Polarization Precession

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1 Problem

Deduce an expression for the rate of precession of the spin polarization unit vector $\hat{\mathbf{P}}$ of a relativistic charged particle in laboratory electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. The polarization vector is to be measured in the so-called comoving rest frame of the particle obtained by boosting from the lab frame along the direction $\hat{\mathbf{l}}$ that is tangent to the particle's laboratory trajectory, and whose axes are defined by the (lab-frame) triad $\hat{\mathbf{l}}$, $d\hat{\mathbf{l}}/dt$ and $\hat{\mathbf{l}} \times d\hat{\mathbf{l}}/dt$. However, express the time dependence of this polarization vector in terms of lab-frame quantities. Take into account the so-called Thomas precession [2, 3] of vectors in the rest frame of an accelerating particle when the (instantaneous) rest frame is defined by a boost along the instantaneous lab-frame velocity.

2 Solution

Many high-energy-physics experiments measure the spin polarization or other quantities depending on the polarization of a charged particle. If the particle passes through an electromagnetic field (such as that of a particle accelerator or a magnetic spectrometer) the spin may precess. We derive an expression for the spin precession for arbitrary fields, polarizations and particle velocities. Among the many previous discussions of the precession [2]-[9], most restricted their examples to velocities either normal or parallel to the magnetic field.[1]

Rather than use the Dirac equation [6]-[8] (or some relativistic wave equation for higher spin) to calculate the precession, we first consider the (instantaneous) rest frame of the particle, where nonrelativistic wave equations apply [2]-[5, 9]. The results may then be transformed to the lab with complete generality. However, if we construct a polarization 4-vector and transform it to the lab frame, the meaning of the various lab components of the 4-vector is unclear. One usually returns to the rest frame to understand the polarization vector. Our approach is to deal always with the physically meaningful rest-frame polarization vector, but to use the laboratory values of all other quantities.[2]

[1] The earliest discussion, that of Thomas [2, 3], was of considerable generality – although he did not use the comoving rest frame, and assumed that the g factor of the electron was 2. However, it appears that the common application of the Thomas precession to atomic systems (the emphasis of the Nature article [2]) resulted in it being nearly forgotten that his results also apply to free electrons in macroscopic electromagnetic fields.

[2] Let us illustrate that the rest-frame polarization is indeed the physically meaningful quantity to a lab observer. Consider the decay of a polarized particle. Calculations of the correlation between the polarization and the decay angular distribution are always made in the rest frame of the particle. As another example, consider the two-body scattering involving a spin-1/2 particle whose polarization is to be measured. The only parity-conserving operator in the scattering amplitude involving spin is $\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$, where $\boldsymbol{\sigma}$ is the Pauli spin matrix vector and $\hat{\mathbf{n}}$ is the normal to the scattering plane. Thus, only the component of the spin transverse

If the particle is not accelerated then its rest frame is a single inertial frame, and the polarization 3-vector $\hat{\mathbf{P}}$ obeys,

$$\left.\frac{d\hat{\mathbf{P}}}{d\tau}\right|_{\text{nonaccel.}} = g\frac{e}{2m}\hat{\mathbf{P}}\times\mathbf{B}^\star, \tag{1}$$

where

$$\begin{aligned}
\tau &= \text{proper time,}\\
g &= \text{gyromagnetic ratio } (=2(1+\alpha/2\pi+...) \text{ for electrons}, =2(2.79) \text{ for protons}),\\
e &= \text{electric charge,}\\
m &= \text{mass,}\\
\mathbf{B}^\star &= \text{magnetic field strength in the rest frame,}
\end{aligned}$$

and we work in units where the speed of light is unity. This equation follows from classical electromagnetism using the correspondence principle.[3]

Before turning to the important issue of accelerated motion, we first express τ and $\mathbf{B}^\star$ in terms of lab-frame quantities. Let the instantaneous 3-velocity of the particle be $\boldsymbol{\beta}$ with respect to the lab frame, and let,

$$\beta = |\boldsymbol{\beta}|\,, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \text{and} \qquad \hat{\mathbf{l}} = \frac{\boldsymbol{\beta}}{\beta}\,, \tag{5}$$

such that $\hat{\mathbf{l}}$ is the unit tangent to the particle's trajectory. Then, the Lorentz transformation from the (instantaneous) rest frame to the lab frame gives,

$$d\tau = \frac{dt}{\gamma}\,, \tag{6}$$

where t is the time in the lab frame. The transformation of the magnetic field is,

$$\mathbf{B}^\star = \mathbf{B}_\parallel + \gamma(\mathbf{B}_\perp - \boldsymbol{\beta}\times\mathbf{E}) = \gamma\mathbf{B} + (\gamma-1)(\mathbf{B}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}} - \gamma\beta\,\hat{\mathbf{l}}\times\mathbf{E}, \tag{7}$$

to the particle's direction is significant. This component is the same in the rest, lab and center of mass frames, so that calculation of the spin direction in the particle's rest frame is sufficient.

[3] We illustrate this for spin-1/2 particles, for which $\mathbf{P} = \langle\boldsymbol{\sigma}\rangle$. Now,

$$\frac{d\langle\boldsymbol{\sigma}\rangle}{dt} = \left\langle\frac{d\boldsymbol{\sigma}}{dt}\right\rangle = \frac{i}{\hbar}\langle[H,\boldsymbol{\sigma}]\rangle\,, \tag{2}$$

where H is the Hamiltonian. Since we work in the rest frame, we may use the nonrelativistic Hamiltonian. The only part of the Hamiltonian for a free particle in an electromagnetic field that does not commute with $\boldsymbol{\sigma}$ is,

$$H' = -\frac{g}{2}\frac{e\hbar}{2m}\boldsymbol{\sigma}\cdot\mathbf{B}^\star. \tag{3}$$

Hence,

$$\frac{d\langle\boldsymbol{\sigma}\rangle}{dt} = -i\frac{g}{2}\frac{e}{2m}[\boldsymbol{\sigma}\cdot\mathbf{B}^\star,\boldsymbol{\sigma}] = g\frac{e}{2m}\boldsymbol{\sigma}\times\mathbf{B}^\star, \tag{4}$$

whose expectation value is eq. (1).

where $B_\parallel = (\mathbf{B}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}}$ is the longitudinal part of the lab magnetic field, $\mathbf{B}_\perp = \mathbf{B} - \mathbf{B}_\parallel$, and $\mathbf{E}$ is the lab electric field.

