# Radiation Pressure
# of a Monochromatic Plane Wave on a Flat Mirror

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## 1 Problem

Discuss the radiation pressure of a monochromatic plane wave on a flat, perfectly conducting mirror when the angle of incidence of the wave is $\theta$. Consider the various perspectives of the momentum density in the wave, the Maxwell stress tensor, the Lorentz force, and the radiation reaction force on the oscillating charges on the surface of the mirror. Compare the idealized case of a perfectly conducting mirror with that of a mirror with finite conductivity, using a classical model for inelastic collisions of electrons with the lattice ions. Comment on the kinetic energy of the conduction electrons, as well as the energy they lose to Joule heating.

## 2 Solution for a Perfectly Conducting Mirror

The topic of radiation pressure seems to have been first considered by Balfour Stewart in 1871 [1] for the case of thermal radiation. Maxwell took up this theme in Arts. 792-793 of his *Treatise* in 1873 [2], arguing on the basis of his stress tensor. The first convincing experimental evidence for the radiation pressure of light was given by Lebedev in 1901 [3]. (The famous Crookes radiometer does not demonstrate electromagnetic radiation pressure.)

We avoid the interesting topic of momentum in dielectric media [4, 5, 6] by supposing that the space outside the mirror is vacuum.

*The quickest solution is given in sec. 2.2.*

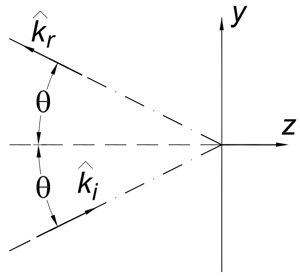
### 2.1 Pressure via the Stress Tensor

According to Maxwell (chap. XI, part IV of his *Treatise*), the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ just outside a surface element $d\mathbf{Area}$ lead to mechanical stresses $\mathsf{T}\cdot\, d\mathbf{Area}$ on that surface, where the stress tensor $\mathsf{T}$ in vacuum is given by,

$$\begin{aligned}\mathsf{T}_{mn} &= \epsilon_0\left(E_mE_n - \frac{\delta_{mn}}{2}E^2\right) + \frac{1}{\mu_0}\left(B_mB_n - \frac{\delta_{mn}}{2}B^2\right) \\ &= \epsilon_0\left(E_mE_n - \frac{\delta_{mn}}{2}E^2 + (cB_m)(cB_n) - \frac{\delta_{mn}}{2}(cB)^2\right). \qquad (1)\end{aligned}$$

See, for example, sec. 8.2.2 of [7].

We consider the case that the mirror is in the plane $z = 0$ and the wave is incident from $z < 0$ with its wave vector $\mathbf{k}_i$ in the $y$-$z$ plane, making angle $\theta$ to the $z$-axis as shown in the figure on the next page.




Then, unit area on the mirror has element d**Area** $= -\hat{\mathbf{z}}$ (with a minus sign since the outward normal from the mirror is in the $-z$-direction), and the time-average radiation pressure $P$ is given by,

$$P = \langle \mathsf{T}_{zn} \rangle \cdot \mathrm{d}\mathbf{Area}_n = -\langle \mathsf{T}_{zz} \rangle = \frac{\epsilon_0}{2}\left(-\left\langle E_z^2 \right\rangle + \left\langle (cB_\parallel)^2 \right\rangle\right), \tag{2}$$

noting that the electric field is normal to, and the magnetic field is tangential to, the surface of a perfect conductor.

