamma_0)^2 = 1$$

The conjugate equation is $$-i\,\psi\frac{\partial\psi^+}{\partial t} - i\,\bar{\nabla}\psi^+\cdot\gamma_0\bar{\gamma}\psi = m\,\psi^+\gamma_0\psi$$

Subtracting: $$i\frac{\partial}{\partial t}(\psi^+\psi) + i\,\bar{\nabla}\cdot(\psi^+\gamma_0\bar{\gamma}\psi) = 0$$

So $j_\mu = \psi^+\gamma_0\gamma_\mu\psi$ is the probability current, and

$\rho = j_0 = \psi^+\psi$.

IN GENERAL, THE COMBINATION $\psi^{+}\gamma_0$ LEADS TO SIMPLER PHYSICAL INTERPRETATIONS THAN ψ^{+} ALONE, SO WE DEFINE

$\bar{\psi} = \psi^{+}\gamma_0$ AS THE <u>ADJOINT</u> OF ψ

MATRIX ELEMENTS OF A DIRAC OPERATOR $f(\gamma_\mu)$ WILL BE TAKEN AS $(\bar{\psi}_2 | f(\gamma_\mu) | \psi_1)$ RATHER THAN $(\psi_2^{+} | f(\gamma_\mu) | \psi_1)$, AS THE FORMER DEFINITION LEADS TO QUANTITIES WITH WELL DEFINED LORENTZ TRANSFORMATIONS (I.E. SCALAR, 4-VECTOR, TENSOR, ETC)

THE ADJOINT OF AN OPERATOR IS THEN $\bar{f} = \gamma_0 f^{+} \gamma_0$. THAT IS

$$(\bar{\psi}_2\ f\psi_1)^{+} = (\psi_1^{+} f^{+} \gamma_0^{+} \psi_2) = (\psi_1^{+}\gamma_0\gamma_0 f^{+}\gamma_0\psi_2) = (\bar{\psi}_1\ \bar{f}\ \psi_2)$$

RETURNING TO OUR PROBABILITY DENSITY,

$$\rho = \psi^{+}\psi = \bar{\psi}\gamma_0\psi$$

FOR THE SPINOR $u = \begin{pmatrix} \chi \\ \frac{\bar{p}\cdot\bar{\sigma}}{E+m}\chi \end{pmatrix}$, WE HAVE $\bar{u} = \left(\chi^{*},\ -\frac{\bar{p}\cdot\bar{\sigma}}{E+m}\chi^{*}\right)$

$$\text{SO} \quad \rho = \bar{u}\gamma_0 u = \chi^{*}\left(1 + \frac{(\bar{p}\cdot\bar{\sigma})(\bar{p}\cdot\bar{\sigma})}{(E+m)^2}\right)\chi$$

RECALL SOME FACTS: $(\bar{a}\cdot\bar{\sigma})(\bar{b}\cdot\bar{\sigma}) = \bar{a}\cdot\bar{b} + i\,\bar{\sigma}\cdot\bar{a}\times\bar{b}$

$$(\bar{\sigma}\cdot\bar{a})\bar{\sigma} = \bar{a} + i\,\bar{\sigma}\times\bar{a} \qquad \bar{\sigma}(\bar{\sigma}\cdot\bar{b}) = \bar{b} - i\,\bar{\sigma}\times\bar{b}$$

$$\text{SO} \quad \rho = \left[1 + \frac{p^2}{(E+M)^2}\right]\chi^{*}\chi = \frac{2E}{E+M} \qquad \left(\text{SUPPOSING } \chi^{*}\chi = 1\right)$$

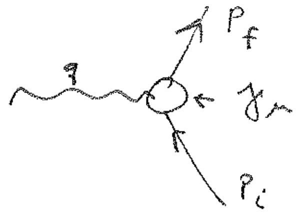
TO MAINTAIN THE NORMALIZATION CONVENTION USED IN OUR CROSS-SECTION 'GOLDEN RULE' (P 79) WE WANT $\rho = 2E$ NOT $\rho = 1$

WITH THIS CHOICE $u = \sqrt{E+m}\begin{pmatrix} \chi \\ \frac{\bar{p}\cdot\bar{\sigma}}{E+m}\chi \end{pmatrix}$ AND $\bar{u}u = E+m\left(1 - \frac{p^2}{(E+m)^2}\right) = 2M$

<u>WARNING!</u> NOT EVERYBODY USES THIS NORMALIZATION. SOME PREFER $\bar{u}u = 1$

2. THE FORM OF THE SPIN 1/2 ELECTROMAGNETIC CURRENT

WE WISH TO DETERMINE THE FORM OF THE ELECTROMAGNETIC CURRENT, $\mathcal{J}_\mu$ AT THE VERTEX



IN GENERAL $\mathcal{J}_\mu = \bar{u}_f f_\mu u_i$

WHERE THE 4-VECTOR OPERATOR IS A FUNCTION OF THE DIRAC MATICES AS WELL AS P_i AND P_f. ANTICIPATING THAT THE PARTICLE WITH THE FORM FACTOR WILL BE LABELLED 2 IN OUR APPLICATIONS, WE DEFINE

$q = P_f - P_i$ (NOT $P_i - P_f$). AND $P = P_i + P_f$

AS BEFORE, THE ONLY POSSIBLE SCALAR VARIABLE IS q^2.