Using eqs. (6) and (7) in (1), the polarization in the rest frame (still assuming the particle is not accelerated) obeys the equation of motion,

$$\left.\frac{d\hat{\mathbf{P}}}{dt}\right|_{\text{nonaccel.}} = \hat{\mathbf{P}}\times\frac{e}{m}\frac{g}{2}\left[\mathbf{B} - \left(1-\frac{1}{\gamma}\right)(\mathbf{B}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}} - \beta\,\hat{\mathbf{l}}\times\mathbf{E}\right]. \tag{8}$$

In general, if the particle is subject to fields that can precess the polarization, its trajectory is also deflected. Then, the rest frame of the particle is an accelerated frame, and care is required when using Lorentz transformations between the lab frame and the instantaneous rest frame. The latter frame is, by construction, an inertial frame, but leads to the concept of the rest frame as a sequence of inertial frames each with a different boost from the lab frame. The subtle consequence of this construction is that there is an additional precession, the Thomas precession [2, 3], of a rest-frame vector such as the polarization.

At this point we should distinguish between two different "rest" frames, both of whose origins are at the position of the particle. The first, to be called the rest frame, is the frame obtained by a boost from the lab frame along the particle's velocity.[4] We have been using this rest frame so far.

The second frame, to be called the comoving frame, differs from the rest frame defined above by a rotation of the coordinate axes onto the lab-frame triad $\hat{\mathbf{l}}$, $d\hat{\mathbf{l}}/dt$ and $\hat{\mathbf{l}}\times d\hat{\mathbf{l}}/dt$.

We report the precession of the polarization in the comoving frame, rather than that in the lab frame, since the geometry of the comoving frame is more closely related to the geometry of the lab trajectory of the particle.

In Appendix A it is shown that a vector $\mathbf{s}$ that has no explicit time dependence in the rest frame obeys,

$$\left.\frac{d\mathbf{s}}{dt}\right|_{\text{comoving}} = \mathbf{s}\times\boldsymbol{\omega} \tag{9}$$

relative to the comoving rest frame, where,

$$\boldsymbol{\omega} = \gamma\,\hat{\mathbf{l}}\times\frac{d\hat{\mathbf{l}}}{dt}\,. \tag{10}$$

That is, vector $\mathbf{s}$ precesses with frequency $\boldsymbol{\omega}$ with respect to the triad form by $\hat{\mathbf{l}}$, $d\hat{\mathbf{l}}/dt$ and $\boldsymbol{\omega}$ in the comoving frame. This is the so-called Thomas precession [2, 3]. The equation of motion of the polarization in the comoving frame is therefore,

$$\left.\frac{d\hat{\mathbf{P}}}{dt}\right|_{\text{comoving}} = \left.\frac{d\hat{\mathbf{P}}}{dt}\right|_{\text{nonaccel.}} + \hat{\mathbf{P}}\times\boldsymbol{\omega}, \tag{11}$$

where (8) is the equation of motion in the rest frame for nonaccelerated motion.

To calculate the vector $\boldsymbol{\omega}$ using eq. (10) we need $d\hat{\mathbf{l}}/dt$. For this, we use the relativistic form of the Lorentz force,

$$\frac{du}{dt} = \frac{e}{m}F\cdot u, \tag{12}$$

[4] This frame is sometimes called the Fermi-Walker rest frame.

where $u = (\gamma, \gamma\boldsymbol{\beta})$ is the 4-velocity and F is the electromagnetic field tensor,

$$F = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}. \tag{13}$$

The well-known resulting equations are,

$$\frac{d\gamma}{dt} = \frac{e}{m}\beta\,\mathbf{E}\cdot\hat{\mathbf{l}}, \tag{14}$$

$$\frac{d(\gamma\beta\,\hat{\mathbf{l}})}{dt} = \frac{e}{m}(\mathbf{E} + \beta\,\hat{\mathbf{l}}\times\mathbf{B}). \tag{15}$$

Differentiating the identity $\gamma^2\beta^2 = \gamma^2 - 1$, we also have,

$$\frac{d(\gamma\beta)}{dt} = \frac{1}{\beta}\frac{d\gamma}{dt} = \frac{e}{m}\mathbf{E}\cdot\hat{\mathbf{l}}. \tag{16}$$

Solving for $d\hat{\mathbf{l}}/dt$ we find,[5]

$$\frac{d\hat{\mathbf{l}}}{dt} = \frac{1}{\gamma\beta}\frac{d(\gamma\beta\hat{\mathbf{l}})}{dt} - \frac{\hat{\mathbf{l}}}{\gamma\beta}\frac{d(\gamma\beta)}{dt} = \frac{e}{m}\left[\frac{\mathbf{E} - (\mathbf{E}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}}}{\gamma\beta} + \frac{\hat{\mathbf{l}}\times\mathbf{B}}{\gamma}\right]. \tag{18}$$

Then, from eq. (10) we have,

$$\omega = \gamma\,\hat{\mathbf{l}}\times\frac{d\hat{\mathbf{l}}}{dt} = \frac{e}{m}\left[\frac{\hat{\mathbf{l}}\times\mathbf{E}}{\beta} - \mathbf{B} + (\mathbf{B}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}}\right], \tag{19}$$

Inserting eq. (19) in (11) we have,

$$\left.\frac{d\hat{\mathbf{P}}}{dt}\right|_{\text{comoving}} = \hat{\mathbf{P}}\times\frac{e}{m}\left\{\left(\frac{g}{2}-1\right)\mathbf{B} - \left(\frac{g}{2}-1-\frac{g}{2\gamma}\right)(\mathbf{B}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}} - \left(\frac{g}{2}-1-\frac{g}{2\gamma^2}\right)\frac{\hat{\mathbf{l}}\times\mathbf{E}}{\beta}\right\}. \tag{20}$$

This is the main result for the precession of the polarization vector with respect to the comoving frame.

If $d\hat{\mathbf{P}}/dt = 0$ in eq. (20), then $\hat{\mathbf{P}}$ is constant with respect to the triad $\hat{\mathbf{l}}$, $d\hat{\mathbf{l}}/dt$ and $\boldsymbol{\omega}$.

[5]Equation (18) is strictly true only in homogeneous field [10]. In an inhomogeneous magnetic field there is an additional force on the particle due to the magnetic moment interaction $\boldsymbol{\nabla}(\boldsymbol{\mu}\cdot\mathbf{B})$ (plus other terms for higher multipole moments). However, this correction is extremely small compared to the usual magnetic force $e\boldsymbol{\beta}\times\mathbf{B}$. For example, suppose $\boldsymbol{\mu}$ is parallel to $\mathbf{B}$ and normal to $\boldsymbol{\beta}$. Then, the ratio of these two terms is,

$$\frac{F_{\text{mag dipole}}}{F_{\text{Lorentz}}} = \frac{\mu}{e\beta}\frac{\nabla B}{B} \approx \frac{\hbar}{m\beta}\frac{\nabla B}{B} = \frac{\lambda\!\!\!^{-}_C}{\beta}\frac{\nabla B}{B}, \tag{17}$$

where $\lambda\!\!\!^{-}_C = \hbar/m$ is the Compton wavelength of the particle. Even for a relativistic electron in a rapidly varying magnetic field with $\nabla B/B \approx 1/\text{cm}$, the ratio is 10^{-11}. Hence, the results of this problem are applicable to inhomogeneous fields on any reasonable laboratory scale.