### 2.1.1 Polarization Perpendicular to the Plane of Incidence

If the incident electric field $\mathbf{E}_i$ is polarized in the $x$-direction (*i.e.*, perpendicular to the plane of incidence) and has angular frequency $\omega$, the incident fields can be written,

$$\mathbf{E}_i = E_0\, e^{i(\mathbf{k}_i\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{x}} = E_0\, e^{i(k_0\, y\sin\theta + k_0\, z\cos\theta - \omega t)}\,\hat{\mathbf{x}}, \tag{3}$$

$$c\mathbf{B}_i = \hat{\mathbf{k}}_i \times \mathbf{E}_i = E_0\, e^{i(k_0\, y\sin\theta + k_0\, z\cos\theta - \omega t)}(\cos\theta\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}), \tag{4}$$

where, $\mathbf{k}_i = k_0(0, \sin\theta, \cos\theta)$, $k_0 = \omega/c$ and $c$ is the speed of light in vacuum. The reflected field also polarized in this direction, such that the total electric field is zero at the surface of the mirror. That is,

$$\mathbf{E}_r = -E_0\, e^{i(\mathbf{k}_r\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{x}} = -E_0\, e^{i(k_0\, y\sin\theta - k_0\, z\cos\theta - \omega t)}\,\hat{\mathbf{x}}, \tag{5}$$

$$c\mathbf{B}_r = \hat{\mathbf{k}}_r \times \mathbf{E}_r = E_0\, e^{i(k_0\, y\sin\theta - k_0\, z\cos\theta - \omega t)}(\cos\theta\,\hat{\mathbf{y}} + \sin\theta\,\hat{\mathbf{z}}), \tag{6}$$

where, $\mathbf{k}_r = k(0, \sin\theta, -\cos\theta)$.

The magnitude $B$ of the magnetic field at the surface of the mirror is twice that of the tangential component of the incident wave,

$$cB_y = cB_{iy}(z=0) + cB_{ry}(z=0) = 2E_0\cos\theta\cos(k_0\, y\sin\theta - \omega t). \tag{7}$$

The radiation pressure on the mirror follows from eq. (2) as,

$$P = \epsilon_0 E_0^2\cos^2\theta. \tag{8}$$

### 2.1.2 Polarization Parallel to the Plane of Incidence

If the incident electric field is polarized in the $y$-$z$ plane (*i.e.*, parallel to the plane of incidence), then the incident magnetic field is in the $-x$-direction,

$$c\mathbf{B}_i = -E_0\, e^{i(\mathbf{k}_i\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{x}} = -E_0\, e^{i(k_0\, y\sin\theta + k_0\, z\cos\theta - \omega t)}\,\hat{\mathbf{x}}, \tag{9}$$

$$\mathbf{E}_i = -\hat{\mathbf{k}}_i \times c\mathbf{B}_i = E_0\, e^{i(k_0\, y\sin\theta + k_0\, z\cos\theta - \omega t)}(\cos\theta\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}). \tag{10}$$

The reflected magnetic field is also in the $x$-direction, with sign such that the $y$-component of the electric field at $z = 0$ is opposite to that of the incident electric field there. That is,

$$c\mathbf{B}_r = -E_0\, e^{i(\mathbf{k}_r \cdot \mathbf{r} - \omega t)}\, \hat{\mathbf{x}} = -E_0\, e^{i(k_0\, y \sin\theta - k_0\, z \cos\theta - \omega t)}\, \hat{\mathbf{x}}, \tag{11}$$

$$\mathbf{E}_r = -\hat{\mathbf{k}}_r \times c\mathbf{B}_r = -E_0\, e^{i(k_0\, y \sin\theta - k_0\, z \cos\theta - \omega t)}(\cos\theta\, \hat{\mathbf{y}} + \sin\theta\, \hat{\mathbf{z}}). \tag{12}$$

The total electric and magnetic fields at the surface of the mirror have only the nonzero components,

$$E_z = -2E_0 \sin\theta \cos(k_0\, y \sin\theta - \omega t), \qquad cB_x = -2E_0\, \cos(k_0\, y \sin\theta - \omega t), \tag{13}$$

The radiation pressure on the mirror follows from eq. (2) as,

$$P = \epsilon_0 E_0^2 \cos^2\theta, \tag{14}$$

so that the radiation pressure on the mirror is independent of the polarization of the incident wave.[1]

## 2.2 Pressure Calculated from Momentum Density/Flow

A shorter argument can be given by noting that a plane electromagnetic wave has a (time-average) momentum density,

$$\langle \mathbf{p} \rangle = \frac{\langle \mathbf{S} \rangle}{c^2} = \frac{\langle u \rangle}{c}\, \hat{\mathbf{k}} = \frac{\epsilon_0 E_0^2}{2c}\, \hat{\mathbf{k}} \tag{15}$$

where $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0 = c\,\epsilon_0 \mathbf{E} \times c\mathbf{B}$ is the Poynting vector, and $u = \epsilon_0 E^2/2 + B^2/2\mu_0 = \epsilon_0[E^2 + (cB)^2]/2$ is the volume density of electromagnetic field energy.