SINCE THE DIRAC MATRICES ARE 4×4, THERE ARE 16 IN ALL. THEY ALL CAN BE EXPRESSED AS PRODUCTS OF THE 4 γ_μ MATRICES.

MATRIX Γ	# OF MATRICES	TRANSFORMATION LAW OF $\bar{u}_f \Gamma u_i$
$1 = \gamma_0\gamma_0$	1	SCALAR (S)
γ_μ	4	VECTOR (V)
$\sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$	6	TENSOR (T)
$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$	1	PSEUDOSCALAR (P)
$\gamma_5\gamma_\mu$	4	PSEUDOVECTOR, or AXIAL VECTOR (A)
	16	

NOT SELF-EVIDENT FROM DISCUSSION ABOVE!

WE KNOW THAT ELECTROMAGNETIC INTERACTIONS CONSERVE PARITY, SO WE CAN EXCLUDE TERMS WITH γ_5 AND $\gamma_5\gamma_\mu$ FROM $\mathcal{J}_\mu$.

THERE ARE 12 POSSIBLE 4-VECTOR OPERATORS WHICH CONSERVE PARITY:

$$\gamma_\mu,\ q_\mu,\ P_\mu,\ i\sigma_{\mu\nu}q^\nu,\ i\sigma_{\mu\nu}P^\nu,\ \not{q}\, q_\mu,\ \not{P} q_\mu \ \ldots,$$

HOWEVER, ONLY 3 OF THE 12 GIVE INDEPENDENT MATRIX ELEMENTS

$(\bar{u}_f | \Gamma_\mu | u_i)$ IF $M_i = M_f$ (ELASTIC SCATTERING)

FOR EXAMPLE: $(\bar{u}_f | \not{P} q_\mu | u_i) = (\bar{u}_f | (\not{p}_i + \not{p}_f) q_\mu | u_i) = 2M \; (\bar{u}_f | q_\mu | u_i)$

THE HISTORICAL CHOICE FOR THE 3 INDEPENDENT OPERATORS IS

$$\gamma_\mu = (\bar{u}_f | \; F_1(q^2)\, \gamma_\mu + i\, F_2(q^2)\, \sigma_{\mu\nu} q^\nu + F_3(q^2)\, q_\mu | u_i)$$

CURRENT CONSERVATION, $\gamma_\mu q^\mu = 0 \Rightarrow F_3 = 0$

HERMITICITY, $\gamma_\mu^+ = \gamma_\mu$, $\Rightarrow$ F_1, F_2 ARE REAL.

F_1 IS SOMETIMES CALLED THE 'DIRAC' FORM FACTOR; F_2 = 'PAULI' FORM FACTOR.

THE EXISTENCE OF 2 FORM FACTORS HAS TO DO WITH THE EXISTENCE OF A MAGNETIC MOMENT FOR SPIN 1/2 PARTICLES (SEE SEC 3)

IT IS USUAL TO REARRANGE TERMS, NOTING THAT

$$i\sigma_{\mu\nu} q^\nu = -\frac{(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)}{2} q = \frac{\not{q}\, \gamma_\mu - \gamma_\mu \not{q}}{2}$$

$$= \frac{(\not{p}_f - \not{p}_i)\gamma_\mu - \gamma_\mu(\not{p}_f - \not{p}_i)}{2} = \not{p}_f \gamma_\mu + \gamma_\mu \not{p}_i - (P_f + P_i)_\mu$$

SO $(\bar{u}_f | i\sigma_{\mu\nu} q^\nu | u_i) = (\bar{u}_f | 2M \gamma_\mu - P_\mu | u_i)$

AND
$$\underline{\gamma_\mu = (\bar{u}_f | [F_1(q^2) + 2M\, F_2(q^2)]\, \gamma_\mu - F_2(q^2)\, P_\mu | u_i)} \quad \text{SPIN } 1/2$$

3. <u>NON-RELATIVISITIC LIMIT OF THE CURRENT</u>

TO INTERPRET THE FORM FACTORS F_1, F_2, WE TAKE THE NON-RELATIVISTIC LIMIT: $E \to M$, $\bar{P}^2/M \to 0$, AND $q^2 \to 0$

$$u_i = \sqrt{E_i + M} \begin{pmatrix} \chi_i \\ \frac{\bar{p}_i \cdot \bar{\sigma}}{E_i + M} \chi_i \end{pmatrix} \to \sqrt{2M} \begin{pmatrix} \chi_i \\ \frac{\bar{p}_i \cdot \bar{\sigma}}{2M} \chi_i \end{pmatrix}$$

$$\bar{u}_f = \sqrt{E_f+m}\left(\chi^+ ,\ -\frac{\bar{p}_f\cdot\vec{\sigma}}{E_f+m}\chi^+\right) \rightarrow \sqrt{2m}\left(\chi_f^+ ,\ -\frac{\bar{p}_f\cdot\vec{\sigma}}{2m}\chi_f^+\right)$$