2.1 Thomas' Analysis

Thomas [3] gave a very similar result [his eq. (4.121)], which reported the precession relative to a fixed direction in the lab. As shown in Appendix A, our formalism can be converted to this case by replacing the factor γ in the definition (10) of $\boldsymbol{\omega}$ by the factor $\gamma - 1$. The resulting expression for the polarization precession (which is equivalent to eq. (11.170) of [58]) is,

$$\left.\frac{d\hat{\mathbf{P}}}{dt}\right|_{\rm rest} = \hat{\mathbf{P}} \times \frac{e}{m}\left\{\left(\frac{g}{2} - 1 + \frac{1}{\gamma}\right)\mathbf{B} - \left(\frac{g}{2} - 1\right)\left(1 - \frac{1}{\gamma}\right)(\mathbf{B}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}} - \left(\frac{g}{2} - \frac{1}{1+\gamma}\right)\beta\,\hat{\mathbf{l}}\times\mathbf{E}\right\}. \tag{21}$$

The context of Thomas' effort (1925) was an application of the then-new concept of electron spin [11]-[15] to atomic spectroscopy. Here, an atomic electron was considered to be moving in the electric field $\mathbf{E}$ of an atomic nucleus and of other electrons of the atom, while their magnetic field was ignored. Taking the electron to have $g = 2$, but ignoring the effect of its acceleration, the angular velocity of precession in the rest frame of the electron would be $e\mathbf{v}\times\mathbf{E}/mc$ according to eq. (8), noting that the electron's velocity is $\mathbf{v} = \beta\hat{\mathbf{l}}$. However, this was twice the value inferred from certain experiments. Thomas' result, eq. (21), gives the angular velocity as $e\mathbf{v}\times\mathbf{E}/2mc$ for $\gamma \approx 1$, which brought the new theory into good agreement with experiment, and led to the quick acceptance of the concept of electron spin.[6,7]

It remains surprising that an effect of special relativity yields a factor of two difference in the computation of the precession frequency in the rest frame of a low-velocity electron,

2.2 Discussion

In the rest of this note we emphasize "free" charges in an external magnetic field $\mathbf{B}$, with $\mathbf{E} = 0$.[8]

[6] Thomas' recollections of his notable work are at [16]. Brief surveys of the concept of electron spin have been given by Yang [17] and Pais [18], and more extensive commentary by Tomonaga [19]. The possible relevance of a spinning electron to quantum theory was first noted by Compton (1921) [20], who considered a classical model of an electron as rotating sphere of charge. As recounted by Tomonaga [19], Pauli was the first to argue that spectroscopic evidence implied that the electron has nonclassical, two-valued behavior [21, 22]. As such, he was initially skeptical of the quasiclassical model of Uhlenbeck and Goudsmit, and of the relevance of the classical calculation of Thomas. The latter commented in [16] that Pauli (age 25 in 1925) was an example of how "older people find it difficult to change their ideas". In contrast, Uhlenbeck [15] recalled Pauli as saying (1950): *Ich war so dumm wenn ich jung war!* (I was so stupid when I was young!). See also [23]-[26].

[7] Thomas analysis was anticipated to some extent in [27]-[29]. Its somewhat counterintuitive result has led to many pedagogic discussions, including [30]-[74]. The most unusual/idiosyncratic of these is by Schwinger [40], who defined the Thomas precession to be the negative of that found by Thomas for $g = 2$.

[8] Note added Dec. 2002. In the recent BNL experiment [75] to measure $g-2$ for the muon, it was desired to have electrostatic quadrupole focusing elements in addition to the magnetic field $\mathbf{B} \perp \mathbf{l}$ that cause the muons to execute circular orbits. The precession due to the (spatially varying) electric field was eliminated, following eq. (20), by using muons with $\gamma = \sqrt{(g/2)/(g/2-1)} \approx \sqrt{2\pi/\alpha} \approx 29.3$, noting that $g/2 \approx 1+\alpha/2\pi$, where $\alpha \approx 1/137$ is the fine structure constant.

2.2.1 $\hat{\mathbf{l}} \parallel \mathbf{B}$

In this case,

$$\left.\frac{d\hat{\mathbf{P}}}{dt}\right|_{\rm comoving} = \frac{e}{m}\frac{g}{2}\frac{\hat{\mathbf{P}}\times\mathbf{B}}{\gamma}\,, \qquad (\hat{\mathbf{l}}\perp\mathbf{B}), \tag{22}$$

which is the nonrelativistic result (1) divided by the time-dilation factor γ. If $\mathbf{B}$ is constant, the particle's trajectory is a straight line parallel to $\mathbf{B}$, and the polarization precesses about this fixed direction. This effect has been utilized in measurements of the gyromagnetic ratio of the electron [76].

2.2.2 $\hat{\mathbf{l}} \perp \mathbf{B}$

Here,

$$\left.\frac{d\hat{\mathbf{P}}}{dt}\right|_{\rm comoving} = \frac{e}{m}\left(\frac{g}{2}-1\right)\hat{\mathbf{P}}\times\mathbf{B}, \qquad (\hat{\mathbf{l}}\parallel\mathbf{B}). \tag{23}$$

The polarization vector of a Dirac particle (for which $g = 2$) would not precess at all. From eq. (18), the particle's trajectory obeys,

$$\frac{d\hat{\mathbf{l}}}{dt} = \frac{1}{\gamma\beta}\frac{d(\gamma\beta\hat{\mathbf{l}})}{dt} - \frac{\hat{\mathbf{l}}}{\gamma\beta}\frac{d(\gamma\beta)}{dt} = \frac{e}{m}\frac{\hat{\mathbf{l}}\times\mathbf{B}}{\gamma}\,. \tag{24}$$

Thus, the precession rate of $\hat{\mathbf{P}}$ is $\gamma(g/2-1)$ times that of $\hat{\mathbf{l}}$. Hence, if the trajectory bends through angle θ_l, the polarization vector precesses by angle,

$$\theta_P = \gamma\left(\frac{g}{2}-1\right)\theta_l. \tag{25}$$

This result is useful for quick calculations as to the effect of a bending magnet on polarization. For a slow proton, eq. (24) gives $\theta_P \approx 2\theta_l$. Hence, a 45° bend can interchange transverse and longitudinal polarization (if $\hat{\mathbf{P}}\perp\mathbf{B}$), while a 90° bend flips the polarization [77]. Of course, if $\hat{\mathbf{P}}\parallel\mathbf{B}$, the precession has no physical effect.

