

Alternative Forms of the Poynting Vector

Kirk T. McDonald

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

(July 13, 2018; updated January 7, 2020)

1 Introduction

A prominent feature of Maxwell’s vision of *A Dynamical Theory of the Electromagnetic Field* [7] is that the electromagnetic field contains energy, whose volume density u is, in Gaussian units,

$$u = \frac{\mathbf{E} \cdot \mathbf{D}}{8\pi} + \frac{\mathbf{B} \cdot \mathbf{H}}{8\pi}, \quad (1)$$

for linear media where $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, with ϵ and μ being the (relative) permittivity and permeability of the medium, and c is the speed of light in vacuum.^{1,2,3}

Subsequently, Poynting (1883) [14] promoted the vision that the flux of energy in the electromagnetic field is described by the vector,

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H} \quad (\text{Poynting}), \quad \text{where} \quad \mathbf{E} \cdot \mathbf{J} = -\frac{\partial u}{\partial t} - \nabla \cdot \mathbf{S}, \quad (2)$$

is the rate of work done by electromagnetic fields on electric current density \mathbf{J} .

Perhaps because the implications of eqs. (1)-(2) are sometimes counterintuitive,^{4,5} there has been ongoing doubt as to the physical interpretation of these relations.

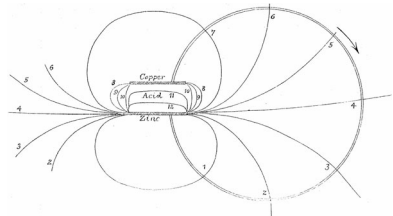
¹In Sec. 74 of [7], Maxwell stated: *In speaking of the Energy of the field ... I wish to be understood literally ... The only question is, Where does it reside? On the old theories it resides in the electrified bodies, conducting circuits, and magnets, in the form of an unknown quality called potential energy, or the power of producing certain effects at a distance. On our theory it resides in the electromagnetic field, in the space surrounding the electrified and magnetic bodies, as well as in those bodies themselves.*

In Art. 631 of his *Treatise* [10], Maxwell wrote: *Hence, the electrostatic (potential) energy of the whole field will be the same if we suppose that it resides in every part of the field where electrical force (\mathbf{E}) and electrical displacement (\mathbf{D}) occur, instead of being confined to places where free electricity is found. In (vector) language (and in Gaussian units) it is $\mathbf{E} \cdot \mathbf{D}/8\pi$. And in Art. 636 he wrote: *According to our hypothesis, we assume the kinetic energy to exist wherever there is magnetic force, that is, in general, in every part of the field. The amount of this energy per unit of volume is $\mathbf{B} \cdot \mathbf{H}/8\pi$, and this energy exists in some form of motion in every portion of space.**

²That the magnetic-field-energy density can be written as $\mathbf{B} \cdot \mathbf{H}/8\pi$ was first deduced by Maxwell in (1856), p. 63 of [5], via a transformation of the magnetic potential energy. Yet, Maxwell called the transformed energy “kinetic”.

³Maxwell’s comments about electromagnetic field energy contrast with his characterization of “mechanical” potential energy in Art. 97 of [12], where he stated: *Rankine [4] introduced the term Potential Energy—a very felicitous expression, since it not only signifies the energy which the system has not in actual possession, but only has the power to acquire.*

⁴One of the first examples of counterintuitive behavior was given by Poynting in Fig. 3 of [14], that the flow of energy from a battery to a resistive wire loop connected to it is not through the wire, but across the “empty” space between the battery and segments of the wire. See also, for example, [97, 127, 148].



⁵Another example was given by Heaviside on p. 94 of [39], where he considered a uniformly magnetized

An early expression of such doubt was by Heaviside (1887) [21], despite his having independently deduced (1884) that the vector \mathbf{S} of eq. (2) can be considered to represent the flow of energy in the electromagnetic field [16, 17, 18]. Heaviside’s comment, p. 93 of [39], was based on an awareness that his and Poynting’s derivation of eq. (2) actually only determine $\nabla \cdot \mathbf{S}$:⁶ *If \mathbf{S} be the vector energy-current density, we may add to it another vector, say \mathbf{s} , provided \mathbf{s} have no convergence anywhere (i.e., $\nabla \cdot \mathbf{s} = 0$). The existence of \mathbf{s} is possible, but there does not appear to be any present means of finding out whether it is real, and how it is to be expressed.*

Among later doubters, the most notable is Feynman.^{7,8}

In the Sec. 3 below note we review the alternatives to the Poynting vector that have been proposed, after some comments (in Sec. 2) about electromagnetic-field momentum and angular momentum.

2 Electromagnetic Momentum and Angular Momentum

2.1 Field Momentum

Maxwell enunciated a conception of electromagnetic momentum in Sec. 57 of [7] as,

$$\mathbf{P}_{\text{EM}}^{(\text{Maxwell})} = \int \frac{\rho \mathbf{A}^{(C)}}{c} d\text{Vol}, \quad (3)$$

where ρ is the electric charge density and $\mathbf{A}^{(C)}$ is the vector potential in the Coulomb gauge (that Maxwell used prior to the explicit recognition of gauge conditions [119]). Maxwell

sphere with a uniform electric surface charge. Here, lines of the Poynting vector \mathbf{S} (Heaviside’s \mathbf{W}) flow in circles outside the sphere, about its magnetic axis. As Heaviside remarked: *What is the use of it? On the other hand, what harm does it do?* See Feynman’s answer in Sec. 3.12.7 below.

⁶Heaviside credited J.J. Thomson for pointing this out. See pp. 151-152 of [19], and also Sec. 280, p. 313 of [37].

⁷Feynman stated, Sec. 27-4 of [70]: *All we did was to find a possible “ u ” and a possible “ \mathbf{S} ”. How do we know that by juggling the terms around some more we couldn’t find another formula for “ u ” and another formula for “ \mathbf{S} ”? The new \mathbf{S} and the new u would be different, but they would still satisfy (the second of eq. 2 of this paper). It’s possible. It can be done, but the forms that have been found always involve various derivatives of the field (and always with second-order terms like a second derivative or the square of a first derivative). There are, in fact, an infinite number of different possibilities for u and \mathbf{S} , and so far no one has thought of an experimental way to tell which one is right! People have guessed that the simplest one is probably the correct one, but we must say that we do not know for certain what is the actual location in space of the electromagnetic field energy. So we too will take the easy way out and say that the field energy is given by eq. (1). Then the flow vector \mathbf{S} must be given by eq. (2).*

⁸This note concerns the notion of flow of field energy in classical electromagnetism. In quantum theory one can also consider the flow of field energy, but here one is led to speak of the probability amplitude and probability density of the flow. For example, in a classical analysis of Young’s double-slit experiment, lines of the Poynting vector (2) go through one slit or the other, whereas in the quantum view a single photon has a nonzero amplitude to go through both slits (unless it is observed at one of the slits).

Although the quantum Poynting vector is subject to “uncertainty” in its interpretation, this has not led to suggestions of alternative forms for it.

regarded the vector potential \mathbf{A} at the location of an electric charge q as providing a measure, $q\mathbf{A}/c$, of electromagnetic momentum, as well as an interpretation of Faraday's electrotonic state (Arts. 60-61 of [2]). That Faraday associated with some kind of momentum with this state is hinted in Art. 1077 of [3].

However, the form (3) seems to associate the momentum with charges rather than with fields.

In 1888, Heaviside, p. 330 of [26], inferred from the Maxwell stress tensor that the volume force density \mathbf{f} could be written as,⁹

$$\mathbf{f} = \frac{d}{dt} \frac{\mathbf{S}}{c^2} = \frac{d}{dt} \frac{\mathbf{E} \times \mathbf{H}}{4\pi c} \quad (\text{Heaviside}), \quad (4)$$

without explicitly identifying $\mathbf{S}/c^2 = \mathbf{E} \times \mathbf{H}/4\pi c$ as the field-momentum density.¹⁰

In 1891, Thomson noted [27] that a sheet of electric displacement \mathbf{D} (parallel to the surface) which moves perpendicular to its surface with velocity \mathbf{v} must be accompanied by a sheet of magnetic field $\mathbf{H} = \mathbf{v}/c \times \mathbf{D}$ according to the free-space Maxwell equation $\nabla \times \mathbf{H} = (1/c) \partial \mathbf{D} / \partial t$.¹¹ Then, the motion of the energy density of these sheets implies there is also a momentum density, eqs. (2) and (6) of [27],

$$\mathbf{p}_{\text{EM}}^{(\text{Thomson})} = \frac{\mathbf{D} \times \mathbf{H}}{4\pi c}. \quad (5)$$

Also in 1891, Heaviside identified the momentum of the *free ether* in Sec. 26 of [30] as,¹²

$$\mathbf{p}_{\text{EM}}^{(\text{Heaviside})} = \frac{\mathbf{D} \times \mathbf{B}}{4\pi c}. \quad (6)$$

This was a clarification of his discussion in 1886, eq. (7a) of [20], of a *magnetolectric force* $\mathbf{D}/4\pi c \times \partial \mathbf{B} / \partial t$.¹³

In 1893, Thomson transcribed much of his 1891 paper into the beginning of *Recent Researches* [37], adding the remark (p. 9) that the momentum density (5) is closely related to the Poynting vector [14, 17],^{14,15}

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H}. \quad (7)$$

⁹The result (4) was anticipated by Heaviside in eqs. (6a) and (7a) of [20] (1886).