SO $(\bar{u}_f|u_i) \rightarrow 2m$ IGNORING TERMS IN $\bar{p}^2/m$

LIKEWISE $(\bar{u}_f|\gamma_0|u_i) \rightarrow 2m$

$$\bar{u}_f|\vec{\gamma}|u_i) \rightarrow 2m\left(1,\ -\frac{\bar{p}_f\cdot\vec{\sigma}}{2m}\right)\begin{pmatrix} 0 & \sigma \\ -\sigma & 0\end{pmatrix}\begin{pmatrix} 1 \\ \frac{\bar{p}_i\cdot\vec{\sigma}}{2m}\end{pmatrix} = (\vec{\sigma}\cdot\bar{p}_f)\sigma + \sigma(\vec{\sigma}\cdot\bar{p}_i)$$

$$= \bar{p}_f + i\vec{\sigma}\times\bar{p}_f + \bar{p}_i - i\vec{\sigma}\times\bar{p}_i = \bar{P} + i\vec{\sigma}\times\bar{q} \qquad (p88)$$

ALSO $p_i \rightarrow (m, \bar{p}_i)$ $\quad p_f \rightarrow (m, \bar{p}_f)$

SO $P = p_i + p_f \rightarrow (2m, \bar{p}_i + \bar{p}_f) = (2m, \bar{P})$

$q = (p_f - p_i) \rightarrow (0, \bar{q})$

THE CHARGE DENSITY IS $\rho = \dfrac{(\bar{u}_f|\gamma_0|u_i)}{(\bar{u}_f|u_i)} \rightarrow \dfrac{(F_1+2mF_2)2m - (2m)^2F_2}{2m} = F_1(0)$

THE CURRENT DENSITY IS $\dfrac{(\bar{u}_f|\vec{\gamma}|u_i)}{(\bar{u}_f|u_i)} \rightarrow \dfrac{(F_1+2mF_2)(\bar{P}+i\vec{\sigma}\times\bar{q}) - 2m\bar{P}F_2}{2m}$

$$= \left(\frac{F_1(0)}{2m} + F_2(0)\right)(i\vec{\sigma}\times\bar{q}) + \frac{F_1(0)\bar{P}}{2m}$$

FROM THE EXPRESSION FOR ρ WE CONCLUDE $F_1(0) = e$, TOTAL CHARGE.
THE TERM $\dfrac{F_1(0)\bar{P}}{2m}$ IS JUST $e\bar{v}$, SINCE $\bar{P} = \bar{p}_i + \bar{p}_f \approx 2\bar{p}_i$.
TO INTERPRET THE OTHER TERM OF CURRENT, WE NOTE THAT NON-RELATIVISTICALLY, THE CURRENT $\bar{j}$ WOULD INTERACT WITH THE EXTERNAL VECTOR POTENTIAL $\bar{A}$ LIKE $\bar{j}\cdot\bar{A}$. MORE PRECISELY, OUR $\bar{j}$ IS THE FOURIER TRANSFORM OF $\bar{j}(\bar{r})$.
SO WRITING $\bar{j}\cdot\bar{A} \sim i(\vec{\sigma}\times\bar{q})\cdot A = i\vec{\sigma}\cdot(\bar{q}\times\bar{A})$, WE RECOGNIZE THIS AS THE FOURIER TRANSFORM OF $\vec{\sigma}\cdot(\bar{\nabla}\times\bar{A}) = \vec{\sigma}\cdot\bar{B}$.
HENCE THE COEFFICIENT, $\dfrac{F_1(0)}{2m} + F_2(0)$ IS THE MAGNETIC MOMENT.

SINCE $\frac{e}{2M}$ IS THE ORDINARY (DIRAC) MAGNETIC MOMENT OF A SPIN ½ PARTICLE, WE INFER THAT $F_2(0)$ = <u>ANOMALOUS MAGNETIC MOMENT</u>

$F_2(q^2)$ THEN REPRESENTS THE FOURIER TRANSFORM OF THE ANOMALOUS MAGNETIC MOMENT DISTRIBUTION.

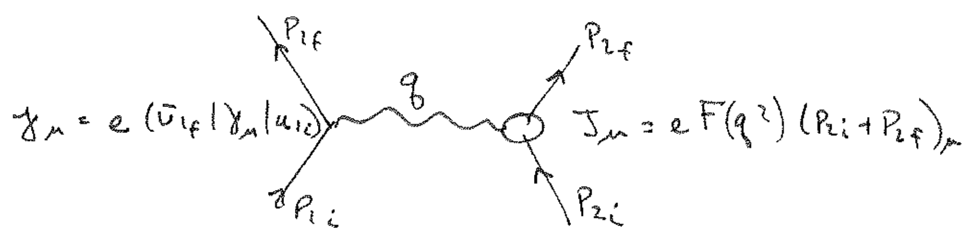
FINALLY, FOR A POINT-LIKE SPIN ½ PARTICLE, SUCH AS THE ELECTRON OR MUON (TO LIMITS OF PRESENT ACCURACY)

$$j_\mu = e(\bar{u}_f|\gamma_\mu|u_i) \qquad \underline{\text{POINTLIKE SPIN } 1/2}$$

FROM P90, THIS COULD BE REWRITTEN, $j_\mu = \frac{e}{2M}(\bar{u}_f|P_\mu + i\sigma_{\mu\nu}q^\nu|u_i) \simeq j_\mu^{\text{ELECTRIC}} + j_\mu^{\text{MAGNETIC}}$.

4. Elastic Scattering of Spin ½ and Spin 0 Particles

ON PP 67 - 73 WE DISCUSSED THE FORM FACTORS OF SPIN 0 PARTICLES IGNORING THE EFFECTS OF ELECTRON SPIN, AND RELATIVITY. WE NOW VERIFY THAT THIS WAS O.K. THE DIAGRAM IS



$q = P_{1i} - P_{1f} = P_{2f} - P_{2i} \qquad \mathcal{P}_2 = P_{2i} + P_{2f}$

NOTE THAT WE REMOVE THE CHARGE e FROM THE FORM FACTOR $F(q^2)$, SO THAT $F(0) = 1$.