The precession frequency is,

$$\Omega_P = \left(\frac{g}{2}-1\right)\frac{eB}{m}\,, \tag{26}$$

which does not depend on γ. Hence, in a given magnetic field a fast muon will undergo more spin precession during its lifetime, which is proportional to γ, than will a slow muon. This allows a more precise determination of the muon gyromagnetic ratio at relativistic energies [78].

2.2.3 $\hat{\mathbf{l}}$ neither $\perp$ nor $\parallel \mathbf{B}$

In this case the particle's trajectory is helical. From eq. (20) we see that the polarization does not precess about $\mathbf{B}$, but rather about,

$$\mathbf{B}' = \mathbf{B}_\perp + \frac{g}{\gamma(g-2)}\mathbf{B}_\parallel, \tag{27}$$

which lies in the plane of $\mathbf{B}$ and $\mathbf{l}$. If we designate the angle between $\mathbf{B}$ and $\mathbf{l}$ by θ, and the angle between $\mathbf{B}'$ and $\mathbf{l}$ by θ', then,

$$\tan\theta' = \left(\frac{g}{2} - 1\right)\frac{2\gamma}{g}\tan\theta \ (= 0.64\gamma\tan\theta \text{ for protons}). \tag{28}$$

When $\gamma = 1.56$ for protons, $\tan\theta' = \tan\theta$ and $\mathbf{B}' = \mathbf{B}$, which occurs at a kinetic energy of 525 MeV.

As $\gamma \to \infty$, $\tan\theta' \to 1$, and $\hat{\mathbf{P}}$ precesses about the transverse part of $\mathbf{B}$. For applications to many bending magnets at finite γ, $\mathbf{l}$ will be nearly normal to $\mathbf{B}$ and the difference between $\mathbf{B}'$ and $\mathbf{B}$ will be small. A first correction to eq. (25) can be made via an "effective edge" approximation to the magnetic field. That is, replace the actual, spatially varying field with a constant field $\mathbf{B}_{\text{eff}}$ that produces the same total bend in the particle's trajectory. Then, $\mathbf{B}_{\text{eff}} \cdot \hat{\mathbf{l}}$ is constant, so that $\mathbf{B}'$ is constant is magnitude and precess about $\mathbf{B}$ at the same frequency as does $\hat{\mathbf{l}}$. The corrected form of eq. (25) is,

$$\theta_P = \gamma\left(\frac{g}{2} - 1\right)\theta_l\frac{|\mathbf{B}'|}{|\mathbf{B}|} = \gamma\left(\frac{g}{2} - 1\right)\theta_l\sqrt{1 + \left(\frac{g^2}{\gamma^2(g-2)^2} - 1\right)\cos^2\theta}, \tag{29}$$

where θ is the angle between $\mathbf{B}$ and $\mathbf{l}$. Note that in eq. (25) both θ_P and θ_l are azimuthal angles measured with respect to $\mathbf{B}$, while in eq. (29) θ_P is with respect to $\mathbf{B}'$.

A Appendix: The Thomas Precession

We wish to find the time dependence in the "comoving" rest frame of a spacelike vector that has no explicit time dependence in the rest frame of an accelerated particle. That is, we seek an expression for the apparent time dependence of the vector that arises because the rest frame is constructed from a sequence of instantaneous inertial frames based on the time-dependent velocity $\boldsymbol{\beta}(t)$ of the particle.

The particle has lab frame 4-velocity u where,

$$u = (\gamma, \gamma\boldsymbol{\beta}) = (\gamma, \gamma\beta\,\hat{\mathbf{l}}). \tag{30}$$

The "rest" frame is obtained from the lab frame by a boost, without rotation of the coordinate axes, along the direction $\hat{\mathbf{l}}$. The "comoving" frame is a rotating frame in which the particle is at rest and whose axes are always aligned with the (lab frame) triad $\hat{\mathbf{l}}$, $d\hat{\mathbf{l}}/dt$, and $\hat{\mathbf{l}}\times d\hat{\mathbf{l}}/dt$. The comoving frame differs from the rest frame by a time-dependent rotation.

Consider a spacelike 4-vector $s = (0, \mathbf{s}_R)$ in the rest frame, that satisfies $\partial\mathbf{s}_R/\partial\tau = 0$, *i.e.*, that has no explicit time dependence in the rest frame. In the rest frame the 4-velocity has components $u = (1, \mathbf{0})$, so the invariant product of s and u vanishes,

$$s \cdot u = 0, \tag{31}$$

which relation is therefore true in all (inertial) frames. The time derivative of eq. (31) also vanishes, so we have,

$$\frac{ds}{d\tau} \cdot u = -s \cdot \frac{du}{d\tau}\,, \tag{32}$$

where τ is the proper time. This suggests that we can write,

$$\frac{ds}{d\tau} = -\left(s \cdot \frac{du}{d\tau}\right) u \tag{33}$$

as the covariant form of the rest-frame condition $\partial \mathbf{s}_R/\partial\tau = 0$.

Equation (33) implies that $d\mathbf{s}_R/d\tau = 0$ if the rest frame is inertial. We extend the use of eq. (33) to the case of accelerated motion by first applying it in the lab frame to find $d\mathbf{s}_L/d\tau$, then transforming $\mathbf{s}_L$ back to the rest frame, followed by taking the time derivative of this transformation. This procedure is sufficient to reveal the Thomas precession.

In the lab frame (indeed, in any frame) where we write $s = (s_{L,0}, \mathbf{s}_L)$, the condition (31) plus the expression (30) tell us that,

$$s_{L,0} = \mathbf{s}_L \cdot \boldsymbol{\beta} = \beta\, \mathbf{s}_L \cdot \hat{\mathbf{l}}. \tag{34}$$

From eq. (33) we find the lab-frame time dependence,

$$\frac{d\mathbf{s}_L}{d\tau} = -\left(s_{L,0}\frac{d\gamma}{d\tau} - \mathbf{s}_L \cdot \frac{d\gamma\beta\,\hat{\mathbf{l}}}{d\tau}\right)\gamma\beta\,\hat{\mathbf{l}} = (\gamma^2-1)\left(\mathbf{s}_L \cdot \frac{d\hat{\mathbf{l}}}{d\tau}\right)\hat{\mathbf{l}} + \frac{1}{\gamma}\frac{d\gamma}{d\tau}(\mathbf{s}_L\cdot\hat{\mathbf{l}})\hat{\mathbf{l}}, \tag{35}$$

recalling eq. (16). Although $d\mathbf{s}_L/d\tau$ is not zero in the lab frame, we would not say that vector $\mathbf{s}_L$ is precessing there. That is, a lab-frame observer does not see the Thomas precession when considering only lab-frame quantities.