For a wave with angle of incidence $\theta$, the $z$-component of the momentum impinging on unit area in the plane $z = 0$ per unit time is $cp_z \cos\theta = \epsilon_0(E_0^2/2)\cos^2\theta$.[2] The reflected wave carries away an equal $z$-component of the momentum per unit time. Hence, there is a reaction force per unit area (a pressure) given by,

$$P = \epsilon_0 E_0^2 \cos^2\theta, \tag{16}$$

as found previously. It is clear from this argument that the radiation pressure should not depend on the polarization of the incident wave.

*This argument is delightfully brief, but it is not entirely Maxwellian.*[3] *Rather, it is more in the Newtonian tradition that light consists of particles, and that the pressure due to light*

[1] A subtle effect of polarization dependence in the emissivity of a black body is discussed in [8].

[2] This argument requires knowledge of the flow of momentum, which equals $c$ times the density of momentum for a plane wave in vacuum. In general, the stress tensor is the entity that describes the flow of electromagnetic momentum, so the argument of sec. 2.2 is, strictly, a short-cut version of the argument of sec. 2.1.

[3] A more complete argument would note that the total momentum density is $\mathbf{p} = \mathbf{S}/c^2 = \epsilon_0 \mathbf{E} \times \mathbf{B} = \epsilon_0(\mathbf{E}_i + \mathbf{E}_r) \times (\mathbf{B}_i + \mathbf{B}_r) = \mathbf{p}_i + \mathbf{p}_r + \mathbf{p}_{\rm int}$, where the interaction momentum density $\mathbf{p}_{\rm int} = \epsilon_0(\mathbf{E}_i \times \mathbf{B}_r + \mathbf{E}_r \times \mathbf{B}_i)$ happens to be parallel to the surface of the mirror and does not contribute to the radiation pressure. Because of this, the naïve argument that neglects the interaction momentum succeeds without an awareness of its somewhat fortuitous success. See [9] for further discussion of the interaction momentum density/energy flow.

*can be calculated similarly to the pressure of atoms bouncing off a wall. As such, Maxwell (a founder of the kinetic theory of gases) did not give this argument, preferring a more completely field-based discussion using the stress tensor.*

> Example: The intensity of sunlight at the Earth's surface is approximately $\langle S\rangle = c\,\epsilon_0 E_0^2 \approx 1$ W/m$^2$, where $\langle S\rangle$ is the magnitude of the Poynting vector. A flat mirror oriented normal to the sun's rays experiences a radiation pressure of $P = \epsilon_0 E_0^2 = \langle S\rangle/c \approx 3\times 10^{-9}$ Pa. Since atmospheric pressure is $10^5$ Pa, the radiation pressure of the sunlight is about $3\times 10^{-14}$ atmospheres. For comparison, the pressure $P_{\rm wind}$ on a flat mirror due to a wind normal to its surface is about $\rho_{\rm air} v^2 \approx v^2$ Pa for air speed $v$ in m/s. A gentle breeze of $v \approx 1$ m/s results in a pressure on the mirror about $10^9$ times larger than the radiation pressure due to sunlight.

## 2.3 The Radiation-Reaction Force

A more microscopic argument in the Maxwellian tradition might be to consider the radiation-reaction force on the oscillating charges on the surface of the perfect conductor.