THE MATRIX ELEMENT IS $\quad M = \dfrac{e^2F(q^2)}{q^2}(\bar{u}_{1f}|\gamma_\mu|u_i)\mathcal{P}_2^{\,\mu}$

IF WE DON'T OBSERVE THE ELECTRONS' SPINS IN OUR EXPERIMENT, WE MUST AVERAGE OVER THE INITIAL STATE SPINS (FOR UNPOLARIZED ELECTRONS) AND SUM OVER THE FINAL STATE SPINS.

$$|M|^2 = \frac{e^4 F^2(q^2)}{q^4} \cdot \frac{1}{2} \sum_{\substack{i,f \\ \text{SPINS}}} \left| (\bar{u}_f | \gamma_\mu | u_i) P_2^\mu \right|^2$$

WE NEED TO EVALUTE THE EXPRESSION $\sum_{\substack{i,f \\ \text{SPINS}}} | \bar{u}_f | X | u_i) |^2$

WHERE X MIGHT BE ANY DIRAC MATRIX.

FEYNMAN HAS GIVEN A CLEVER METHOD FOR THIS

WE EXPAND : $\sum_{\substack{i,f \\ \text{SPINS}}} (\bar{u}_i | \bar{X} | u_f)(\bar{u}_f | X | u_i)$

(RECALL $\bar{X} = \gamma_0 X^+ \gamma_0$)

SINCE $(\bar{u}_f | u_f) = 2M$, WE MIGHT THINK WE COULD REPLACE $\sum_{f \text{ SPINS}} |u_f)(\bar{u}_f|$

BY $2M$. THIS WOULD BE TRUE IF $\sum_{f \text{ SPINS}}$ MEANT $\sum_{\text{ALL 4 POSSIBLE } f \text{ SPINORS}}$

BUT SINCE u_f REPRESENTS ONLY THE 2 SPIN STATES OF THE PARTICLE, THIS TRICK DOESN'T WORK.

HOWEVER, THE ANTI PARTICLE STATES OBEY $(\not{p}_f + m) | v_f) = 0$, WHILE THE PARTICLE STATES OBEY $(\not{p}_f + m) | u_f) = 2M$. HENCE THE OPERATOR $\frac{\not{p}_f + m}{2m}$ GIVES 1 FOR PARTICLES, AND 0 FOR ANTI PARTICLES. HENCE WE CAN INSERT IT IN OUR SUM AND SUM OVER ALL 4 SPINORS, WITHOUT CHANGING THE RESULT!

$\frac{\not{p} + m}{2m}$ IS CALLED THE PARTICLE PROJECTION OPERATOR

$\frac{-\not{p} + m}{2m}$ IS THE ANTIPARTICLE PROJECTION OPERATOR

$$\sum_{\substack{f\\ \text{PARTICLE}\\ \text{SPINS}}} (\bar{u}_i|\bar{X}|u_f)(\bar{u}_f|X|u_i) = \sum_{\substack{\text{ALL 4}\\ f\,\text{SPINORS}}} (\bar{u}_i|X\,\frac{(\not{p}_f+m)}{2m}|u_f)(\bar{u}_f|X|u_i)$$

$$= (\bar{u}_i|X(\not{p}_f+m)X|u_i)$$

USING $\sum_{\substack{\text{ALL}\\ \text{4 SPINORS}}} |u_f)(\bar{u}_f| = 2m$.

WE ARE LEFT WITH AN EXPRESSION LIKE $\sum_{\substack{i\\ \text{PARTICLE SPINS}}} (\bar{u}_i|Y|u_i)$

IF THIS WERE A SUM OVER ALL 4 SPINORS, THE RESULT WOULD BE JUST $2m$ TRACE(Y) (SINCE $(\bar{u}_i|u_i) = 2m$)

AGAIN WE INSERT THE PROJECTION OPERATOR $\frac{\not{p}_i+m}{2m}$ TO CONVERT TO A SUM OVER ALL 4 SPINS:

$$\sum_{\substack{i\ \text{PARTICLE}\\ \text{SPINS}}} (\bar{u}_i|Y|u_i) = \sum_{\substack{\text{ALL}\\ \text{4 SPINORS}}} (\bar{u}_i|\frac{(\not{p}_i+m)}{2m}Y|u_i) = \text{TRACE}\left[(\not{p}_i+m)Y\right]$$

ALLTOGETHER $$\sum_{\substack{i,f\\ \text{PARTICLE SPINS}}} |(\bar{u}_f|X|u_i)|^2 = \text{TRACE}\left[(\not{p}_i+m)\bar{X}(\not{p}_f+m)X\right]$$

SOME USEFUL TRACES ARE

$$\text{TRACE}(1) = 4$$

$$\text{TRACE}(\not{a}\not{b}) = 4ab \quad (= 4a_\mu b_\mu)$$

$$\text{TRACE}(\gamma_\mu \not{a}) = 4a_\mu$$

$$\text{TRACE}(\text{ODD \# OF } \gamma\text{'s}) = 0$$

$$\text{TRACE}(\not{a}\not{b}\not{c}\not{d}) = 4\left[(ab)(cd) - (ac)(bd) + (ad)(bc)\right]$$