¹⁰In the same paper [26], Heaviside gave the first correct derivation of the electromagnetic fields of a uniformly moving charge with any velocity $v < c$.

¹¹Variants of this argument were given by Heaviside in 1891, Sec. 45 of [28], and much later in Sec. 18-4 of [70], where it is noted that Faraday's law, $\nabla \times \mathbf{E} = -(1/c) \partial \mathbf{B} / \partial t$, combined with the Maxwell equation for \mathbf{H} implies that $v = c$ in vacuum, which point seems to have been initially overlooked by Thomson, although noted by him in Sec. 265 of [34].

¹²See also p. 557 of [33] and p. 495 of [29].

¹³Heaviside also mentioned this concept in 1889 on pp. 399-330 of [26].

¹⁴The idea that an energy-flux vector is the product of energy density and energy-flow velocity seems to be due to Umov [11], based on Euler's continuity equation [1] for mass flow, $\nabla \cdot (\rho \mathbf{v}) = -\partial \rho / \partial t$.

¹⁵Thomson argued, in effect, that the field-momentum density (5) is related by $\mathbf{p}_{\text{EM}} = \mathbf{S}/c^2 = \mathbf{uv}/c^2$ [27, 37]. See also eq. (19), p. 79 of [32], and p. 6 of [50].

The form (5) was also used by Poincaré in 1900 [43], following Lorentz’ convention [31] that the force on electric charge q be written $q(\mathbf{D} + \mathbf{v}/c \times \mathbf{H})$, and that the Poynting vector be $(c/4\pi)\mathbf{D} \times \mathbf{H}$. In 1903 Abraham [45] argued for,

$$\mathbf{P}_{\text{EM}}^{(\text{Abraham})} = \frac{\mathbf{E} \times \mathbf{H}}{4\pi c} = \frac{\mathbf{S}}{c^2}, \quad (8)$$

and in 1908 Minkowski [48] advocated the form,^{16,17}

$$\mathbf{P}_{\text{EM}}^{(\text{Minkowski})} = \frac{\mathbf{D} \times \mathbf{B}}{4\pi c}. \quad (9)$$

The forms (5)-(9) all involve either $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ or $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$, where \mathbf{P} and \mathbf{M} are the densities of electric and magnetic dipoles due to electric charges and currents. As such, they involve momentum of the “mechanical” dipoles as well as the of the electromagnetic fields \mathbf{E} and \mathbf{B} , which has led to extensive debate as to the physical interpretation of these various forms [143]. The author’s view that it is often best to avoid this debate, and restrict discussion of “field” momentum to the “electromagnetic-field-only” momentum,

$$\mathbf{P}_{\text{EM}} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}. \quad (10)$$

The integral of the electromagnetic-field-momentum density (10) can be written for static fields in four equivalent forms,

$$\mathbf{P}_{\text{EM}} = \int \frac{\rho \mathbf{A}}{c} d\text{Vol} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} = \int \frac{V \mathbf{J}}{c^2} d\text{Vol} = \int \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \mathbf{r} d\text{Vol} \quad (\text{statics}), \quad (11)$$

where ρ is the (total) electric charge density, $\mathbf{A} = \int \mathbf{J} d\text{Vol}/cr$ is the magnetic vector potential (where $\nabla \cdot \mathbf{A} = 0$ for static fields in both the Coulomb and Lorenz gauges), \mathbf{J} is the electric current density due to electric charges, $\mathbf{E} = \int \rho \hat{\mathbf{r}} d\text{Vol}/r^2$ is the electric field, $\mathbf{B} = \int \mathbf{J} \times \hat{\mathbf{r}} d\text{Vol}/cr^2$ is the magnetic field, and $V = \int \rho d\text{Vol}/r$ is the electric scalar potential.¹⁸ The third form was introduced by Furry [79],¹⁹ and the fourth form is due to Aharonov *et al.* [107].

2.2 Field Angular Momentum

In retrospect, one can recognize that electromagnetic field momentum was first considered by Darboux (1878) [13] as a vector constant of the motion in the interaction of an electric charge with a (hypothetical) magnetic pole. This constant vector was further considered by

¹⁶Minkowski, like Poynting [14], Heaviside [17] and Abraham [45], wrote the Poynting vector as $\mathbf{E} \times \mathbf{H}$. See eq. (75) of [48]. Heaviside wrote the momentum density in the Minkowski form (9) on p. 108 of [32].

¹⁷For some remarks on the “perpetual” Abraham-Minkowski debate see [136].

¹⁸For a review, see [125]. For discussion of alternative forms of electromagnetic energy, momentum and angular momentum for fields with arbitrary time dependence, see, for example [132].

¹⁹The density $V \mathbf{J}/c^2$ is the static limit of an expression for the field-momentum density proposed by Livens (Sec. 3.4 below), bottom of p. 263 of [54]. This density is also the static limit of \mathbf{S}/c^2 for several of the alternative forms of the Poynting vector proposed by Slepian (1942, Sec. 3.6 below) [63].

Poincaré in 1896 [42], but its relation to electromagnetic field momentum was only realized much later. For a review, see [145].

The concept of electromagnetic field momentum was first explicitly mentioned in 1904 by J.J. Thomson [46], who considered an electric charge outside a small solenoid magnet, and then an electric charge q plus a single magnetic pole p . For the latter case, he computed the field angular momentum as,²⁰

$$\mathbf{L}_{\text{EM}} = \int \mathbf{r} \times \mathbf{p}_{\text{EM}} d\text{Vol} = \int \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} = \frac{pq}{c} \hat{\mathbf{z}}, \quad (12)$$

independent of the distance between p and q , where $\hat{\mathbf{z}}$ points from the electric charge q to the magnetic pole p .

The vector (12) is the constant vector considered by Darboux and Poincaré, but this went unnoticed for many years.

In 1904 the notion of quantizing angular momentum was still years away, and the provocative result (12), that the angular momentum of a magnetic pole plus electric charge is independent of their separation, went unremarked until 1931 when Dirac [57] argued that $pq/c = \hbar/2$. See also Sec. 6.12 of [118].

3 Some History of Poynting-Vector Alternatives

3.1 Thomson

In 1885, J.J. Thomson, p. 151 of [19], noted that one could satisfy Poynting's theorem via,

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} + \mathbf{s}, \quad \text{with} \quad \mathbf{s} = \nabla \times \mathbf{f} \quad (\text{Thomson}), \quad (13)$$

since $\nabla \cdot \mathbf{s} = \nabla \cdot (\nabla \times \mathbf{f}) = 0$ for any differentiable vector field \mathbf{f} . Thomson did not give an argument favoring any particular form of \mathbf{f} , but just considered that the energy-flow vector is “indeterminate”.

Heaviside's subsequent comments about this issue were discussed on p. 2 above.

3.2 Birkeland

In 1894, Birkeland [40] argued (in German) that if the Poynting vector is to be a function only of the electromagnetic fields, and not of their derivatives, then the only possible form is the original version of Poynting, our eq. (2).²¹

Birkeland assumed that magnetic monopoles do not exist. If they do, the Poynting vector becomes $\mathbf{S} = \mathbf{D}_m \times \mathbf{H}_e/4\pi c = (\mathbf{E} - 4\pi\mathbf{P}_m) \times (\mathbf{B} - 4\pi\mathbf{M}_e)/4\pi c$ where \mathbf{P}_m is the density of electric-dipole moments due to magnetic-monopole currents and \mathbf{M}_e is the density of magnetic-dipole moments due to electric currents, as reviewed in Sec. 2.3 of [139].

²⁰See also Sec. 2.2 of [141].

²¹Birkeland's argument was mentioned by Feynman, footnote 7 above. For another discussion of Birkeland's argument, see the end of [91].

3.3 Macdonald

Another early proposal for an alternative to the Poynting vector is that by Macdonald (1902), p. 72 of [44],

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} + \mathbf{s}, \quad \text{with} \quad \mathbf{s} = \frac{1}{8\pi} \frac{\partial}{\partial t} (\mathbf{A} \times \mathbf{H}) \quad (\text{Macdonald}). \quad (14)$$

Here, $\nabla \cdot \mathbf{s} \neq 0$, but Macdonald also proposed an alternative form for the energy density u such that the second of eq. (2) is still satisfied.

Macdonald's expressions for \mathbf{S} and u are not gauge invariant,²² so this author considers them not to be valid for a physical description of Nature.^{23,24}

3.3.1 Schott

Macdonald's alternatives were noted by Schott (1912) on pp. 5-6 of [49], who thereafter used Poynting's standard form.

3.3.2 Tralli

Macdonald's vector (14) was presented without attribution in Sec. 9-13 of [71] (1963).

3.4 Livens

In 1917, Livens discussed Poynting's derivation of eq. (2) in Sec. 2 of [51], and then in Sec. 3 he reviewed Macdonald's derivation of eq. (14). In Sec. 4, he proposed a different alternative to the Poynting vector,

$$\mathbf{S} = V \mathbf{J}_{\text{total}} \quad (\text{Livens}), \quad (15)$$

²²Maxwell was aware of the arbitrariness of the electromagnetic potentials Ψ and \mathbf{A} , Sec. 98, pp. 499-500 of [7]. He understood that his preference for $\nabla \cdot \mathbf{A} = 0$ (Coulomb gauge) is a choice, not a physical requirement, and that one can always enforce this condition by what is now called a gauge transformation, eqs. (74) and (77) of [7] on rewriting φ as Ψ' .