Following Lorentz [10, 11, 12], the radiation-reaction force on an oscillating electric charge $e$ can be written as,[4]

$$\mathbf{F}_{\rm rad} = \frac{\mu_0 e^2}{6\pi c}\dot{\mathbf{a}} = -\frac{2\omega^2 r_e}{3c} m\mathbf{v} = -\omega^2\tau_0 m\mathbf{v}, \tag{17}$$

where $r_e = e^2/4\pi\epsilon_0 mc^2 \approx 3\times 10^{-15}$ m is the classical electron radius, and $\tau_0 = 2r_e/3c \approx 7\times 10^{-24}$ s. However, the time average of this force is zero for charges in the present example, so it does not lead to an explanation of the radiation pressure. The reason for this is that while there are charges on the surface of the mirror that are oscillating/accelerating, these charges do not emit any net energy in the form of radiation. Rather, they absorb as much energy as they emit, and no additional mechanical force is required to keep the charges in motion.

In brief, *there is no radiation reaction (17) because there is no (net) radiation.*[5]

Although the discussion in sec. 2.2 is that the radiation pressure is a reaction to the change of momentum in the fields caused by the mirror, which change can be attributed to the "radiation" by the mirror of the reflected wave, the radiation pressure (8) is unrelated to the "radiation reaction force" described by eq. (17).

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[4]For comments on the history of the radiation reaction, see [13].

[5]It was noted as early as 1904 [14] that the oscillating charges in perfect conductors do not emit any net energy, since the flow of electromagnetic energy, described by the Poynting vector $\mathbf{S} = \mathbf{E}\times\mathbf{B}/\mu_0$, is perpendicular to $\mathbf{E}$, and hence parallel to (and outside of) the surface of a perfect conductor. In particular, this result holds for the (good/perfect) conductors of antennas, which has been described as the "radiation paradox" [15] that while the currents in these conductors determine the radiation pattern, the energy in that pattern does not flow out from these conductors. The concept of radiation pressure offers some insight into this "paradox". Namely, since the electromagnetic fields exert a pressure on the conductors, those conductors exert an equal and opposite pressure on the electromagnetic fields, which affects the structure of the fields close to the conductors, and ultimately far from them as well. Thus, while no net energy flows from the conductors, momentum does flow out from them and is the means by which the conductors influence the fields in the Maxwellian view.

## 2.4 Lorentz Force on the Surface Charges and Currents

For an additional understanding of how the electromagnetic fields next to the mirror result in a force/pressure on the mirror, we consider a different insight of Lorentz, namely his force law, which in the present example can be written as,

$$\mathbf{f} = \frac{\sigma \mathbf{E} + \mathbf{K} \times \mathbf{B}}{2}\,, \tag{18}$$

where $\mathbf{f}$ is the normal force per unit area on the surface, $\sigma$ is the surface charge density, $\mathbf{K}$ is the surface current density, $\mathbf{E}$ and $\mathbf{B}$ are the fields just outside the surface (with $\mathbf{E}$ normal to and $\mathbf{B}$ parallel to the surface), and the factor of 1/2 indicates that the average fields on the charge and current densities are 1/2 of their outside values, since the fields drop to zero as they cross the thin surface layer in which the densities exist.

The surface charge density $\sigma$ can be deduced from the first Maxwell equation, leading to,

$$\sigma = \epsilon_0 \mathbf{E} \cdot \hat{\mathbf{n}}, \tag{19}$$

where $\hat{\mathbf{n}}$ $(= -\hat{\mathbf{z}})$ is the outward unit vector normal to the surface. Similarly, the surface current density $\mathbf{K}$ can be deduced from the fourth Maxwell equation, leading to,

$$\mathbf{K} = \frac{\hat{\mathbf{n}} \times \mathbf{B}}{\mu_0}\,. \tag{20}$$

Thus,

$$\mathbf{f} = \frac{\epsilon_0 (\mathbf{E} \cdot \hat{\mathbf{n}})\mathbf{E}}{2} + \frac{(\hat{\mathbf{n}} \times \mathbf{B}) \times \mathbf{B}}{2\mu_0} = \frac{\epsilon_0}{2}\left(E_z^2 - (cB_\parallel)^2\right)\hat{\mathbf{n}}\,, \tag{21}$$

and time-average pressure on the surface is,

$$P = \langle f_z \rangle = \frac{\epsilon_0}{2}\left(-\left\langle E_z^2\right\rangle + \left\langle (cB_\parallel)^2\right\rangle\right), \tag{22}$$

as found previously via the stress tensor.[6]