FOR SPIN 1/2 - SPIN 0 SCATTERING, $\chi = \gamma_\mu$, $\bar{\chi} = \gamma_0 \gamma_\nu^+ \gamma_0 = \gamma_\nu$

SO WE NEED $\text{TRACE}\left((\not{P}_{1i} + M_1)\gamma_\mu (\not{P}_{1f} + M_1)\gamma_\nu \right)$

$$= \text{TRACE}\left(M_1^2 \gamma_\mu \gamma_\nu + \not{P}_{1i}\gamma_\mu \not{P}_{1f}\gamma_\nu \right)$$

$$= 4\left(M_1^2 \delta_{\mu\nu} + P_{1i_\mu} P_{1f_\nu} - (P_{1i} P_{1f})\delta_{\mu\nu} + P_{1i_\nu} P_{1f_\mu} \right)$$

$$= 4\left(\frac{q^2}{2}\delta_{\mu\nu} + P_{1i_\mu} P_{1f_\nu} + P_{1i_\nu} P_{1f_\mu} \right)$$

USING $q^2 = (P_{1i} - P_{1f})^2 = 2M_1^2 - 2P_{1i}P_{1f}$

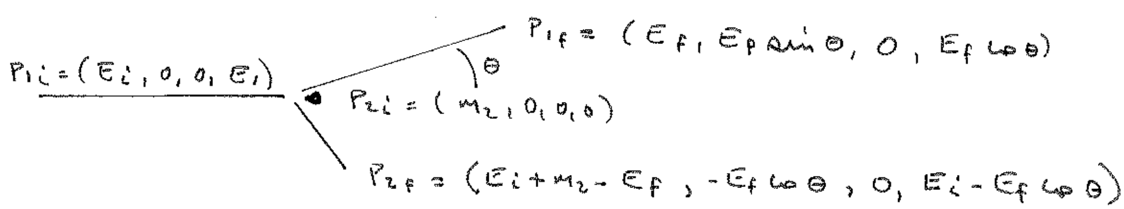
OUR MATRIX ELEMENT SQUARED IS THEN

$$|M|^2 = \frac{2e^4 F^2(q^2)}{q^4}\left[\frac{q^2}{2}\delta_{\mu\nu} + P_{1i_\mu} P_{1f_\nu} + P_{1f_\mu} P_{1i_\nu} \right] \mathbb{P}_{2_\mu} \mathbb{P}_{2_\nu}$$

$$= \frac{2e^4 F^2(q^2)}{q^4}\left[\frac{q^2 \mathbb{P}_2^2}{2} + 2(P_{1i}\mathbb{P}_2)(P_{1f}\mathbb{P}_2) \right]$$

EXPRESSED ENTIRELY IN 4-VECTORS.

SUPPOSE WE EVALUATE THIS IN THE <u>LAB FRAME</u>, WITH PARTICLE 2 AT REST, AND $M_1 = M_e \ll M_2$; $M_e \ll E_1$



$q^2 = (P_{2f} - P_{2i})^2 = 2M_2 - 2P_{2i}P_{2f}$ $\qquad \mathbb{P}_2^2 = (P_{2i} + P_{2f})^2 = 2M_2^2 + 2P_{2i}P_{2f}$

SO $\mathbb{P}_2^2 = 4M_2^2 - q^2$

FROM THE PICTURE, $P_{1i}P_{2i} = E_i M_2$ AND $P_{1f}P_{2i} = E_f M_2$

THEN $(P_{1i} + P_{2i})^2 = (P_{1f} + P_{2f})^2 \Rightarrow P_{1i}P_{2i} = P_{1f}P_{2f} = E_i M_2$

AND $(P_{1i} - P_{2f})^2 = (P_{1f} - P_{2i})^2 \Rightarrow P_{1i}P_{2f} = P_{1f}P_{2i} = E_f M_2$

SO $P_{1i}\mathbb{P}_2 = P_{1i}(P_{2i} + P_{2f}) = M_2(E_i + E_f)$

$P_{1f}\mathbb{P}_2 = P_{1f}(P_{2i} + P_{2f}) = M_2(E_i + E_f)$

so $$|M|^2 = \frac{2e^4F^2(q^2)}{q^4}\left[\frac{q^2(4M_2^2-q^2)}{2} + 2M_2^2(E_i+E_f)^2\right]$$

From this picture, $q^2 = (P_{1i}-P_{1f})^2 = -4E_iE_f \sin^2\theta/2$

and $P_{2f}^2 = M_2^2 \Rightarrow E_i = E_f + \frac{2E_iE_f}{M_2}\sin^2\theta/2 = E_f - \frac{q^2}{2M_2}$ (as on p29)

$$E_i - E_f = \frac{-q^2}{2M_2} \qquad (E_i-E_f)^2 = \frac{q^4}{4M_2^2}$$

so $$(E_i+E_f)^2 = (E_i-E_f)^2 + 4E_iE_f = \frac{q^4}{4M_2^2} - \frac{q^2}{\sin^2\theta/2}$$

$$|M|^2 = \frac{2e^4F^2}{q^4}\left[2M_2^2q^2 - \frac{q^4}{2} + \frac{q^4}{2} - \frac{2M_2^2q^2}{\sin^2\theta/2}\right]$$

$$= -\frac{4e^4M_2^2F^2\cos^2\theta/2}{q^2\sin^2\theta/2} \qquad (q^2<0)$$

Refering to p 82

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{E_f}{E_iM_2^2}\frac{|M|^2}{1+\frac{2E_i}{M_2}\sin^2\theta/2} = \underline{\frac{\alpha^2}{4E_i^2}\frac{\cos^2\theta/2}{\sin^4\theta/2}\frac{F^2(q^2)}{1+\frac{2E_i}{M_2}\sin^2\theta/2}}$$