The Lorentz transformation between the lab frame L and the (instantaneous) rest frame R for the spatial part of 4-vector s is,

$$\mathbf{s}_L = \mathbf{s}_R - (\mathbf{s}_R\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}} + \gamma[(\mathbf{s}_R\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}} + \boldsymbol{\beta} s_{R,0}] = \mathbf{s}_R + (\gamma-1)(\mathbf{s}_R\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}}, \tag{36}$$

$$\mathbf{s}_R = \mathbf{s}_L - (\mathbf{s}_L\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}} + \gamma[(\mathbf{s}_L\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}} - \boldsymbol{\beta} s_{L,0}] = \mathbf{s}_L + \left(\frac{1}{\gamma}-1\right)(\mathbf{s}_L\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}}, \tag{37}$$

since $s_{R,0} = 0$ and $s_{L,0}$ is given by eq. (34).

Taking the time derivative of eq. (37) and using eq. (35) we find,

$$\frac{d\mathbf{s}_R}{d\tau} = (\gamma-1)\left(\mathbf{s}_L\cdot\frac{d\hat{\mathbf{l}}}{d\tau}\right)\hat{\mathbf{l}} + \left(\frac{1}{\gamma}-1\right)(\mathbf{s}_L\cdot\hat{\mathbf{l}})\frac{d\hat{\mathbf{l}}}{d\tau}\,. \tag{38}$$

Inserting eq. (36) in this we find,

$$\frac{d\mathbf{s}_R}{d\tau} = (\gamma-1)\left[\left(\mathbf{s}_R\cdot\frac{d\hat{\mathbf{l}}}{d\tau}\right)\hat{\mathbf{l}} - (\mathbf{s}_R\cdot\hat{\mathbf{l}})\frac{d\hat{\mathbf{l}}}{d\tau}\right] = \mathbf{s}_R \times (\gamma-1)\left(\hat{\mathbf{l}}\times\frac{d\hat{\mathbf{l}}}{d\tau}\right). \tag{39}$$

This is the Thomas precession of the vector $\mathbf{s}_R$ in the rest frame. It is a "relativistic" effect in that $\gamma - 1 \approx v^2/2c^2$, which is negligible for $v \ll c$.

Since $\hat{\mathbf{l}}\cdot d\hat{\mathbf{l}}/d\tau = 0$, we obtain the identity,

$$\frac{d\hat{\mathbf{l}}}{d\tau} = \left(\hat{\mathbf{l}}\times\frac{d\hat{\mathbf{l}}}{d\tau}\right)\times\hat{\mathbf{l}} \tag{40}$$

in the lab frame. Hence, $\hat{\mathbf{l}} \times d\hat{\mathbf{l}}/d\tau$ is the instantaneous angular velocity of the particle's trajectory. Since $d\hat{\mathbf{l}}/d\tau$ is transverse to $\hat{\mathbf{l}}$, eq. (40) holds in the rest frame also. Thus, the part of $d\mathbf{s}_R/d\tau$ coming from the term -1 in the factor $\gamma - 1$ of eq. (39) is due to the rotation of the trajectory in the rest frame. Hence, the precession of $\mathbf{s}$ relative to the precession of the trajectory can be written,

$$\frac{d\mathbf{s}_C}{d\tau} = \mathbf{s}_C \times \gamma \left(\hat{\mathbf{l}} \times \frac{d\hat{\mathbf{l}}}{d\tau} \right), \tag{41}$$

where we use the subscript C to indicate that this expression holds in the comoving rest frame whose axes are aligned with the triad $\hat{\mathbf{l}}$, $d\hat{\mathbf{l}}/dt$ and $\hat{\mathbf{l}} \times d\hat{\mathbf{l}}/dt$.

The covariant description of the precession used in this Appendix can be extended to include the spin precession caused by electromagnetic fields. This is the approach of Bargmann, Michel and Telegdi [9], which is expanded upon by Hagedorn [79]. The specialization to the comoving frame is straightforward, but they do not do it. The result of such analysis is eq. (20).

B Appendix: Precession of an Intrinsic Electric Dipole Moment

Suppose the charged particle had an intrinsic electric dipole moment $\mathbf{p}$. A famous argument (first given a bit indirectly in [80]) is that such a moment would have to be aligned with the spin of the particle, as this the only vector associated with an elementary particle at rest.[9] If we accept this argument, the preceding discussion for polarization precession can be readily extended to include a possible intrinsic electric dipole moment.

If the (spin-1/2) particle is not accelerated then its rest frame is a single inertial frame, and the electric dipole moment 3-vector $\mathbf{p} = p\hat{\mathbf{P}}$ obeys the torque equation,

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}^\star = \left.\frac{d\mathbf{L}}{d\tau}\right|_{\text{nonaccel.}} = \left.\frac{\hbar}{2}\frac{d\hat{\mathbf{P}}}{d\tau}\right|_{\text{nonaccel.}} \tag{42}$$

which adds a contribution to eq. (1) for the polarization vector $\mathbf{P}$, which should now read,

$$\left.\frac{d\hat{\mathbf{P}}}{d\tau}\right|_{\text{nonaccel.}} = g\frac{e}{2m}\hat{\mathbf{P}} \times \mathbf{B}^\star + \frac{2p}{\hbar}\hat{\mathbf{P}} \times \mathbf{E}^\star, \tag{43}$$

The equivalent of eq. (8) including the electric dipole moment is,

$$\begin{aligned} \left.\frac{d\hat{\mathbf{P}}}{dt}\right|_{\text{nonaccel.}} &= \hat{\mathbf{P}} \times \frac{e}{m}\frac{g}{2}\left[\mathbf{B} - \left(1 - \frac{1}{\gamma}\right)(\mathbf{B}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}} - \beta\,\hat{\mathbf{l}} \times \mathbf{E}\right] \\ &\quad + \hat{\mathbf{P}} \times \frac{e}{m}p\lambda\!\!\!^{-}_C\left[\mathbf{E} - \left(1 - \frac{1}{\gamma}\right)(\mathbf{E}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}} + \beta\,\hat{\mathbf{l}} \times \mathbf{B}\right]. \end{aligned} \tag{44}$$

[9] Such an electric dipole moment violates the symmetries of space inversion (parity) and time reversal (see, for example, [81].

where $\lambda\!\!\!^{-}_C = e\hbar/m$ is the Compton wavelength of the particle.