²³Of course, with a particular choice of gauge, valid computations can be made using the electromagnetic potentials.

²⁴One of the first to question the physical significance of the electromagnetic potentials was Heaviside (1888) in the *Postscript* to [23], pp. 47-50, titled *On the Metaphysical Nature of the Propagation of the Potentials*. On p. 47, he wrote: *We make acquaintance, experimentally, not with potentials, but with forces, and we formulate observed facts with the least amount of hypothesis, in terms of the electric force \mathbf{E} and magnetic force \mathbf{H} . In Maxwell's development of Faraday's views, \mathbf{E} and \mathbf{H} actually represent the state of the medium anywhere.*

Such doubt was a topic of debate at the 1888 Bath Meeting of the British Association, where FitzGerald spoke of the "murder of Ψ ", p. 624 of [22]. See also [24] and chap. 7 of [110].

Heaviside indicated an awareness as to the arbitrariness of the electromagnetic potentials Ψ and \mathbf{A} on p. 48 of [23], where he stated: *Thus we have Ψ , \mathbf{A} , and $\dot{\mathbf{A}}$ required, involving seven scalar specifications to find the six in \mathbf{E} and \mathbf{H} .* However, he seems to have followed Maxwell in taking the electromagnetic potentials to be in the Coulomb gauge. At the top of p. 49 of [23], Heaviside noted what Maxwell had missed regarding the Coulomb gauge: *Thus we have instantaneous propagation of Ψ to infinity.* Then he added: *I prefer, however, to say that this is only a mathematical fiction.*

where V is the electrical scalar potential, and $\mathbf{J}_{\text{total}} = \mathbf{J} + (1/4\pi)\partial\mathbf{D}/\partial t$ is the total current density as first considered by Maxwell in eq. (112), p. 19, of [6]. Like Maxwell, Livens tacitly used the Coulomb gauge.

The form (15) has the appeal that in case of a steady conduction current the flow of energy follows the flow of electric current, in contrast to Poynting's vision (footnote 4 above).

Livens also made this argument in secs. 627-628, pp. 555-556 of his textbook [52], and in Sec. 229, pp. 242-244 of the 2nd edition [54].

Livens briefly mentioned electromagnetic field momentum in the Abraham form (8) on p. 598 of [52], and gave greater discussion in secs. 239-242, pp. 258-264 of [54]. In particular, he proposed an alternative to electromagnetic-field-momentum density on p. 263 of [54] as,

$$\mathbf{p}_{\text{EM}} = \frac{V\mathbf{J}_{\text{total}}}{c^2} + \left(-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right) \times \frac{\mathbf{B}}{4\pi c} \quad (\text{Livens}). \quad (16)$$

Livens' forms are subject to the general objection that physical quantities must be gauge invariant. For example, in the Gibbs gauge [144], $V = 0$ everywhere.

It is, however, noteworthy that in case of static fields (where $\mathbf{J}_{\text{total}} = \mathbf{J}$), Livens' form (16) for the field-momentum density reduces to the form of eq. (11), as later advocated by Furry [79].

3.4.1 Carpenter

Livens' vector (15) was advocated by Carpenter [111, 112] in 1989.

3.4.2 Haus and Melcher (July 29, 2022)

On pp. 469-470 of [113] (1989) it was argued that if the magnetic field is independent of time, then we can take $\mathbf{E} = -\nabla V$,²⁵ and $\int \mathbf{E} \times \mathbf{H} \cdot d\mathbf{Area} = \int V[(4\pi/c)\mathbf{J} + (1/c)\partial\mathbf{D}/\partial t] d\mathbf{Area}$. In this case we could identify a Poynting vector as $\mathbf{S} = V(\mathbf{J} + (1/4\pi)\partial\mathbf{D}/\partial t)$, which reduces to $\mathbf{S} = V\mathbf{J}$ if both \mathbf{B} and \mathbf{D} are independent of time.

An issue (besides the lack of gauge invariance of the electric potential V) is then whether one wants to consider that the Poynting vector has one form for static examples and a very different form for dynamic cases. *The present author prefers that there be a single form of the Poynting vector, $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{H}$, in all examples.*

3.5 Bateman

In 1922, Bateman [53] proposed a variant of the Poynting vector as part of an effort to explain the quantum behavior of atoms via classical electromagnetism,

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H} - \frac{\psi}{2\pi}\frac{\partial}{\partial t}\nabla\psi, \quad \text{where} \quad \psi(\mathbf{r}, t) = \frac{1}{4\pi} \int \frac{\rho(\mathbf{r}', t')}{\gamma' |\mathbf{r} - \mathbf{r}'|} d\text{Vol}' \quad (\text{Bateman}), \quad (17)$$

$t' = t - |\mathbf{r} - \mathbf{r}'|/c$ is the retarded time, and $\gamma' = 1/\sqrt{1 - v^2(\mathbf{r}', t')/c^2}$ is the Lorentz factor of the moving source charges at the retarded time.

This peculiar suggestion depends on a potential ψ , and is not gauge invariant.

²⁵This is an implicit choice of gauge, which excludes use of the Gibbs gauge, *i.e.*, $V = 0$ everywhere.

3.6 Slepian

In 1942, Slepian (who had a delightful sense of humor²⁶) discussed Poynting’s theorem [62] and offered 8 alternatives to standard Poynting vector [63]. One of these, eq. (13) of [63], involves the magnetization density \mathbf{M} which is not typically considered explicitly in the context of the Poynting vector, and the other seven involve the electromagnetic potentials as well as the fields.²⁷

Slepian’s third form, eq. (19) of [63] is,²⁸

$$\mathbf{S}_3 = V\mathbf{J}_{\text{total}} + \frac{c}{4\pi} \left(-\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) \times \mathbf{H} \quad (\text{Slepian}), \quad (19)$$

where the electric current density $\mathbf{J}_{\text{total}}$ includes the displacement current (density) $\partial \mathbf{D} / \partial t$.²⁹

Slepian remarked after his eq. (19): *The first term $V\mathbf{J}_{\text{total}}$ is the formula for energy flow used extensively by electrical engineers of power systems, and where displacement currents are only a small part of the total current, $|V\mathbf{J}| = VI/\text{Area}$ may be determined directly by wattmeters.*^{30,31}

The seven alternatives of Slepian which involve the electromagnetic potentials are unphysical, in that they are not gauge invariant.

3.6.1 Carter

Slepian’s third form, eq. (19) of [63], was called the “Slepian vector” on p. 321 of [76] (and was later reinvented by Lai [87], Sec. 3.11 below).

²⁶For examples of Slepian’s wry humor regarding the Poynting vector, see [64, 65, 129].

²⁷The variant involving the magnetization \mathbf{M} is a precursor to considerations by the author [140] that the standard Poynting vector can be rewritten in 729 different ways using different combinations of auxiliary fields. These variants ultimately contain the same physics as the standard Poynting vector, but the notation used makes them appear to be very different.

²⁸

$$\begin{aligned} \mathbf{S} &= \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} = \left(-\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) \times \frac{c\mathbf{H}}{4\pi} = V\nabla \times \frac{c\mathbf{H}}{4\pi} - \nabla \times V \frac{c\mathbf{H}}{4\pi} - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \times \frac{c\mathbf{H}}{4\pi} \\ &= V\mathbf{J}_{\text{total}} + \frac{c}{4\pi} \left(-\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) \times \mathbf{H} - \nabla \times V \frac{c\mathbf{H}}{4\pi} = \mathbf{S}_3 - \nabla \times \frac{cV\mathbf{H}}{4\pi}, \end{aligned} \quad (18)$$

such that \mathbf{S} and \mathbf{S}_3 differ only by the vector $\mathbf{s} = \nabla \times cV\mathbf{H}/4\pi$, whose divergence is zero.

²⁹Although this differs from Livens’ form (15), note that \mathbf{S}_3/c^2 is the same as Livens’ expression (16) for the field-momentum density \mathbf{p}_{EM} , if Livens’ \mathbf{B} were changed to \mathbf{H} .

³⁰For more extended comments by Slepian on the use of $V\mathbf{J}_{\text{total}}$, see [62].

³¹In the electric-power industry there is interest in scalar measures of electric power, such as the product VI of voltage and current. The merits of a vector measure of energy flow, such as the standard Poynting vector \mathbf{S} , or even the alternative $V\mathbf{J}_{\text{total}}$, are not always obvious. Some of the ongoing debate on this theme is at [124, 128, 130, 142]. The Feynman cylinder paradox [98] (Sec. 2.6) illustrates the advantage of the standard form of the Poynting vector over the form $V\mathbf{J}$ in a static example.

3.7 Hines

In 1951, Hines [66] reviewed the alternatives of Macdonald, eq. (14) and Livens, eq. (15), and then proposed the hybrid variant,

$$\mathbf{S} = \frac{1}{8\pi} \left(\mathbf{A} \times \frac{\partial \mathbf{H}}{\partial t} - \frac{\partial \mathbf{A}}{\partial t} \times \mathbf{H} \right) + V \mathbf{J}_{\text{total}} \quad (\text{Hines}), \quad (20)$$

which differs slightly from Slepian’s sixth form, eq. (26) of [63].