For point-like spin-0 particles, $F(q^2)=1$, this is called

<u>Mott Scattering</u>

$$\left.\frac{d\sigma}{d\Omega}\right|_{Mott} = \frac{\alpha^2}{4E_i^2}\frac{\cos^2\theta/2}{\sin^4\theta/2}\frac{1}{1+\frac{2E_i}{M_2}\sin^2\theta/2}$$

This differs from Rutherford scattering by 2 factors

a) $\frac{1}{1+\frac{2E_i}{M_2}\sin^2\theta/2}$, the target recoil factor, which favors small angle scattering.

b) $\cos^2\theta/2$. This is the effect of spin. Large angle scattering is suppressed!

But the relation, $\frac{d\sigma}{d\Omega}$ = kinematic factors $\times F^2(q^2)$ remains, as in the non-relativistic case.

5. SCATTERING OF SPIN ZERO PARTICLES OFF ELECTRONS

ON p 70 WE DISCUSSED AN EXPERIMENT IN WHICH A PION OR KAON BEAM HIT ELECTRONS AT REST, AND RECOIL ELECTRONS WERE DETECTED.

TO CALCULATE THIS LET 1 LABEL THE ELECTRON AS BEFORE. NOW $\vec{p}_{1i} = 0$, WE REQUIRE $E_{1f} \gg m_e$ SO THAT THE RECOIL ELECTRON CAN BE DETECTED. FOR HIGH ENERGY BEAMS, $E_{2i} \gg m_2$.

THEN IN THE LAB FRAME, $\dfrac{d\sigma}{d\Omega} = \dfrac{1}{128\pi^2} \dfrac{E_{1f}}{E_{2i}^2 m_1} \dfrac{|M|^2}{\sin^2\theta/2}$

SOME STEPS IN THE CALCULATION INCLUDE

$E_{1f} \approx \dfrac{m_1}{2\sin^2\theta/2}$, SO NO RECOIL ELECTRON

CAN BE DETECTED EXCEPT AT VERY SMALL LAB ANGLES

$q^2 \approx -\dfrac{m_1^2}{\sin^2\theta/2} = -2m_1 E_{1f}$

SO EVEN FOR LARGE E_{1f}, q^2 WILL BE RATHER SMALL.

THEN $\dfrac{d\sigma}{d\Omega} \to \dfrac{\alpha^2}{16} \dfrac{F^2(q^2)}{m_1^2}$

WHICH IS ISOTROPIC OVER THE ANGLES FOR WHICH $E_{1f} \gg m_1$

PERHAPS A BETTER WAY IS TO CONSIDER THE CROSS SECTION AS A FUNCTION OF q^2:

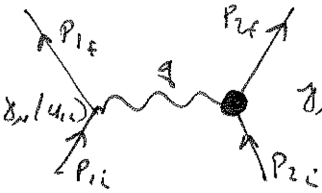
$$\frac{d\sigma}{dq^2} = \frac{\pi\alpha^2}{4} \frac{F^2(q^2)}{q^4}$$

AGAIN WE INFER THAT ONLY THE SMALLER VALUES OF q^2 WILL BE EASILY ACCESIBLE.

6. ELECTRON - PROTON ELASTIC SCATTERING

AT LENGTH WE COME TO THE CASE OF RELATIVISTIC ELECTRON-PROTON SCATTERING, WHICH SHOULD ALLOW US TO EXTEND OUR KNOWLEDGE OF THE PROTON'S STRUCTURE TO VERY SMALL DISTANCES.

$\gamma_\mu^e = e\,(\bar{u}_{1f}|\gamma_\mu|u_{1i})$



$\gamma_\mu^P = e\left(\bar{u}_{2f}\left|\left(F_1(q^2)+2M_2F_2(q^2)\right)\gamma_\mu - F_2\,P_{2\mu}\right|u_{2i}\right)$

AGAIN WE PULL THE FACTOR e OUT OF F_1 AND F_2.

THE MATRIX ELEMENT IS, OF COURSE $\dfrac{\gamma_\mu^e\,\gamma^{\mu,P}}{q^2}$, SO

$$|M|^2 = \frac{e^4}{q^2}\cdot\frac{1}{4}\;\text{TRACE}\left[(\not{p}_{1i}+m_1)\gamma^\mu(\not{p}_{1f}+m_1)\gamma^\nu\right]\text{TRACE}\left[(\not{P}_{2i}+M_2)\left(A\gamma_\mu - F_2P_{2\mu}\right)(\not{P}_{2f}+M_2)\left(A\gamma_\nu - F_2P_{2\nu}\right)\right]$$

$\uparrow$ AVERAGE OVER INITIAL SPINS

WHERE $A \equiv F_1 + 2M_2F_2$

THE ELECTRON TRACE WAS CALCULATED ON p95 : $4\left(\frac{q^2}{2}\delta_{\mu\nu} + P_{1i_\mu}P_{1f\nu} + P_{1f_\mu}P_{1i\nu}\right)$