*While today we might prefer using the Lorentz force law rather than the stress tensor to deduce eq. (22), the Lorentz force law is based on an understanding of electric charge not available in Maxwell's time. Hence, it is all the more impressive that the concept of radiation pressure was understood before the Lorentz force law was developed.*

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[6]The form (18) gives the initial impression that the Lorentz force is linear in the fields $\mathbf{E}$ and $\mathbf{B}$, and hence amenable to the decomposition $\mathbf{E} = \mathbf{E}_i + \mathbf{E}_r$, $\mathbf{B} = \mathbf{B}_i + \mathbf{B}_r$, but we see in eq. (21) that the force is actually quadratic in the fields. Therefore, the decomposition of the fields in eq. (22) into incident and reflected terms leads to an expression for the radiation pressure of the form $P_i + P_r + P_{\rm int}$, where $P_i = P_r = P/4 = P_{\rm int}/2$. We could formally write $P = \tilde{P}_i + \tilde{P}_r$ where $\tilde{P}_i = P_i + P_{\rm int}/2$ and $\tilde{P}_r = P_r + P_{\rm int}/2$, but the stress tensor and the Lorentz force law provide no physical basis for this identification. We have seen in footnote 2 how a fortuitous aspect of the momentum density vector $\mathbf{p}$ associated with plane waves incident on a flat mirror permits a justification for this procedure.

## 2.5 Flow of Energy

The time-average flow of electromagnetic energy outside the mirror is described by the Poynting vector,

$$\langle \mathbf{S}(z<0) \rangle = \frac{c\,\epsilon_0}{2} Re(\mathbf{E} \times c\mathbf{B}^\star) = 2c\,\epsilon_0 E_0^2 \sin\theta \sin^2(kz\cos\theta)\,\hat{\mathbf{y}}, \tag{23}$$

for both polarizations of the electric field. This corresponds to a (steady) flow of energy in the $y$-direction, parallel to the mirror for any nonzero value of the angle of incidence $\theta$. This flow is modulated in $z$ according to $\sin^2(kz\cos\theta)$.

We could also decompose the total Poynting vector as,

$$\begin{aligned} \langle \mathbf{S} \rangle &= \frac{c\,\epsilon_0}{2} Re(\mathbf{E} \times c\mathbf{B}^\star) = \frac{c\,\epsilon_0}{2} Re[(\mathbf{E}_i + \mathbf{E}_r) \times (c\mathbf{B}_i^\star + c\mathbf{B}_r^\star)] \\ &= \frac{c\,\epsilon_0}{2} Re(\mathbf{E}_i \times c\mathbf{B}_i^\star) + \frac{c\,\epsilon_0}{2} Re(\mathbf{E}_r \times c\mathbf{B}_r^\star) + \frac{c\,\epsilon_0}{2} Re[(\mathbf{E}_i \times c\mathbf{B}_r^\star) + (\mathbf{E}_r \times c\mathbf{B}_i^\star)] \\ &= \langle \mathbf{S}_i \rangle + \langle \mathbf{S}_r \rangle + \langle \mathbf{S}_{\rm int} \rangle\,, \end{aligned} \tag{24}$$

where,

$$\langle \mathbf{S}_i \rangle = \frac{c\,\epsilon_0}{2} E_0^2\,\hat{\mathbf{k}}_i\,, \qquad \langle \mathbf{S}_r \rangle = \frac{c\,\epsilon_0}{2} E_0^2\,\hat{\mathbf{k}}_r\,, \qquad \langle \mathbf{S}_i \rangle + \langle \mathbf{S}_r \rangle = \frac{c\,\epsilon_0}{2} E_0^2 \sin\theta\,\hat{\mathbf{y}}, \tag{25}$$

and the interaction Poynting vector is,

$$\langle \mathbf{S}_{\rm int}(z<0) \rangle = -c\,\epsilon_0 E_0^2 \sin\theta \cos(2kz\cos\theta)\,\hat{\mathbf{y}} = c\,\epsilon_0 E_0^2 \sin\theta [2\sin^2(kz\cos\theta) - 1]\,\hat{\mathbf{y}}, \tag{26}$$

so the total energy flow is again given by eq. (23).