In 1979, Wallace and McConnell [84] argued that use of Hines’ form (20) does not lead to the correct rate of radiation by an accelerated, nonrelativistic charge. But, see also [94, 101].

3.8 Ohmura

In 1956, Ohmura [67] made a brief discussion (in biquaternion notation) of a possible theory of electromagnetism that included magnetic monopoles, as well as an additional scalar field e and a pseudoscalar field p .

This theory was later revived by van Vlaenderen [123], who only considered the new scalar field (which he unfortunately called S). In Ohmura’s notation,

$$e = -\frac{1}{c} \frac{\partial V}{\partial t} - \nabla \cdot \mathbf{A}, \quad (21)$$

such that the scalar field e is not gauge invariant, and is zero in the Lorenz gauge [8]. For nonzero e , two of the four Maxwell equations are modified,

$$\nabla \cdot \mathbf{E} = 4\pi\rho + \frac{1}{c} \frac{\partial e}{\partial t}, \quad \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} - \nabla e, \quad (22)$$

and the (non gauge invariant) Poynting vector is now,

$$\mathbf{S} = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B} + e\mathbf{E}). \quad (23)$$

These equations admit “free space” wave solutions for \mathbf{E} and \mathbf{B} with longitudinal polarization in the far zone,³² and as such are popular with fans of Tesla (whose studies of wave phenomena with longitudinal polarization in the near zone have been interpreted by some as evidence of such waves in the far zone as well).

While most variants of Ohmura’s theory remain obscure [114, 120, 121, 131, 133, 137], one has recently appeared in a book [149] from a major scientific publisher (where the scalar field e is called β).

3.9 Wolf

The electromagnetic fields \mathbf{E} and \mathbf{B} can be deduced from other potentials than the usual scalar and vector potentials V and \mathbf{A} . This was first done by Hertz (1888) [25], who introduced the so-called polarization (vector) potential. In 1904, Whittaker [47] noted that Hertz’ polarization-vector potential could be replaced by two scalar potentials, and in 1959 Wolf, Sec. 3 of [69], advocated expressing the Poynting vector in terms of these scalar potentials. This form is, of course, not gauge invariant.

³²Guided waves include longitudinal components, but such waves are always in the near zone of associated conductors. Waves with only a longitudinal electric field, and no magnetic field, can exist in plasmas [122].

3.10 Butler

In 1969, Butler [80] argued that the momentum density $\mathbf{p} = \mathbf{S}/c^2$ should be part of an energy-momentum-density 4-vector, which (he claimed) could be arranged with the form,

$$\mathbf{S} = \frac{E^2 - B^2}{8\pi} \gamma^2 \mathbf{v}, \quad (\text{Butler}), \quad (24)$$

where \mathbf{v} is the velocity of the electric charge in a Universe that contains only a single charge, and $\gamma = 1/\sqrt{1 - v^2/c^2}$. He did not explain what happens in a Universe with more than one charge.

3.11 Lai

In 1980, Lai [87] proposed a variant on the form (20) of Hines,

$$\mathbf{S} = -\frac{1}{4\pi} \frac{\partial A}{\partial t} \times \mathbf{H} + V \mathbf{J}_{\text{total}} \quad (\text{Lai}). \quad (25)$$

3.11.1 Kobe

In 1981, Kobe [92] pointed out that Lai's form (25) is the same as Slepian's third form, eq. (19) of [63], our eq. (19), as well as objecting to all forms that are not gauge invariant.

3.11.2 Peters

In 1981, Peters [90] also objected to Lai's form (25) on the grounds that it is not gauge invariant, and added a second objection that, with this form, \mathbf{S}/c^2 is not a reasonable expression for the field-momentum density.

3.11.3 Romer

In 1981, Romer [91] seconded the objections by Kobe and Peters to Lai's form (25), and added a comment about Birkeland's analysis (mentioned in Sec. 3.2 above).

3.11.4 Puthoff

Lai's form (25) was advocated by Puthoff (2010) [134].

3.12 Other Commentaries

3.12.1 Jeans

In the 5th edition of his text, p. 519 of [55], Jeans (1927) added cautionary remarks about the Poynting vector similar to those of Heaviside (p. 93 of [39]): *The integral of the Poynting Flux over a closed surface gives the total flow of energy into or out of a surface, but it has not been proved, and we are not entitled to assume, that there is an actual flow of energy at every point equal to the Poynting Flux. For instance if an electrified sphere is placed near to*

a bar magnet, this latter assumption would require a perpetual flow of energy at every point in the field except the special points at which the electric and magnetic lines of force are tangential to one another. It is difficult to believe that this predicted circulation of energy can have any physical reality. On the other hand it is to be noticed that such a circulation of energy is almost meaningless. The circulation of a fluid is a definite conception because it is possible to identify the different particles of a fluid; we can say for instance whether or not the particles entering a small element of volume are identical or not with an equal number of particles coming out, but the same is not true of energy.

3.12.2 Mason and Weaver

Mason and Weaver (1929), Sec. 54 of [56], reviewed Poynting's theorem, and on p. 268 remarked that its integral form *suggests the interpretation given, rather than demands it. This is, of course, recognized by every careful reader of electrodynamics.*

3.12.3 Sumpner

In 1934, Sumpner, an enthusiast of Heaviside [58], published an article [59] which cast doubt on all the work of Poynting because an argument in Poynting's paper [15] supposedly implied that the speed of the flow of electromagnetic energy is $\approx 10^8$ times the speed of light.

See also Sec. 3.3 of [68].

3.12.4 O'Rahilly

Although he offered no alternatives to the Poynting vector, O'Rahilly (1938) gave, on pp. 275-323 of [60], a very extensive criticism of perceived inconsistencies in the use and interpretation of the Poynting vector, and of the electromagnetic-field energy and momentum densities.

3.12.5 Stratton

In Sec. 2.19 of [61], Stratton (1941) noted: *The classical interpretation of Poynting's theorem appears to rest to a considerable degree on hypothesis. Various alternative forms of the theorem have been offered from time to time, but none of these has the advantage of greater plausibility or greater simplicity to recommend it, and it is significant that thus far no other interpretation has contributed anything of value to the theory. The hypothesis of an energy density in the electromagnetic field and a flow of intensity $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ has, on the other hand, proved extraordinarily fruitful. A theory is not an absolute truth but a self-consistent analytical formulation of the relations governing a group of natural phenomena. By this standard there is every reason to retain the Poynting-Heaviside viewpoint until a clash with new experimental evidence shall call for its revision.*

3.12.6 Hammond

In 1958, Hammond [68] reviewed the physics of the Poynting vector and commented on several proposed alternatives.³³

3.12.7 Feynman

In addition to the comments on the Poynting vector mentioned in Sec. 1 above, Feynman posed the now-famous disk paradox related to field angular momentum in Sec. 17-4 of [70]. This paradox was perhaps inspired by a comment of J.J. Thomson, p. 348 of [46], and has led to extensive additional commentary, including [73, 74, 75, 85, 89, 93, 96, 98, 99, 100, 102, 103, 104, 105, 106, 108, 109, 115, 117, 135, 147, 150, 151, 152].

At the end of Sec. 27-6, Feynman gave a verbal resolution of the paradox: *Do you remember the paradox we described in Section 17-4 about a solenoid and some charges mounted on a disc? It seemed that when the current turned off, the whole disc should start to turn. The puzzle was: Where did the angular momentum come from? The answer is that if you have a magnetic field and some charges, there will be some angular momentum in the field. It must have been put there when the field was built up. When the field is turned off, the angular momentum is given back. So the disc in the paradox would start rotating. This mystic circulating flow of energy, which at first seemed so ridiculous, is absolutely necessary. There is really a momentum flow. It is needed to maintain the conservation of angular momentum in the whole world.*

3.12.8 Romer

In 1966, Romer [73, 75] commented on field angular momentum in another paradox [72, 126].

3.12.9 Shockley

In 1968, Shockley [78] argued that certain thought experiments involving pulsed electric currents “prove” that the standard form (2) of the Poynting vector is the only valid one.³⁴

3.12.10 Lahoz and Graham (added May 19, 2021)

Starting in 1978, Lahoz and Graham [82, 83, 86, 95] published several papers on the momentum density when a permanent magnet is inside an electric field, arguing that their experiments favored our eq. (10), $\mathbf{p}_{\text{EM}} = \mathbf{E} \times \mathbf{B}/4\pi c$, rather than the Abraham form (8), $\mathbf{p}_{\text{EM}}^{(\text{Abraham})} = \mathbf{E} \times \mathbf{H}/4\pi c$ (which latter tacitly assumes linear media).

³³Hammond seems to have been misled by Slepian’s subtle humor into supposing, Sec. 3.5 of [68], that the latter favored an alternative form involving the electromagnetic potentials.

³⁴Shockley is also notable for popularizing the term “hidden momentum” [77], inspired by examples (first given in [46]) in which the static electromagnetic fields of a system “at rest” have nonzero net field momentum. As alternative forms of the Poynting vector have played little role in discussions of “hidden momentum,” we do not consider it further here (but see [151]). A review by the author on this topic is at [138].