THE PROTON TRACE IS

$$4\left[P_{2\mu}P_{2\nu}\left(M_2^2F_2^{\,2} + F_2^{\,2}(P_{2i}P_{2f}) - 2M_2AF_2\right) + A^2\left(\frac{q^2\delta_{\mu\nu}}{2} + P_{2i_\mu}P_{2f\nu} + P_{2f_\mu}P_{2f\nu}\right)\right]$$

$$|M|^2 = \frac{e^4}{q^4}\left[\left(-2M_2F_1F_2 - 2M_2^2F_2^{\,2} - \frac{q^2F_2^{\,2}}{2}\right)\left(\frac{q^2P_2^2}{2} + 2(P_{1i}P_2)(P_{1f}P_2)\right)\right.$$
$$\left. + A^2\left(q^2M_2^2 + 2(P_{1i}P_{2i})(P_{1f}P_{2f}) + 2(P_{1i}P_{2f})(P_{1f}P_{2i})\right)\right]$$

IN THE <u>LAB FRAME</u> $\quad \dfrac{d\sigma}{d\Omega} = \dfrac{1}{64\pi^2}\,\dfrac{E_{1f}}{E_{1i}M_1^2}\,\dfrac{|M|^2}{1+\frac{2E_{1i}}{M_2}\sin^2\theta/2}$

$$\frac{d\sigma}{d\Omega} = \underbrace{\frac{\alpha^2}{4E_{1i}^2}\,\frac{\cos^2\theta/2}{\sin^4\theta/2}\,\frac{1}{1+\frac{2E_{1i}}{M_2}\sin^2\theta/2}}_{\left.\frac{d\sigma}{d\Omega}\right|_{\text{MOTT}}}\left\{F_1^{\,2} - q^2F_2^{\,2} - \frac{q^2}{2M_2^2}(F_1+2MF_2)^2\tan^2\theta/2\right\}$$

$$q^2 = -\frac{4E_{1i}^2\sin^2\theta/2}{1+\frac{2E_{1i}}{M_2}\sin^2\theta/2}$$

This is the so-called Rosenbluth Formula P.R. 79, 615 (1950)

It was one of the first applications of Feynman's method by someone besides himself.

Recall that if the proton were point-like, $F_1 \to 1$, $F_2 \to 0$, so

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{pointlike}} = \left.\frac{d\sigma}{d\Omega}\right|_{\text{Mott}} \left(1 - \frac{q^2}{2M_2^2} \tan^2 \theta/2\right) \qquad [q^2<0]$$

From the interpretation of the form factors, we recognize the extra term as due to magnetic scattering. Classically, the dipole potential falls off like $1/r^2$ so magnetic scattering should dominate at small distances $\Rightarrow$ large q^2. The Fourier transform of $1/r^2$ is $\sim 1/q$; the pointlike magnetic dipole strength is $e/2M_2$, so the magnetic term should be $\sim \left(\frac{q}{2M_2}\right)^2$ stronger than the electric term (whose amplitude $\sim e/q^2$)

In the very high energy limit, $E_{1i} \to \infty$ and

$$d\sigma/d\Omega \to \left.\frac{d\sigma}{d\Omega}\right|_{\text{Rutherford}} \frac{2\cos^4\theta/2}{\left(\frac{M_2}{2E_{1i}} + \sin^2\theta/2\right)^2} \qquad [\text{compare p 83}]$$

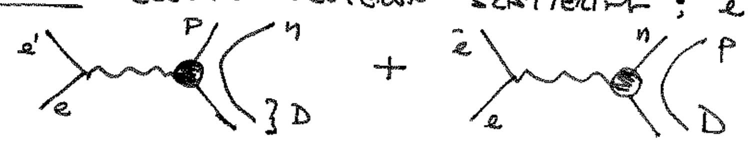
Nowadays it is common to use different definitions of the form factors:

$$G_E \equiv F_1 + \frac{q^2}{2M_2^2} F_2 = \text{electric} \qquad G_M \equiv F_1 + 2M_2 F_2 = \text{magnetic}$$

In the non-relativisitic limit, $q^2 \to 0$, $G_E \to 1$; $G_M \to \frac{\mu}{\mu_{\text{Bohr}}}$

$$\boxed{\frac{d\sigma}{d\Omega} = \left.\frac{d\sigma}{d\Omega}\right|_{\text{Mott}} \left\{ \frac{G_E^2 - \frac{q^2}{4M_2^2} G_M^2}{1 - q^2/4M_2^2} - \frac{q^2}{2M_2^2} G_M^2 \tan^2\theta/2 \right\}} \qquad [q^2<0]$$

Data on the neutron form factors are obtained from inelastic electron-deuteron scattering; $e + D \to e' + n + p$



To a good approximation: $\sigma_{D,\text{inelastic}} = \sigma_{p,\text{elastic}} + \sigma_{n,\text{elastic}}$.

Small corrections are made using a model of the deuteron wave function.

THIS USE OF INELASTIC SCATTERING AS A MEANS OF STUDYING THE ELASTIC SCATTERING OF AN OTHERWISE INACCESSIBLE CONSTITUENT WAS LATER ADAPTED TO THE STUDY OF QUARKS INSIDE HADRONS, AS WE SHALL SEE.

