"Hidden" Momentum in a Plane Electromagnetic Wave?

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

The term "hidden" momentum was popularized by Shockley [1] in considerations of an electromechanical example, and essentially all subsequent use of this term has been for such examples, where one considers the system to consist of matter plus electromagnetic fields.

Recently, a definition of "hidden" momentum has been proposed [2] (see also [3]) which can be applied to mechanical systems as well, where a subsystem has a specified volume and can interact with the rest of the system via contact forces and/or transfer of mass/energy across its surface (which can be in motion),

$$\mathbf{P}_{\text{hidden}} \equiv \mathbf{P} - M\mathbf{v}_{\text{cm}} - \oint_{\text{boundary}} (\mathbf{x} - \mathbf{x}_{\text{cm}})\,(\mathbf{p} - \rho\mathbf{v}_b)\cdot d\mathbf{Area} = -\int \frac{f^0}{c}(\mathbf{x} - \mathbf{x}_{\text{cm}})\,d\text{Vol}, \tag{1}$$

where $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total "mass", U is its total energy, $\mathbf{x}_{\text{cm}}$ is its center of mass/energy, $\mathbf{v}_{\text{cm}} = d\mathbf{x}_{\text{cm}}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its "mass" density, u is its energy density, $\mathbf{v}_b$ is the velocity (field) of its boundary, and

$$f^\mu = \frac{\partial T^{\mu\nu}}{\partial x^\nu}\,, \tag{2}$$

is the 4-force density exerted on the subsystem by the rest of the system, with $T^{\mu\nu}$ being the stress-energy-momentum 4-tensor of the subsystem.

Does a plane electromagnetic wave contain "hidden" momentum according to the above definition?

Discuss also the case where the plane wave is normally incident on an infinite slab of thickness L and index of refraction $n = \sqrt{\epsilon\mu}$ where ϵ and μ are the (relative) permittivity and permeability of the slab.

2 Solution

2.1 Plane Wave in Vacuum

We consider a linearly polarized plane electromagnetic wave in vacuum, for which the electric and magnetic fields are,

$$\mathbf{E} = E_0\,\cos(kz - \omega t)\,\hat{\mathbf{x}}, \qquad \mathbf{B} = E_0\,\cos(kz - \omega t)\,\hat{\mathbf{y}}, \tag{3}$$

in Gaussian units, where $\omega = kc$ and c is the speed of light. The energy density u is,

$$u = \frac{E^2 + B^2}{8\pi} = \frac{E_0^2 \cos^2(kz - \omega t)}{4\pi}\,, \tag{4}$$

and the density $\mathbf{p}$ of electromagnetic momentum in this wave is,

$$\mathbf{p} = \frac{\mathbf{S}}{c^2} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} = \frac{E_0^2 \cos^2(kz - \omega t)}{4\pi c}\,\hat{\mathbf{z}} = \frac{u}{c^2}\,\mathbf{c} = \rho\,\mathbf{c}, \tag{5}$$

where $\rho = u/c^2$ is the (effective) mass/energy density, and $\mathbf{c} = c\,\hat{\mathbf{z}}$ is the velocity of the wave.

Consider a rectangular parallelepiped of length L along the z-direction, with area A perpendicular to $\hat{\mathbf{z}}$, and volume $V = AL$. We take the length L to be an integral number of wavelengths.

The total effective mass inside the parallelepiped is,

$$M = \langle \rho \rangle\, V = \frac{E_0^2 V}{8\pi c^2}\,, \tag{6}$$

since its length is an integer number of wavelengths. The total momentum $\mathbf{P}$ inside the parallelepiped is,

$$\mathbf{P} = \langle \mathbf{p} \rangle\, V = M\mathbf{c}. \tag{7}$$

The center of mass/energy of the wave inside the parallelepiped has z-coordinate,

$$\begin{aligned} z_{\rm cm} &= \frac{\int_{z_0}^{z_0+L} z\rho\, dz}{\int_{z_0}^{z_0+L} \rho\, dz} = \frac{\int_{z_0}^{z_0+L} z\cos^2(kz - \omega t)\, dz}{L/2} = \frac{1}{L}\int_{z_0}^{z_0+L} z[1 + \cos 2(kz - \omega t)]\, dz \\ &= z_0 + \frac{L}{2} + \frac{\sin 2(kz_0 - \omega t)}{2k}\,, \end{aligned} \tag{8}$$

where $z_0(t)$ is the z-coordinate of the "left" end of the parallelepiped, so the velocity of the center of mass/energy is,

$$\mathbf{v}_{\rm cm} = \mathbf{v} - \cos 2(kz_0 - \omega t)\,\mathbf{c} = \mathbf{v} + [1 - 2\cos^2(kz_0 - \omega t)]\,\mathbf{c}. \tag{9}$$