3.12.11 Lorrain

In 1982, Lorrain [88] made a brief comment, in reference to [87], that the forms (19) and (20) could not be correct because they fail to predict the existence of density of field angular momentum,

$$\mathbf{l}_{\text{EM}} = \mathbf{r} \times \mathbf{p}_{\text{EM}} = \mathbf{r} \times \frac{\mathbf{S}}{c^2}, \quad (26)$$

in certain static examples. However, this complaint is not actually valid, in that any example with nonzero field angular momentum must include a nonzero electric current density \mathbf{J} , contrary to a claim of Lorrain.

3.12.12 Backhaus and Schäfer

In 1984, Backhaus and Schäfer [101] noted that some arguments against alternatives to the Poynting vector are not as decisive as claimed, without commenting on the key issue of gauge invariance.

3.12.13 Nelson

In 1995, Nelson [116] made an extension of the Poynting vector to include certain quantum phenomena in a semiclassical model.

3.12.14 Bossavit

A recent (2018) paper by Bossavit [146] argued that with use of differential forms one can “prove” the uniqueness of the standard version of the Poynting vector.

References

- [1] L. Euler, *Principes généraux du mouvement des fluides*, Acad. Roy. Sci. Belles Lett. (Berlin) **11**, 274 (1757), http://kirkmcd.princeton.edu/examples/fluids/euler_e226.pdf
kirkmcd.princeton.edu/examples/fluids/euler_fluids_75_english.pdf
- [2] M. Faraday, *Experimental Researches in Electricity*, Phil. Trans. Roy. Soc. London **122**, 125 (1832), kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_122_163_32.pdf
- [3] M. Faraday, *Experimental Researches in Electricity.—Ninth Series*, Phil. Trans. Roy. Soc. London **122**, 125 (1832), kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_125_41_35.pdf
- [4] W.J.M. Rankine, *On the General Law of the Transformation of Energy*, Phil. Mag. **5**, 106 (1853), kirkmcd.princeton.edu/examples/statmech/rankine_pm_5_106_53.pdf
- [5] J.C. Maxwell, *On Faraday’s Lines of Force*, Trans. Camb. Phil. Soc. **10**, 27 (1858), kirkmcd.princeton.edu/examples/EM/maxwell_tcps_10_27_58.pdf

- [6] J.C. Maxwell, *On Physical Lines of Force. Part III.—The Theory of Molecular Vortices applied to Statical Electricity*, Phil. Mag. **23**, 12 (1862), kirkmcd.princeton.edu/examples/EM/maxwell_pm_23_12_62.pdf
- [7] J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459 (1865), http://kirkmcd.princeton.edu/examples/EM/maxwell_ptrs1_155_459_65.pdf
- [8] L. Lorenz, *Ueber die Identität der Schwingungen des Lichts mit den elektrischen Strömen*, Ann. d. Phys. **207**, 243 (1867), http://kirkmcd.princeton.edu/examples/EM/lorenz_ap_207_243_67.pdf
On the Identity of the Vibration of Light with Electrical Currents, Phil. Mag. **34**, 287 (1867), http://kirkmcd.princeton.edu/examples/EM/lorenz_pm_34_287_67.pdf
- [9] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 1 (Clarendon Press, 1873), kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v1_73.pdf
- [10] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 2 (Clarendon Press, 1873), kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v2_73.pdf
- [11] N. Umow, *Ableitung der Bewegungsgleichungen der Energie in continuirlichen Körpern*, Zeit. Math. Phys. **19**, 418 (1874), kirkmcd.princeton.edu/examples/EM/umow_zmp_19_97_418_74.pdf
kirkmcd.princeton.edu/examples/EM/umov_theorem.pdf
- [12] J.C. Maxwell, *Matter and Motion* (Society for the Promotion of Christian Knowledge, 1876), kirkmcd.princeton.edu/examples/mechanics/maxwell_matter_motion_76.pdf
kirkmcd.princeton.edu/examples/mechanics/maxwell_matter_motion.pdf
- [13] G. Darboux, *Problème de Mécanique*, Bull. Sci. Math. Astro. **2**, 433 (1878), kirkmcd.princeton.edu/examples/EM/darboux_bsma_2_433_78.pdf
- [14] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884), kirkmcd.princeton.edu/examples/EM/poynting_ptrs1_175_343_84.pdf
- [15] J.H. Poynting, *On the connexion between Electric Current and the Electric and Magnetic Inductions in the surrounding field*, Phil. Trans. Roy. Soc. London **176**, 277 (1885), kirkmcd.princeton.edu/examples/EM/poynting_ptrs1_176_277_85.pdf
- [16] O. Heaviside, *The Induction of Currents in Cores*, Electrician **13**, 133 (1884), kirkmcd.princeton.edu/examples/EM/heaviside_eip4_electrician_13_133_84.pdf
 Also on pp. 377-378 of [38].
- [17] O. Heaviside, *Electromagnetic Induction and Its Propagation*, part 2, Electrician **14**, 178 (1885), kirkmcd.princeton.edu/examples/EM/heaviside_eip4_electrician_14_178_85.pdf
 Also on p. 438 of [38].

- [18] O. Heaviside, *Electromagnetic Induction and Its Propagation*, part 4, *Electrician* **14**, 306 (1885), kirkmcd.princeton.edu/examples/EM/heaviside_eip4_electrician_14_306_85.pdf
Also on pp. 449-450 of [38].
- [19] J.J. Thomson, *Report on Electric Theories*, British Assoc. Adv. Sci. 55th Meeting, 97 (1885), http://kirkmcd.princeton.edu/examples/EM/thomson_baas_55_97_86.pdf
- [20] O. Heaviside, *Electromagnetic Induction and Its Propagation*, part 22, *Electrician* **16**, 186 (1886), kirkmcd.princeton.edu/examples/EM/heaviside_electrician_16_186_86.pdf
Also on pp. 542-548 of [38].
- [21] O. Heaviside, *Electromagnetic Induction and Its Propagation*, part 35, *Electrician* **18**, 211 (1887), kirkmcd.princeton.edu/examples/EM/heaviside_electrician_18_211_87.pdf
Also on pp. 93-94 of [39].
- [22] O.J. Lodge, *Sketch of the Electrical Papers in Section A of the Recent Bath Meeting of the British Association*, *Electrician* **21**, 622 (1888),
kirkmcd.princeton.edu/examples/EM/lodge_electrician_21_622_88.pdf
- [23] O. Heaviside, *The General Solution of Maxwell's Electromagnetic Equations in a Homogeneous Isotropic Medium, especially in regard to the Derivation of special Solutions, and the Formulæ for Plane Waves*, *Phil. Mag.* **27**, 29 (1889),
kirkmcd.princeton.edu/examples/EM/heaviside_pm_27_29_89.pdf
See also pp. 468-485 of [39].
- [24] O. Heaviside, *Electromagnetic Waves and the Propagation of Potential*, *Electrician* **22**, 23 (1888), kirkmcd.princeton.edu/examples/EM/heaviside_electrician_22_23_88.pdf
Also on pp. 490-492 of [39].
- [25] H. Hertz, *The Forces of Electrical Oscillations Treated According to Maxwell's Theory*, *Ann. d. Phys.* **36**, 1 (1889),
http://kirkmcd.princeton.edu/examples/EM/hertz_ap_36_1_89.pdf
reprinted in chap. 9 of H. Hertz, *Electric Waves* (Macmillan, 1900),
http://kirkmcd.princeton.edu/examples/EM/hertz_electric_waves.pdf
A translation by O. Lodge appeared in *Nature* **39**, 402 (1889),
<http://physics.princeton.edu/~mcdonald/examples/EM/hertz.pdf>
- [26] O. Heaviside, *On the Electromagnetic Effects due to the Motion of Electrification through a Dielectric*, *Phil. Mag.* **27**, 324 (1889),
kirkmcd.princeton.edu/examples/EM/heaviside_pm_27_324_89.pdf
See also pp. 504-518 of [39].
- [27] J.J. Thomson, *On the Illustration of the Properties of the Electric Field by Means of Tubes of Electrostatic Induction*, *Phil. Mag.* **31**, 149 (1891),
kirkmcd.princeton.edu/examples/EM/thomson_pm_31_149_91.pdf
- [28] O. Heaviside, *Electromagnetic Theory—VII*, *Electrician* **26**, 663 (1891),
http://kirkmcd.princeton.edu/examples/EM/heaviside_electrician_27_493_91.pdf
See also pp. 42-43 of [32].