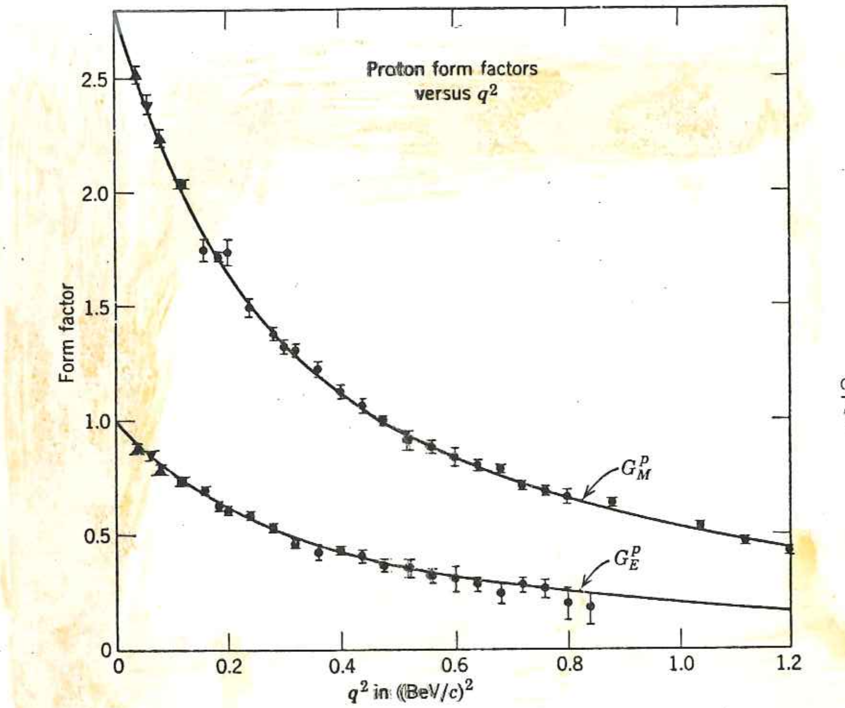
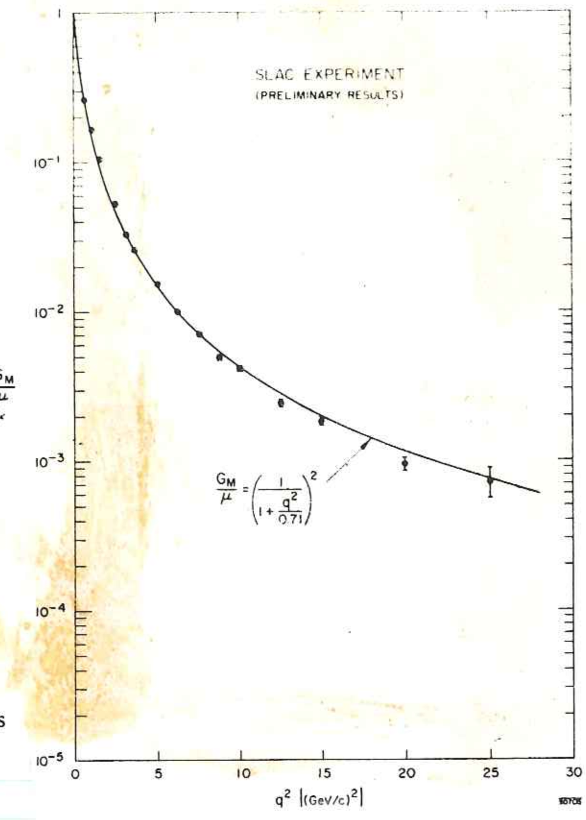


Fig. 26.2. The proton form factors as a function of $q^2 = -t$ in (BeV/c)^2. [From Hughes et al., *Phys. Rev.* **139,** B458 (1965).]



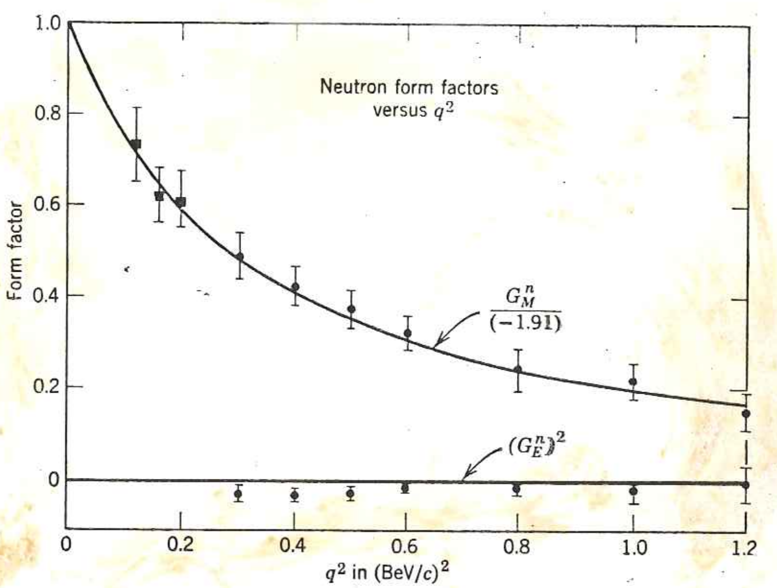


Fig. 26.3. The neutron form factors as a function of $q^2 = -t$ in (BeV/c)^2. (From Hughes et al., *loc. cit.*)