The boundary integral in eq. (1) is (for $kL = 2n\pi$),

$$\begin{aligned} &\oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm})\,(\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area} \\ &= A\,\hat{\mathbf{z}}\left[(z_0 + L) - \left(z_0 + \frac{L}{2} + \frac{\sin 2(kz_0 - \omega t)}{2k}\right)\right]\left[\frac{E_0^2}{4\pi c}\cos^2[k(z_0 + L) - \omega t] - \frac{E_0^2 v}{8\pi c^2}\right] \\ &\quad -A\,\hat{\mathbf{z}}\left[z_0 - \left(z_0 + \frac{L}{2} + \frac{\sin 2(kz_0 - \omega t)}{2k}\right)\right]\left[\frac{E_0^2}{4\pi c}\cos^2[kz_0 - \omega t] - \frac{E_0^2 v}{8\pi c^2}\right] \\ &= M[2\cos^2(kz_0 - \omega t)\,\mathbf{c} - \mathbf{v}]. \end{aligned} \tag{10}$$

Then, according to the first version of definition (1), the parallelepiped has "hidden" momentum,

$$\begin{aligned} \mathbf{P}_{\rm hidden} &= \mathbf{P} - M\mathbf{v}_{\rm cm} - \oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm})\,(\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area} \qquad (11) \\ &= M\mathbf{c} - M\{\mathbf{v} + [1 - 2\cos^2(kz_0 - \omega t)]\,\mathbf{c}\} - M[2\cos^2(kz_0 - \omega t)\mathbf{c} - \mathbf{v}] = 0. \end{aligned}$$

2.1.1 Alternative Analysis

According to definition (1) the "hidden" momentum (of a subsystem) can also be written as,

$$\mathbf{P}_{\rm hidden} = -\int \frac{f^0}{c}(\mathbf{x} - \mathbf{x}_{\rm cm})\, d{\rm Vol}, \tag{12}$$

where,

$$f^\mu = \frac{\partial T^{\mu\nu}}{\partial x^\nu}\,. \tag{13}$$

is the 4-force density that the subsystem exerts on all other subsystems, and $T^{\mu\nu}$ is the stress-energy-momentum tensor of the subsystem (which is zero outside its bounding surface).

In the present example, the stress tensor of the electromagnetic wave is,

$$\mathsf{T}^{\mu\nu} = \left(\begin{array}{c|ccc} u & 0 & 0 & cp_z \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ cp_z & 0 & 0 & -u \end{array}\right). \tag{14}$$

noting that $T^{0z} = cp_z = u$, and that $T^{zz} = -E_0^2 \cos^2(kz - \omega t)/4\pi = -u$.

Then, $f^0 = \partial T^{0\nu}/\partial x^\nu = \partial u/\partial ct + \partial cp_z/\partial z$ vanishes inside the parallelepiped, while having δ-function terms at its ends. We consider that the integral in eq. (12) is taken over only the interior of the volume of the subsystem (*i.e.*, of the parallelepiped), in which case we find $\mathbf{P}_{\rm hidden} = 0$.

2.1.2 Comment

The momentum carried by a plane electromagnetic wave is not considered to be "hidden" according to both forms of definition (1).

2.2 Plane Wave Plus Slab of Index n

We now consider a slab of thickness L of index of refraction $n = \sqrt{\epsilon\mu}$, where ϵ and μ are the relative permittivity and permeability of the slab. We ignore absorption by the slab. We work in the frame in which the slab is at rest.[1]

The electric and magnetic fields inside the slab are taken to have the form,

$$\mathbf{E} = E_0 \cos(kz - \omega t)\,\hat{\mathbf{x}}, \qquad \mathbf{B} = B_0 \cos(kz - \omega t)\,\hat{\mathbf{y}}, \tag{15}$$

where $\omega = kc/n$. Faraday's law tells us that $B_0 = (kc/\omega)E_0 = nE_0$.

[1] If the slab were at rest in the lab frame before the wave arrived, the radiation pressure of the wave on the slab would set it in motion along the direction of incidence. In this case, our analysis is not in the lab frame.

We can also consider the fields $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \epsilon\mathbf{E}$, and $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = \mathbf{B}/\mu$, where $\mathbf{P}$ and $\mathbf{M}$ are the densities of electric and magnetic dipoles in the slab.

We take the slab to have two subsystems (within the same volume), "matter" and "electromagnetic fields". The densities $\mathbf{P}$ and $\mathbf{M}$ are taken to belong to the "matter" subsystem. We consider the "electromagnetic fields" to be $\mathbf{E}$ and $\mathbf{B}$, while the fields $\mathbf{D}$ and $\mathbf{H}$ combine aspects of both the "matter" and "electromagnetic field" subsystems.