The motion of the charge in laboratory $\mathbf{E}$ and $\mathbf{B}$ fields is again given by eqs. (14)-(17), if we again ignore the tiny force on the dipole moment due to field inhomogeneity. Then, the extension of eq. (20) to include the electric dipole moment is,

$$\left.\frac{d\hat{\mathbf{P}}}{dt}\right|_{\rm comoving} = \hat{\mathbf{P}} \times \frac{e}{m}\left\{\left(\frac{g}{2}-1\right)[\mathbf{B}-(\mathbf{B}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}}] + \frac{g}{2\gamma}(\mathbf{B}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}} - \left(\frac{g}{2}-1-\frac{g}{2\gamma^2}\right)\frac{\hat{\mathbf{l}}\times\mathbf{E}}{\beta}\right.$$
$$\left. + p\lambda\!\!\!^{-}_C\left[\mathbf{E}-\left(1-\frac{1}{\gamma}\right)(\mathbf{E}\cdot\hat{\mathbf{l}})\,\hat{\mathbf{l}} + \beta\,\hat{\mathbf{l}}\times\mathbf{B}\right]\right\}. \tag{45}$$

References

[1] K.T. McDonald, *Polarization Precession*, CALT-68-236 (Jan. 14, 1970), http://kirkmcd.princeton.edu/papers/mcdonald_calt-68-236.pdf

[2] L.H. Thomas, *The Motion of the Spinning Electron*, Nature **117**, 514 (1926), http://kirkmcd.princeton.edu/examples/QED/thomas_nature_117_514_26.pdf

[3] *The Kinematics of an Electron with an Axis*, Phil. Mag. **3**, 1 (1927), http://kirkmcd.princeton.edu/examples/QED/thomas_pm_3_1_27.pdf

[4] J. Frenkel, *Die Elektrodynamik des rotierenden Elektrons*, Z. Phys. **37**, 243 (1926), http://kirkmcd.princeton.edu/examples/QED/frenkel_zp_37_243_26.pdf

[5] H.A. Kramers, *Quantum Mechanics* (North Holland, Amsterdam, 1957), p. 226 ff., http://kirkmcd.princeton.edu/examples/EP/kramers_57_sec57.pdf

[6] H.A. Tolhoek and S.R. DeGroot, *On the Theory of Beta-Radioactivity III. The Influence of Electric and Magnetic Fields on Polarized Electron Beams*, Physica **17**, 17 (1951), http://kirkmcd.princeton.edu/examples/QED/tolhoek_physica_17_17_51.pdf
H.A. Tolhoek, *Electron Polarization, Theory and Experiment*, Rev. Mod. Phys. **28**, 277 (1956), http://kirkmcd.princeton.edu/examples/QED/tolhoek_rmp_28_277_56.pdf

[7] H. Mendlowitz and K.M. Case, *Double Scattering of Electrons with Magnetic Interaction*, Phys. Rev. **97**, 33 (1955), http://kirkmcd.princeton.edu/examples/QED/mendlowitz_pr_97_33_55.pdf

[8] M. Carrassi, *The Influence of the Anomalous Magnetic Moment on the Spin Kinematics of Electrons in a Uniform Magnetic Field*, Nuovo Cim. **7**, 525 (1958), http://kirkmcd.princeton.edu/examples/QED/carrassi_nc_7_525_58.pdf

[9] V. Bargmann, L. Michel and V.L. Telegdi, *Precession of the Polarization of Particles Moving in a Homogeneous Electromagnetic Field*, Phys. Rev. Lett. **2**, 435 (1959), http://kirkmcd.princeton.edu/examples/QED/bargmann_prl_2_435_59.pdf

[10] R.H. Good, Jr, *Classical Equations of Motion for a Polarized Particle in an Electromagnetic Field*, Phys. Rev. **125**, 2112 (1962), http://kirkmcd.princeton.edu/examples/EM/good_pr_125_2112_62.pdf

[11] G.E. Uhlenbeck and S. Goudsmit, *Ersetzung der Hypothese vom unmechanischen Zwang durch eine Forderung bezüglich des inneren Verhaltens jedes einzelnen Elektrons*, Naturwiss. **13**, 953 (1925),
http://kirkmcd.princeton.edu/examples/QM/uhlenbeck_naturw_13_953_25.pdf

[12] G.E. Uhlenbeck and S. Goudsmit, *Spinning Electrons and the Structure of Spectra*, Nature **117**, 264 (1926),
http://kirkmcd.princeton.edu/examples/QM/uhlenbeck_nature_117_264_26.pdf

[13] S.A. Goudsmit, *The discovery of the electron spin* (April, 1971),
http://kirkmcd.princeton.edu/examples/QM/goudsmit_71.pdf

[14] S.A. Goudsmit, *It might as well be spin*, Phys. Today **29**(6), 40 (1976),
http://kirkmcd.princeton.edu/examples/QM/goudsmit_pt_29-6_40_76.pdf

[15] G.E. Uhlenbeck, *Personal reminiscences*, Phys. Today **29**(6), 43 (1976),
http://kirkmcd.princeton.edu/examples/QM/goudsmit_pt_29-6_40_76.pdf

[16] L.H. Thomas, *Recollections of the Discovery of the Thomas Precessional Frequency*, AIP Conf. Proc. **95**, 4 (1983),
http://kirkmcd.princeton.edu/examples/GR/thomas_aipcp_95_4_83.pdf

[17] C.N. Yang, *The Spin*, AIP Conf. Proc. **95**, 1 (1983),
http://kirkmcd.princeton.edu/examples/GR/yang_aipcp_95_1_83.pdf

[18] A. Pais, *George Uhlenbeck and the Discovery of Electron Spin*, Phys. Today. **42**(12), 34 (1989), http://kirkmcd.princeton.edu/examples/QM/pais_pt_42-12_34_89.pdf

[19] S.-I. Tomonaga, *The Story of Spin* (U. Chicago Press, 1997),
http://kirkmcd.princeton.edu/examples/QM/tomonaga_spin_97.pdf

[20] A.H. Compton, *The Magnetic Electron*, J. Franklin Inst. **192**, 145 (1921),
http://kirkmcd.princeton.edu/examples/QM/compton_jfi_192_145_21.pdf

[21] W. Pauli, *Über den Einfluβ der Geschwindigkeitsabhängigkeit der Elektronenmasse auf den Zeemaneffekt*, Z. Phys. **31**, 373 (1925),
http://kirkmcd.princeton.edu/examples/QM/pauli_zp_31_373_25.pdf

[22] W. Pauli, *Über den Zusammenhang des Abschlusses der Elektronengruppen im Atom mit der Komplexstruktur der Spektren*, Z. Phys. **31**, 765 (1925),
http://kirkmcd.princeton.edu/examples/QM/pauli_zp_31_765_25.pdf

[23] W. Pauli, *Remarks on the History of the Exclusion Principle*, Science **103**, 213 (1946),
http://kirkmcd.princeton.edu/examples/QM/pauli_science_103_213_46.pdf

[24] B.L. van der Waerden, *Exclusion Principle and Spin*, in *Theoretical Physics in the Twentieth Century*, M. Fierz and V.F. Weiskopf, eds. (Interscience, 1960), p. 199,
http://kirkmcd.princeton.edu/examples/QM/fierz_60.pdf

[25] D. Giulini, *Electron spin or "classically non-describable two-valuedness"*, Stud. Hist. Phil. Mod. Phys. **39**, 557 (2008), http://kirkmcd.princeton.edu/examples/QM/giulini_shpmp_39_557_08.pdf