- [29] O. Heaviside, *Electromagnetic Theory*–XV, *Electrician* **27**, 493 (1891),
http://kirkmcd.princeton.edu/examples/EM/heaviside_electrician_27_493_91.pdf
 See also pp. 107-108 of [32].
- [30] O. Heaviside, *On the Forces, Stresses and Fluxes of Energy in the Electromagnetic Field*, *Phil. Trans. Roy. Soc. London* **183**, 423 (1892),
http://kirkmcd.princeton.edu/examples/EM/heaviside_ptrsla_183_423_92.pdf
- [31] H.A. Lorentz, *La Théorie Électromagnétique Maxwell et Son Application aux Corps Mouvants*, (Brill, Leiden, 1892),
kirkmcd.princeton.edu/examples/EM/lorentz_theorie_electromagnetique_92.pdf
- [32] O. Heaviside, *Electromagnetic Theory*, Vol. 1 (Electrician Publishing Co., 1893),
kirkmcd.princeton.edu/examples/EM/heaviside_electromagnetic_theory_1.pdf
- [33] O. Heaviside, *Electrical Papers*, Vol. 2 (Macmillan, 1894),
http://kirkmcd.princeton.edu/examples/EM/heaviside_electrical_papers_2.pdf
- [34] J.J. Thomson, *Elements of the Mathematical Theory of Electricity and Magnetism* (Cambridge U. Press, 1895),
http://kirkmcd.princeton.edu/examples/EM/thomson_EM_1st_ed_95.pdf
- [35] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 2, 3rd ed. (Clarendon Press, 1892, 1904), kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v1_04.pdf
- [36] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 2, 3rd ed. (Clarendon Press, 1892), kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v2_92.pdf
- [37] J.J. Thomson, *Recent Researches in Electricity and Magnetism* (Clarendon Press, 1893),
kirkmcd.princeton.edu/examples/EM/thomson_recent_researches_in_electricity.pdf
- [38] O. Heaviside, *Electrical Papers*, Vol. 1 (Macmillan, 1894),
kirkmcd.princeton.edu/examples/EM/heaviside_electrical_papers_1.pdf
- [39] O. Heaviside, *Electrical Papers*, Vol. 2 (Macmillan, 1894),
kirkmcd.princeton.edu/examples/EM/heaviside_electrical_papers_2.pdf
- [40] K. Birkeland, *Über die Strahlung electromagnetischer energie in Raume*, *Ann. d. Phys.* **52**, 537 (1894), kirkmcd.princeton.edu/examples/EM/birkeland_ap_52_357_94.pdf
- [41] J.W. Gibbs, *Velocity of Propagation of Electrostatic Forces*, *Nature* **53**, 509 (1896),
kirkmcd.princeton.edu/examples/EM/gibbs_nature_53_509_96.pdf
- [42] H. Poincaré, *Remarques sur une expérience de M. Birkeland*, *Comptes Rendus Acad. Sci.* **123**, 530 (1896), kirkmcd.princeton.edu/examples/EM/poincare_cras_123_530_96.pdf
kirkmcd.princeton.edu/examples/EM/poincare_cras_123_530_96_english.pdf

- [43] H. Poincaré, *La Théorie de Lorentz et la Principe de Réaction*, Arch. Neer. **5**, 252 (1900), kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00.pdf
Translation: *The Theory of Lorentz and the Principle of Reaction*, kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00_english.pdf
- [44] H.M. Macdonald, *Electric Waves* (Cambridge U. Press, 1902), pp. 32, 72, kirkmcd.princeton.edu/examples/EM/macdonald_electric_waves_02.pdf
- [45] M. Abraham, *Prinzipien der Dynamik des Elektrons*, Ann. d. Phys. **10**, 105 (1903), physics.princeton.edu/~mcdonald/examples/EM/abraham_ap_10_105_03.pdf
kirkmcd.princeton.edu/examples/EM/abraham_ap_10_105_03_english.pdf
- [46] J.J. Thomson, *On Momentum in the Electric Field*, Phil. Mag. **8**, 331 (1904), kirkmcd.princeton.edu/examples/EM/thomson_pm_8_331_04.pdf
- [47] E.T. Whittaker, *On an Expression of the Electromagnetic Field Due to Electrons by means of Two Scalar Potential Functions*, Proc. London Math. Soc. **1**, 367 (1904), http://kirkmcd.princeton.edu/examples/EM/whittaker_plms_1_367_04.pdf
- [48] H. Minkowski, *Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körper*, Nachr. Ges. Wiss. Göttingen **1**, 53 (1908), kirkmcd.princeton.edu/examples/EM/minkowski_ngwg_53_08.pdf
kirkmcd.princeton.edu/examples/EM/minkowski_ngwg_53_08_english.pdf
- [49] G.A. Schott, *Electromagnetic Radiation and the Mechanical Reactions Arising from it* (Cambridge U. Press, 1912), kirkmcd.princeton.edu/examples/EM/schott_radiation_12.pdf
- [50] H. Bateman, *The Mathematical Analysis of Electrical and Optical Wave Motion*, (Cambridge U. Press, 1915), kirkmcd.princeton.edu/examples/EM/bateman_wave_motion_15.pdf
- [51] G.H. Livens, *On the Flux of Energy in the Electrodynamic Field*, Phil. Mag. **34**, 385 (1917), kirkmcd.princeton.edu/examples/EM/livens_pm_34_385_17.pdf
- [52] G.H. Livens, *The Theory of Electricity* (Cambridge U. Press, 1918), kirkmcd.princeton.edu/examples/EM/livens_electricity_18.pdf
- [53] H. Bateman, *The Stress-Energy Tensor in Electromagnetic Theory and a New Law of Force*, Phys. Rev. **20**, 243 (1922), kirkmcd.princeton.edu/examples/EM/bateman_pr_20_243_22.pdf
- [54] G.H. Livens, *The Theory of Electricity*, 2nd ed. (Cambridge U. Press, 1926), kirkmcd.princeton.edu/examples/EM/livens_26.pdf
- [55] J. Jeans, *The Mathematical Theory of Electricity and Magnetism*, 5th ed. (Cambridge U. Press, 1927), kirkmcd.princeton.edu/examples/EM/jeans_em_27.pdf
- [56] M. Mason and W. Weaver, *The Electromagnetic Field* (Dover, 1929), pp. 266-268, kirkmcd.princeton.edu/examples/EM/mason_emf_29.pdf

- [57] P.A.M. Dirac, *Quantised Singularities in the Electromagnetic Field*, Proc. Roy. Soc. London A **133**, 60 (1931), kirkmcd.princeton.edu/examples/QED/dirac_prsla_133_60_31.pdf
- [58] W.E. Sumpner, *The Work of Oliver Heaviside*, J. IEE **71**, 837 (1932), kirkmcd.princeton.edu/examples/EM/sumpner_jiee_71_837_32.pdf
- [59] W.E. Sumpner, *Radiation and Electrical Power Transmission*, J. IEE **75**, 512 (1934), kirkmcd.princeton.edu/examples/EM/sumpner_jiee_75_512_34.pdf
- [60] A. O’Rahilly, *Electromagnetic Theory* (Longmans, Green, 1938), kirkmcd.princeton.edu/examples/EM/orahilly_EM.pdf
- [61] J.A. Stratton, *Electromagnetic Theory* (McGraw-Hill, 1941), Sec. 2.19, kirkmcd.princeton.edu/examples/EM/stratton_electromagnetic_theory.pdf
- [62] J. Slepian, *Energy Flow In Electric Systems—the V_i Energy-Flow Postulate*, Trans. Am. Inst. EE **61**, 835 (1942), kirkmcd.princeton.edu/examples/EM/slepian_taiee_61_835_42.pdf
- [63] J. Slepian, *Energy and Flow in the Electromagnetic Field*, J. Appl. Phys. **13**, 512 (1942), kirkmcd.princeton.edu/examples/EM/slepian_jap_13_512_42.pdf
- [64] J. Slepian, *Electromagnetic Space-Ship*, Electrical Engineering **68**, 145 (1949), kirkmcd.princeton.edu/examples/EM/slepian_ee_68_145_49
- [65] J. Slepian, *Answer to Previous Essay*, Electrical Engineering **68**, 245 (1949), kirkmcd.princeton.edu/examples/EM/slepian_ee_68_245_49
- [66] C.O. Hines, *Electromagnetic energy density and flux*, Can. J. Phys. **30**, 123 (1952), kirkmcd.princeton.edu/examples/EM/hines_cjp_30_123_52.pdf
- [67] T. Ohmura, *A New Formulation on the Electromagnetic Field*, Prog. Theor. Phys. **16**, 684 (1956), kirkmcd.princeton.edu/examples/EM/ohmura_ptp_16_684_56.pdf
- [68] P. Hammond, *Electromagnetic Energy Transfer*, Proc. IEE **105**, 352 (1958), kirkmcd.princeton.edu/examples/EM/hammond_piee_105_352_58.pdf
- [69] E. Wolf, *A Scalar Representation of Electromagnetic Fields: II*, Proc. Phys. Soc. **A74**, 269 (1959), http://kirkmcd.princeton.edu/examples/EM/wolf_pps_a74_269_59.pdf
- [70] R.P. Feynman, R.B. Leighton and M. Sands, *The Feynman Lectures on Physics*, Vol. 2 (Addison-Wesley, 1963), http://www.feynmanlectures.caltech.edu/II_17.html#Ch17-S4
http://www.feynmanlectures.caltech.edu/II_18.html#Ch18-S2
http://www.feynmanlectures.caltech.edu/II_27.html
- [71] N. Tralli, *Classical Electromagnetic Theory* (McGraw-Hill, 1963), kirkmcd.princeton.edu/examples/EM/tralli_9-13.pdf
- [72] R.P. McKenna, *Case of the Paradoxical Invention*, Analog **75**, No. 1, 14 (1965), kirkmcd.princeton.edu/examples/EM/mckenna_analog_75_1_14_65.pdf