The energy density in the electromagnetic fields is then,

$$u_{\rm EM} = \frac{E^2 + B^2}{8\pi} = \frac{E_0^2(1+n^2)\cos^2(kz - \omega t)}{8\pi}\,, \tag{16}$$

and the density $\mathbf{p}$ of electromagnetic-field momentum in this wave is,

$$\mathbf{p}_{\rm EM} = \frac{\mathbf{S}}{c^2} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi c} = \frac{nE_0^2\cos^2(kz-\omega t)}{4\pi c}\,\hat{\mathbf{z}} = \frac{2n}{n^2+1}u_{\rm EM}\,\mathbf{c}. \tag{17}$$

We again take the length L to be an integral number of wavelengths.

The total effective mass of the electromagnetic field inside the slab is,

$$M_{\rm EM} = \langle\rho\rangle\,V = \frac{(n^2+1)E_0^2 V}{16\pi c^2}\,, \tag{18}$$

and the total electromagnetic-field momentum inside the slab is,

$$\mathbf{P}_{\rm EM} = \langle\mathbf{p}_{\rm EM}\rangle\,V = \frac{2n}{n^2+1}M_{\rm EM}\,\mathbf{c}. \tag{19}$$

The velocity of the center of mass/energy of the electromagnetic field inside the slab depends on the energy density and not on the momentum density, so its form remains that same as eq. (9) except that the velocity $\mathbf{v}$ of the slab is zero,

$$\mathbf{v}_{\rm cm,EM} = -\cos 2(kz_0 - \omega t)\,\mathbf{c}. \tag{20}$$

The boundary integral in eq. (1) is, noting that the velocity $\mathbf{v}_b$ of the boundary is zero and that $kL = 2m\pi$,

$$\begin{aligned}
&\oint_{\rm boundary}(\mathbf{x}-\mathbf{x}_{\rm cm})\,\mathbf{p}\cdot d\mathbf{Area} \\
&= A\,\hat{\mathbf{z}}\left[(z_0+L) - \left(z_0 + \frac{L}{2} + \frac{\sin 2(kz_0-\omega t)}{2k}\right)\right]\frac{nE_0^2}{4\pi c}\cos^2[k(z_0+L)-\omega t] \\
&\quad - A\,\hat{\mathbf{z}}\left[z_0 - \left(z_0 + \frac{L}{2} + \frac{\sin 2(kz_0-\omega t)}{2k}\right)\right]\frac{nE_0^2}{4\pi c}\cos^2[kz_0-\omega t] \\
&= \frac{4n}{n^2+1}M\cos^2(kz_0-\omega t)\,\mathbf{c} = \frac{2n}{n^2+1}M[1+\cos 2(kz_0-\omega t)]\,\mathbf{c}.
\end{aligned} \tag{21}$$

Then, according to the first version of definition (1), the slab has "hidden" momentum,

$$\begin{aligned}
\mathbf{P}_{\rm hidden,EM} &= \mathbf{P}_{\rm EM} - M_{\rm EM}\mathbf{v}_{\rm cm,em} - \oint_{\rm boundary}(\mathbf{x}-\mathbf{x}_{\rm cm})\,\mathbf{p}\cdot d\mathbf{Area} \qquad (22)\\
&= \frac{2n}{n^2+1}M_{\rm EM}\mathbf{c} + M_{\rm EM}\cos^2(kz_0-\omega t)]\,\mathbf{c} - \frac{2n}{n^2+1}M_{\rm EM}[1+\cos 2(kz_0-\omega t)]\,\mathbf{c} \\
&= \frac{(n-1)^2}{n^2+1}M_{\rm EM}\cos 2(kz_0-\omega t)\,\mathbf{c}.
\end{aligned}$$

When the index n differs from unity, the slab does contain nonzero "hidden" momentum in its electromagnetic field. The time average of this "hidden" momentum is zero. The possible movement of the slab due to the radiation pressure of the plane wave is unaffected by the "hidden" momentum.

The "matter" of the slab also contains "hidden" momentum, which is equal and opposite to that of eq. (22), at least for the case that $L = 2m\pi$.

References

[1] W. Shockley and R.P. James, *"Try Simplest Cases" Discovery of "Hidden Momentum" Forces on "Magnetic Currents"*, Phys. Rev. Lett. **18**, 876 (1967), http://kirkmcd.princeton.edu/examples/EM/shockley_prl_18_876_67.pdf

[2] D. Vanzella, Private communication (June 29, 2012).

[3] K.T. McDonald, *On the Definition of "Hidden" Momentum* (July 9, 2012), http://kirkmcd.princeton.edu/examples/hiddendef.pdf