[26] E.D. Commins, *Electron Spin and Its History*, Ann. Rev. Nucl. Part. Phys. **62**, 133 (2012), http://kirkmcd.princeton.edu/examples/QM/commins_arnps_62_133_12.pdf

[27] A. Sommerfeld, *Über die Zusammensetzung der Geschwindigkeiten in der Relativtheorie*, Phys. Z. **10**, 826 (1909), http://kirkmcd.princeton.edu/examples/GR/sommerfeld_pz_10_826_09.pdf

[28] É. Borel, *La théorie de la relativité et la cinématique*, Compte Rend. Acad. Sci. **156**, 215 (1913), http://kirkmcd.princeton.edu/examples/GR/borel_cras_156_215_13.pdf

[29] L. Silberstein, *The Theory of Relativity*, (Macmillan, 1914), Chap. 6, particularly pp. 168-170, http://kirkmcd.princeton.edu/examples/GR/silberstein_relativity_14.pdf

[30] A. Sommerfeld, *Atombau und Spektrallinien*, 5th ed., Vol. 1 (Braunschweig, 1931), pp. 707-711.

[31] C. Møller, *The Theory of Relativity* (Clarendon Press, 1952), sec. 22, http://kirkmcd.princeton.edu/examples/GR/moller_relativity_52.pdf

[32] W. Furry, *Lorentz Transformation and the Thomas precession*, Am. J. Phys. **23**, 517 (1955), http://kirkmcd.princeton.edu/examples/GR/furry_ajp_23_517_55.pdf

[33] H. Zatzkis, *The Thomas Precession*, J. Franklin Inst. **269**, 268 (1960), http://kirkmcd.princeton.edu/examples/GR/zatzkis_jfi_269_268_60.pdf

[34] P. Nyborg, *Thomas Precession and Classical Theories of Spinning Particles*, Nuovo Cim. **23**, 1057 (1962), http://kirkmcd.princeton.edu/examples/GR/nyborg_nc_23_1057_62.pdf

[35] E.G. Taylor and J.A. Wheeler, *Space-Time Physics*, 1st ed. (W.H. Freeman, 1966), pp. 169-174, http://kirkmcd.princeton.edu/examples/GR/taylor_92.pdf

[36] P.S. Farago, *Derivation of the Spin-Orbit Interaction*, Am. J. Phys. **35**, 246 (1967), http://kirkmcd.princeton.edu/examples/GR/farago_ajp_35_246_67.pdf

[37] J.T. Cushing, *Vector Lorentz Transformations*, Am. J. Phys. **35**, 858 (1967), http://kirkmcd.princeton.edu/examples/GR/cushing_ajp_35_858_67.pdf

[38] G.P. Fisher, *The Thomas Precession*, Am. J. Phys. **40**, 1772 (1972), http://kirkmcd.princeton.edu/examples/GR/fisher_ajp_40_1772_72.pdf

[39] K.R. MacKenzie, *Thomas Precession and the Clock Paradox*, Am. J. Phys. **40**, 1661 (1972), http://kirkmcd.princeton.edu/examples/GR/mackenzie_ajp_40_1661_72.pdf

[40] J. Schwinger, *Spin Precession—A Dynamical Discussion*, Am. J. Phys. **42**, 510 (1974), http://kirkmcd.princeton.edu/examples/GR/schwinger_ajp_42_510_74.pdf

[41] G.H. Goedecke, *Geometry of the Thomas precession*, Am. J. Phys. **46**, 1055 (1978), http://kirkmcd.princeton.edu/examples/GR/goedecke_ajp_46_1055_78.pdf

[42] H.A. Farach *et al.*, *Application of the nonlinear vector product to Lorentz transformations*, Am. J. Phys. **47**, 247 (1979), http://kirkmcd.princeton.edu/examples/GR/farach_ajp_47_247_79.pdf

[43] J.D. Hamilton, *The rotation and precession of relativistic reference frames*, Can. J. Phys. **59**, 213 (1981), http://kirkmcd.princeton.edu/examples/GR/hamilton_cjp_59_213_81.pdf

[44] A. Ben-Menahem, *Wigner's rotation revisited*, Am. J. Phys. **53**, 62 (1985), http://kirkmcd.princeton.edu/examples/GR/ben-menaham_ajp_53_62_85.pdf

[45] C.B. van Wyk, *Rotation associated with the product of two Lorentz transformations*, Am. J. Phys. **52**, 583 (1984), http://kirkmcd.princeton.edu/examples/GR/vanwyk_ajp_52_853_84.pdf

[46] E.G.P. Rowe, *The Thomas precession*, Eur. J. Phys. **5**, 40 (1984), http://kirkmcd.princeton.edu/examples/GR/rowe_ejp_5_40_84.pdf

[47] L. Belloni and C. Reina, *Sommerfeld's way to the Thomas precession*, Eur. J. Phys. **7**, 55 (1986), http://kirkmcd.princeton.edu/examples/GR/belloni_ejp_7_55_86.pdf

[48] H. Urbantke, *Physical holonomy, Thomas precession, and Clifford algebra*, Am. J. Phys. **58**, 747 (1990), http://kirkmcd.princeton.edu/examples/GR/urbantke_ajp_58_747_90.pdf

[49] W.Rindler and V. Perlick, *Rotating Coordinates as Tools for Calculating Circular Geodesics and Gyroscopic Precession*, Gen. Rel. Grav. **22**, 1067 (1990), http://kirkmcd.princeton.edu/examples/GR/rindler_grg_22_1067_90.pdf

[50] A.A. Ungar, *The Relativistic Velocity Composition Paradox and the Thomas Rotation*, Found. Phys. **19**, 1385 (1989), http://kirkmcd.princeton.edu/examples/GR/ungar_fp_19_1385_89.pdf

[51] A.A. Ungar, *Thomas precession and its associated grouplike structure*, Am. J. Phys. **59**, 824 (1991), http://kirkmcd.princeton.edu/examples/GR/ungar_ajp_59_824_91.pdf

[52] R.A. Muller, *Thomas precession: where is the torque?* Am. J. Phys. **60**, 313 (1992), http://kirkmcd.princeton.edu/examples/GR/muller_ajp_60_313_92.pdf

[53] H.C. Corben, *Factors of 2 in magnetic moments, spin-orbit coupling, and Thomas precession*, Am. J. Phys. **61**, 551 (1993), http://kirkmcd.princeton.edu/examples/GR/corben_ajp_61_551_93.pdf

[54] R.J. Philpott, *Thomas precession and the Liénard-Wiechert field*, Am. J. Phys. **64**, 552 (1996), http://kirkmcd.princeton.edu/examples/GR/philpott_ajp_64_552_96.pdf