- [73] R.H. Romer, *Angular Momentum of Static Electromagnetic Fields*, Am. J. Phys. **34**, 772 (1966), kirkmcd.princeton.edu/examples/EM/romer_ajp_34_772_66.pdf
- [74] E.M. Pugh and G.E. Pugh, *Physical Significance of the Poynting Vector in Static Fields*, Am. J. Phys. **35**, 153 (1967), kirkmcd.princeton.edu/examples/EM/pugh_ajp_35_153_67.pdf
- [75] R.H. Romer, *Electromagnetic Angular Momentum*, Am. J. Phys. **35**, 445 (1967), kirkmcd.princeton.edu/examples/EM/romer_ajp_35_445_67.pdf
- [76] G.W. Carter, *The Electromagnetic Field in its Engineering Aspects* (Longmans, Green, 1967), kirkmcd.princeton.edu/examples/EM/carter_chap13_67.pdf
- [77] W. Shockley and R.P. James, “Try Simplest Cases” Discovery of “Hidden Momentum” Forces on “Magnetic Currents”, Phys. Rev. Lett. **18**, 876 (1967), kirkmcd.princeton.edu/examples/EM/shockley_prl_18_876_67.pdf
- [78] W. Shockley, *S-ambiguity of Poynting’s integral theorem eliminated by conceptual experiments with pulsed current distributions*, Phys. Lett. A **28**, 185 (1968), kirkmcd.princeton.edu/examples/EM/shockley_pl_28a_185_68.pdf
- [79] W.H. Furry, *Examples of Momentum Distributions in the Electromagnetic Field and in Matter*, Am. J. Phys. **37**, 621 (1969), kirkmcd.princeton.edu/examples/EM/furry_ajp_37_621_69.pdf
- [80] J.W. Butler, *A Proposed Electromagnetic Momentum-Energy 4-Vector for Charged Bodies*, Am. J. Phys. **37**, 1258 (1969), kirkmcd.princeton.edu/examples/EM/butler_ajp_37_1258_69.pdf
- [81] D. Villarroel, *On Redefinitions of the Energy Momentum Tensor*, Ann. Phys. **89**, 241 (1974), kirkmcd.princeton.edu/examples/EM/villarroel_ap_89_241_74.pdf
- [82] D.G. Lahoz and G.M. Graham, *Observation of Electromagnetic Angular Momentum within Magnetite*, Phys. Rev. Lett. **285**, 155 (1980), http://kirkmcd.princeton.edu/examples/EM/lahoz_prl_42_1137_79.pdf
- [83] G.M. Graham and D.G. Lahoz, *Observation of static electromagnetic angular momentum in vacuo*, Nature **285**, 155 (1980), http://kirkmcd.princeton.edu/examples/EM/graham_nature_285_155_80.pdf
- [84] G.L. Wallace and R.F. O’Connell, *Energy flow vector of the electromagnetic field*, Can. J. Phys. **58**, 744 (1980), kirkmcd.princeton.edu/examples/EM/wallace_cjp_58_744_80.pdf
- [85] E. Corinaldesi, *Angular momentum of a static electromagnetic field*, Am. J. Phys. **48**, 83 (1980), kirkmcd.princeton.edu/examples/EM/corinaldesi_ajp_48_83_80.pdf
- [86] D.G. Lahoz and G.M. Graham, *The phenomenological electromagnetic force density*, Can. J. Phys. **59**, 1 (1981), http://kirkmcd.princeton.edu/examples/EM/lahoz_cjp_59_1_81.pdf
- [87] C.S. Lai, *Alternative choice for the energy flow vector of the electromagnetic field*, Am. J. Phys. **49**, 841 (1981), kirkmcd.princeton.edu/examples/EM/lai_ajp_49_841_81.pdf

- [88] P. Lorrain, *Alternative choice for the energy flow vector of the electromagnetic field*, Am. J. Phys. **50**, 492 (1982), kirkmcd.princeton.edu/examples/EM/lorrain_ajp_50_492_82.pdf
- [89] J.M. Aguirregabiria and A. Hernández, *The Feynman paradox revisited*, Eur. J. Phys. **2**, 168 (1981), kirkmcd.princeton.edu/examples/EM/aguirregabiria_ejp_2_168_81.pdf
- [90] P.C. Peters, *Objections to an alternate energy flow vector*, Am. J. Phys. **50**, 1165 (1982), kirkmcd.princeton.edu/examples/EM/peters_ajp_50_1165_82.pdf
- [91] R.H. Romer, *Alternatives to the Poynting vector for describing the flow of electromagnetic energy*, Am. J. Phys. **50**, 1166 (1982), kirkmcd.princeton.edu/examples/EM/romer_ajp_50_1166_82.pdf
- [92] D.H. Kobe, *Energy flux vector for the electromagnetic field and gauge invariance*, Am. J. Phys. **50**, 1162 (1982), kirkmcd.princeton.edu/examples/EM/kobe_ajp_50_1162_82.pdf
- [93] G.G. Lombardi, *Feynman's disk paradox*, Am. J. Phys. **51**, 213 (1983), physics.princeton.edu/~mcdonald/examples/EM/lombardi_ajp_51_213_83.pdf
- [94] W. Gough, *Poynting in the wrong direction?* Eur. J. Phys. **3**, 83 (1982), kirkmcd.princeton.edu/examples/EM/gough_ejp_3_83_82.pdf
- [95] D.G. Lahoz and G.M. Graham, *Experimental decision on the electromagnetic momentum expression for magnetic media*, J. Phys. A **15**, 303 (1982), http://kirkmcd.princeton.edu/examples/EM/lahoz_jpa_15_303_82.pdf
- [96] F.L. Boos, Jr, *More on the Feynman's Disk Paradox*, Am. J. Phys. **52**, 756 (1984), kirkmcd.princeton.edu/examples/EM/boos_ajp_52_756_84.pdf
- [97] M. Heald, *Electric fields and charges in elementary circuits*, Am. J. Phys. **52**, 522 (1984), kirkmcd.princeton.edu/examples/EM/heald_ajp_52_522_84.pdf
- [98] J. Belcher and K.T. McDonald, *Feynman Cylinder Paradox* (Fall, 1983), kirkmcd.princeton.edu/examples/feynman_cylinder.pdf
- [99] T. Bahder and J. Sak, *Elementary solution to Feynman's disk paradox*, Am. J. Phys. **53**, 495 (1985), kirkmcd.princeton.edu/examples/EM/bahder_ajp_53_495_85.pdf
- [100] O.B. Keyes, *Comments on "Feynman's Disk Paradox"*, Am. J. Phys. **52**, 680 (1984), kirkmcd.princeton.edu/examples/EM/keyes_ajp_52_680_84.pdf
- [101] U. Backhaus and K. Schäfer, *On the uniqueness of the vector for energy flow density in electromagnetic fields*, Am. J. Phys. **54**, 279 (1986), kirkmcd.princeton.edu/examples/EM/backhaus_ajp_54_279_86.pdf
- [102] R.H. Romer, *Electromagnetic Angular Momentum*, Am. J. Phys. **53**, 15 (1985), kirkmcd.princeton.edu/examples/EM/romer_ajp_53_15_85.pdf
- [103] T.-C.E. Ma, *Field angular momentum in Feynman's disk paradox*, Am. J. Phys. **54**, 949 (1986), kirkmcd.princeton.edu/examples/EM/ma_ajp_54_949_86.pdf

- [104] H.S.T. Driver, *Angular momentum in static electric and magnetic fields: A simple case*, Am. J. Phys. **55**, 755 (1987), kirkmcd.princeton.edu/examples/EM/driver_ajp_55_755_87.pdf
- [105] N.L. Sharma, *Field versus action at a distance in a static situation*, Am. J. Phys. **56**, 420 (1988), kirkmcd.princeton.edu/examples/EM/sharma_ajp_56_420_88.pdf
- [106] J. Higbie, *Angular momentum in the field of an electron*, Am. J. Phys. **56**, 378 (1988), http://kirkmcd.princeton.edu/examples/EM/higbie_ajp_56_378_88.pdf
- [107] Y. Aharonov, P. Pearle and L. Vaidman, *Comment on "Proposed Aharonov-Casher effect: Another example of an Aharonov-Bohm effect arising from a classical lag,"* Phys. Rev. A **37**, 4052 (1988), kirkmcd.princeton.edu/examples/QM/aharonov_pra_37_4052_88.pdf
- [108] D.J. Griffiths, *Note on "Field versus action-at-a-distance in a static situation" by N.L. Sharma*, Am. J. Phys. **57**, 558 (1989), kirkmcd.princeton.edu/examples/EM/griffiths_ajp_57_558_89.pdf
- [109] A.S. de Castro, *Electromagnetic angular momentum for a rotating charged shell*, Am. J. Phys. **59**, 180 (1991), kirkmcd.princeton.edu/examples/EM/decastro_ajp_59_180_91.pdf
- [110] B.J. Hunt, *The Maxwellians* (Cornell U. Press, 1991), kirkmcd.princeton.edu/examples/EM/hunt_chap7.pdf
- [111] C.J. Carpenter, *Electromagnetic energy and power in terms of charges and potentials instead of fields*, Proc. IEE **136** A, 55 (1989), kirkmcd.princeton.edu/examples/EM/carpenter_piee_136_55_89.pdf
- [112] C.J. Carpenter, *Comparison of the practical advantages of alternative descriptions of electromagnetic momentum*, Proc. IEE **136** A, 101 (1989), http://kirkmcd.princeton.edu/examples/EM/carpenter_piee_136a_101_89.pdf
- [113] H.A. Haus and J.R. Melcher, *Electromagnetic Fields and Energy*, (Prentice Hall, 1999), http://kirkmcd.princeton.edu/examples/EM/haus_89.pdf
- [114] N.P. Khvorostenko, *Longitudinal Electromagnetic Waves*, Russ. Phys. J. **35**, 223 (1992), kirkmcd.princeton.edu/examples/EM/khvorostenko_rpj_35_223_92.pdf
- [115] F.S. Johnson, B.L. Cragin and R.L. Hodges, *Electromagnetic momentum density and the Poynting vector in static fields*, Am. J. Phys. **62**, 33 (1994), kirkmcd.princeton.edu/examples/EM/johnson_ajp_62_33_94.pdf
- [116] D.F. Nelson, *Generalizing the Poynting Vector*, Phys. Rev. Lett. **76**, 4713 (1996), http://kirkmcd.princeton.edu/examples/EM/nelson_prl_76_4713_96.pdf
- [117] O. Rojo, *On the Energy Flow Vector of the E-M Field*, Rev. Bras. Ens. Fis. **19**, 396 (1997), kirkmcd.princeton.edu/examples/EM/rojo_rbef_19_396_97.pdf
- [118] J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, 1999), http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf

- [119] J.D. Jackson and L.B. Okun, *Historical roots of gauge invariance*, Rev. Mod. Phys. **73**, 663 (2001), kirkmcd.princeton.edu/examples/EM/jackson_rmp_73_663_01.pdf
- [120] K.J. van Vlaenderen and A. Waser, *Generalisation of Classical Electrodynamics to Admit a Scalar Field and Longitudinal Waves*, Hadr. J. **24**, 609 (2001), kirkmcd.princeton.edu/examples/EM/vanvlaenderen_hj_24_609_01.pdf
- [121] C. Monstein and J.P. Wesley, *Observation of scalar longitudinal electrodynamic waves*, Europhys. Lett. **59**, 514 (2002), kirkmcd.princeton.edu/examples/EM/monstein_epl_59_514_02.pdf
- [122] K.T. McDonald, *An Electrostatic Wave* (July 28, 2002), kirkmcd.princeton.edu/examples/bernstein.pdf
- [123] K.J. van Vlaenderen, *A generalisation of classical electrodynamics for the prediction of scalar field effects*, (submitted May 23, 2003), <https://arxiv.org/abs/physics/0305098>
- [124] L.S. Czarnecki, *Could Power Properties of Three-Phase Systems Be Described in Terms of the Poynting Vector?* IEEE Trans. Pow. Del. **21**, 339 (2007), kirkmcd.princeton.edu/examples/EM/czarnecki_ieeetpd_21_339_07.pdf
- [125] K.T. McDonald, *Four Expressions for Electromagnetic Field Momentum* (Apr. 10, 2006), kirkmcd.princeton.edu/examples/pem_forms.pdf
- [126] K.T. McDonald, *McKenna's Paradox: Charged Particle Exiting the Side of A Solenoid Magnet* (Apr. 12, 2006), kirkmcd.princeton.edu/examples/mckenna.pdf
- [127] K.T. McDonald, *Momentum in a DC Circuit* (May 26, 2006), kirkmcd.princeton.edu/examples/loop.pdf
- [128] P.E. Sutherland, *On the Definition of Power in an Electrical System*, IEEE Trans. Pow. Del. **22**, 1100 (2007), kirkmcd.princeton.edu/examples/EM/sutherland_ieeetpd_22_1100_07.pdf
- [129] K.T. McDonald, *Slepian's Electromagnetic Spaceship* (June 18, 2006), kirkmcd.princeton.edu/examples/slepian.pdf
- [130] A.E. Emanuel, *About the Rejection of Poynting Vector in Power System Analysis*, J. Elec. Pow. Qual. Util. **13**, 43 (2007), kirkmcd.princeton.edu/examples/EM/emanuel_epquj_13_43_07.pdf
- [131] A.I. Arbab and Z.A. Satti, *On the Generalized Maxwell Equations and Their Prediction of Electroscalar Wave*, Prog. Phys. **5**(2), 8 (2009), [kirkmcd.princeton.edu/examples/EM/arbap_pp_5\(2\)_8_09.pdf](http://kirkmcd.princeton.edu/examples/EM/arbap_pp_5(2)_8_09.pdf)
- [132] K.T. McDonald, *Orbital and Spin Angular Momentum of Electromagnetic Fields* (Mar. 12, 2009), kirkmcd.princeton.edu/examples/spin.pdf
- [133] D.V. Podgany and O.A. Zaimidoroga, *Nonrelativistic theory of electroscalar field and Maxwell electrodynamics* (May 28, 2010), <https://arxiv.org/abs/1005.3130>
- [134] H.E. Puthoff, *Electromagnetic Potentials Basis for Energy Density and Power Flux* (Sept. 26, 2010), <https://arxiv.org/abs/0904.1617>

- [135] B.Y.-K. Hu, *Introducing electromagnetic field momentum*, Eur. J. Phys. **33**, 873 (2012), kirkmc.d.princeton.edu/examples/EM/hu_ejp_33_873_12.pdf
- [136] K.T. McDonald, *Abraham, Minkowski and “Hidden” Mechanical Momentum* (June 6, 2012), kirkmc.d.princeton.edu/examples/abraham.pdf
- [137] L.H. Hively and G.C. Giakos, *Toward a more complete electrodynamic theory*, Int. J. Sig. Imag. Syst. Eng. **5**, 3 (2012), kirkmc.d.princeton.edu/examples/EM/hively_ijsism_5_3_12.pdf
- [138] K.T. McDonald, *On the Definition of “Hidden” Momentum* (July 9, 2012), kirkmc.d.princeton.edu/examples/hiddendef.pdf
- [139] K.T. McDonald, *Poynting’s Theorem with Magnetic Monopoles* (Mar. 23, 2013), kirkmc.d.princeton.edu/examples/poynting.pdf
- [140] K.T. McDonald, *729 Variants of Poynting’s Theorem* (May 21, 2013), kirkmc.d.princeton.edu/examples/variants.pdf
- [141] K.T. McDonald, *J.J. Thomson and “Hidden” Momentum* (Apr. 30, 2014), kirkmc.d.princeton.edu/examples/thomson.pdf
- [142] N. Calamaro, Y. Beck and D. Shmilovitz, *A review and insights on Poynting vector theory and periodic averaged electric energy transport theories*, Renew. Sust. Energy Rev. **42**, 1279 (2015), kirkmc.d.princeton.edu/examples/EM/calamaro_rser_42_1279_15.pdf
- [143] K.T. McDonald, *Bibliography on the Abraham-Minkowski Debate* (Feb. 17, 2015), kirkmc.d.princeton.edu/examples/ambib.pdf
- [144] K.-H. Yang and K.T. McDonald, *Formal Expressions for the Electromagnetic Potentials in Any Gauge* (Feb. 25, 2015), kirkmc.d.princeton.edu/examples/gauge.pdf
- [145] K.T. McDonald, *Birkeland and Poincaré: Motion of an Electric Charge in the Field of a Magnetic Pole* (Apr. 15, 2015), kirkmc.d.princeton.edu/examples/birkeland.pdf
- [146] A. Bossavit, *A note on the uniqueness of the Poynting vector and of the Maxwell tensor*, Int. J. Num. Mod. **31**, c2214 (2018), kirkmc.d.princeton.edu/examples/EM/bossavit_ijnm_31_c2214_18.pdf
- [147] G. Pantazis and L. Perivolaropoulos, *A general realistic treatment of the disk paradox*, Eur. J. Phys. **38**, 015207 (2017), http://kirkmc.d.princeton.edu/examples/EM/pantazis_ejp_38_015204_17.pdf
- [148] M. Matar and R. Welti, *Surface charges and J.H. Poynting’s disquisitions on energy transfer in electrical circuits*, Eur. J. Phys. **38**, 065201 (2017), http://kirkmc.d.princeton.edu/examples/EM/matar_ejp_38_065201_17.pdf
- [149] B. Zohuri, *Scalar Wave Driven Energy Applications* (Springer, 2018), kirkmc.d.princeton.edu/examples/EM/zohuri_19b.pdf

- [150] K. Fukushima and S. Pu, *Relativistic Decomposition of the Orbital and the Spin Angular Momentum in Chiral Physics and Feynman's Angular Momentum Paradox*, Lect. Notes Phys. **987**, 381 (2021), http://kirkmcd.princeton.edu/examples/EM/fukushima_lnp_987_381_21.pdf
- [151] L. Jiménez, I. Campos and J.A.E. Roa-Neri, *The Feynman paradox and hidden momentum*, Eur. J. Phys. **43**, 055202 (2022), http://kirkmcd.princeton.edu/examples/EM/jimenez_ejp_43_055202_22.pdf
- [152] O.D. Johns, *A spherical version of Feynman's static field momentum example*, Eur. J. Phys. **45**, 015201 (2024), http://kirkmcd.princeton.edu/examples/EM/johns_ejp_45_015201_24.pdf