[55] J.D. Hamilton, *Relativistic precession*, Am. J. Phys. **64**, 1197 (1996), http://kirkmcd.princeton.edu/examples/GR/hamilton_ajp_64_1197_96.pdf

[56] E.G.P. Rowe, *Rest frames for a point particle in special relativity*, Am. J. Phys. **64**, 1184 (1996), http://kirkmcd.princeton.edu/examples/GR/rowe_ajp_64_1184_96.pdf

[57] P.K. MacKeown, *Question #57. Thomas precession*, Am. J. Phys. **65**, 105 (1997), http://kirkmcd.princeton.edu/examples/GR/mackeown_ajp_65_105_97.pdf

[58] J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, 1999), pp. 548-553 and 561-565, http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf

[59] G.B. Malykin, *The relation of Thomas precession to Ishlinskiĭ's theorem as applied to the rotating image of a relativistically moving body*, Phys. Usph. **42**, 505 (1999), http://kirkmcd.princeton.edu/examples/GR/malykin_pu_42_505_99.pdf

[60] J.P. Costella *et al.*, *The Thomas rotation*, Am. J. Phys. **69**, 837 (2001), http://kirkmcd.princeton.edu/examples/GR/costella_ajp_69_837_01.pdf

[61] G. Muñoz, *Spin-orbit interaction and the Thomas precession: A comment on the lab frame point of view*, Am. J. Phys. **69**, 554 (2001), http://kirkmcd.princeton.edu/examples/GR/munoz_ajp_69_554_01.pdf

[62] W.L. Kennedy, *Thomas rotation: a Lorentz matrix approach*, Eur. J. Phys. **23**, 235 (2002), http://kirkmcd.princeton.edu/examples/GR/kennedy_ejp_23_235_02.pdf

[63] H. Kroemer, *The Thomas precession factor in spin-orbit interaction*, Am. J. Phys. **72**, 51 (2004), http://kirkmcd.princeton.edu/examples/GR/kroemer_ajp_72_51_04.pdf

[64] J.A. Rhodes and M.D. Semon, *Relativistic velocity space, Wigner rotation, and Thomas precession*, Am. J. Phys. **72**, 943 (2004), http://kirkmcd.princeton.edu/examples/GR/rhodes_ajp_72_943_04.pdf

[65] R.M. Jonsson, *Gyroscope precession in special and general relativity from basic principles*, Am. J. Phys. **75**, 463 (2007), http://kirkmcd.princeton.edu/examples/GR/jonsson_ajp_75_463_07.pdf

[66] G.B. Malykin, *Thomas precession: correct and incorrect solutions*, Phys. Usph. **49**, 837 (2006). 287 references. http://kirkmcd.princeton.edu/examples/GR/malykin_pu_49_837_06.pdf

[67] K. Rębilas, *Thomas Precession and the Bargmann-Michel-Telegdi Equation*, Found. Phys. **41**, 1800 (2011), http://kirkmcd.princeton.edu/examples/GR/rebilas_fp_41_1800_11.pdf

[68] K. Rębilas, *Simple approach to relativistic spin dynamics*, Am. J. Phys. **79**, 1064 (2011), http://kirkmcd.princeton.edu/examples/GR/rebilas_ajp_79_1064_11.pdf

[69] K. O'Donnell1 and M. Visser, *Elementary analysis of the special relativistic combination of velocities, Wigner rotation and Thomas precession*, Eur. J. Phys. **32**, 1033 (2011), http://kirkmcd.princeton.edu/examples/GR/odonnell_ejp_32_1033_11.pdf

[70] A. Dragan and T. Odrzygóźdź, *A half-page derivation of the Thomas precession*, Am. J. Phys. **81**, 631 (2013), http://kirkmcd.princeton.edu/examples/GR/dragan_ajp_81_631_13.pdf

[71] K. Rębilas, *Thomas precession and torque*, Am. J. Phys. **83**, 199 (2015), http://kirkmcd.princeton.edu/examples/GR/rebilas_ajp_83_199_15.pdf

[72] K. Rębilas, *Subtleties of the Thomas precession*, Eur. J. Phys. **36**, 045007 (2015), http://kirkmcd.princeton.edu/examples/GR/rebilas_ejp_36_045007_15.pdf

[73] J.P. Lambare, *Fermi-Walker transport and Thomas precession*, Eur. J. Phys. **38**, 045602 (2017), http://kirkmcd.princeton.edu/examples/GR/lambare_ejp_38_045602_17.pdf

[74] P. Lewulis and A. Dragan, *Three-line derivation of the Thomas precession*, Am. J. Phys. **87**, 674 (2019), http://kirkmcd.princeton.edu/examples/GR/lewulis_ajp_87_674_19.pdf

[75] G.W. Bennett *et al.*, *Measurement of the Positive Muon Anomalous Magnetic Moment to 0.7 ppm*, Phys. Rev. Lett. **89**, 101804 (2002), http://kirkmcd.princeton.edu/examples/QED/bennett_prl_89_101804_02.pdf

[76] W.H. Louisell *et al.*, *An Experimental Measurement of the Gyromagnetic Ratio of the Free Electron*, Phys. Rev. **94**, 7 (1954), http://kirkmcd.princeton.edu/examples/QED/louisell_pr_94_7_54.pdf
A.A. Schupp *et al.*, *Measurement of the g Factor of Free, High-Energy Electrons*, Phys. Rev. **121**, 1 (1961), http://kirkmcd.princeton.edu/examples/QED/schupp_pr_121_1_61.pdf

[77] D.E. Lundquist *et al.*, *Polarization of Recoil Protons from Neutral Pion Photoproduction*, Phys. Rev. **168**, 1527 (1968), http://kirkmcd.princeton.edu/examples/QED/lundquist_pr_168_1527_68.pdf

[78] F.J.M. Farley *et al.*, *The Anomalous Magnetic Moment of the Negative Muon*, Nuovo Cim. **45**, 281 (1966), http://kirkmcd.princeton.edu/examples/QED/farley_nc_45_281_66.pdf

[79] R. Hagedorn, *Relativistic Kinematics* (Benjamin, New York, 1963), chap. 9, http://kirkmcd.princeton.edu/examples/EP/hagedorn_63_ch9.pdf
http://kirkmcd.princeton.edu/examples/EP/hagedorn_64.pdf

[80] E.M. Purcell and N.F. Ramsey, *On the Possibility of Electric Dipole Moments for Elementary Particles and Nuclei*, Phys. Rev. **78**, 807 (1950), http://kirkmcd.princeton.edu/examples/EP/purcell_pr_78_807_50.pdf

[81] N.F. Ramsey, *Electric Dipole Moments of Particles*, Ann. Rev. Nucl. Part. Sci. **32**, 211 (1982), http://kirkmcd.princeton.edu/examples/EP/ramsey_arnps_32_211_82.pdf