The existence of a nontrivial interaction term (26) in this simple example illustrates that some aspects of the description of energy flow via the Poynting vector are not very intuitive. For example, while the flow of energy in the incident or reflected beams, considered by themselves, has only a positive $y$-component, the direction of the interaction flow (26) oscillates in $z$ with period $\lambda/(2\cos\theta)$. That is, the interaction energy flows in loops that are infinite in $y$ and about $\lambda$ thick in $z$. See [9] for a discussion of these flow loops for the more realistic case of an incident beam of finite transverse extent.

## 2.6 The Mirror as an Antenna

It may be instructive to consider reflection from a mirror a kind of radiation from the mirror induced by the incident wave. In this view, the mirror is an antenna, with known drive fields.

In principle, if the currents in the mirror/antenna can be determined from the drive fields, then the electromagnetic fields due those currents can be calculated, and combined with the drive fields to obtain the total fields. However, the usual issue with this approach is that the currents in the mirror/antenna are not simply due to the drive fields, but are also affected by the fields due to the currents. This effect of the currents on themselves can be expressed mathematically as an integral equation (due to Pocklington [16]) for the currents which takes into account the good-conductor boundary condition at the surface of the mirror/antenna. For an introduction to this integral equation, and its solution, see [17].

As remarked in footnote 3, this approach leads to a good understanding of the fields, and the flow of energy, associated with the mirror/antenna, but the good-conductor boundary condition implies that the flow of energy from the drive fields, as described by the Poynting vector, has flow streamlines that never touch the mirror/antenna. Further, since energy is quadratic in the fields, the separation of the fields into incident and reflected components does not lead to a similar separation of energy flow in incident and reflected components; an interaction flow term is always present that in general does not have a simple physical interpretation.

### 2.6.1 Polarization Parallel to the Plane of Incidence

We will not illustrate the analysis of a mirror as an antenna by use of an integral equation. Rather, we start from knowledge of the surface currents according to eq. (20), based on a complete solution for the reflected fields via the boundary-value problem in sec. 2.1. Then, we pretend not to have the complete solution already, and deduce the fields of the reflected wave from the currents.

If the incident electric field is polarized in the $y$-$z$ plane (*i.e.*, parallel to the plane of incidence), then the incident and reflected magnetic fields are in the $-x$-direction. The magnetic field at the surface of the mirror follows from eqs. (9) and (11) as,

$$\mathbf{B}(z=0) = -\frac{2E_0}{c}\, e^{i(ky\sin\theta - \omega t)}\,\hat{\mathbf{x}}, \tag{27}$$

and the corresponding surface currents are given by,

$$\mathbf{K} = -\frac{\hat{\mathbf{z}} \times \mathbf{B}(z=0)}{\mu_0} = -\frac{2E_0}{\mu_0 c}\, e^{i(ky\sin\theta - \omega t)}\,\hat{\mathbf{y}}, \tag{28}$$

We calculate the retarded vector potential due to the surface currents,[7]

$$\begin{aligned} \mathbf{A}(x,y,z,t) &= \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(x',y',t'=t-R/c)}{R}\,dx'dy' \\ &= -\frac{E_0}{2\pi c}\,e^{-i\omega t}\,\hat{\mathbf{y}}\int \frac{e^{i(ky'\sin\theta + kR)}}{R}\,dx'dy' \\ &= \frac{E_0}{ck\cos\theta}\,e^{i(ky\sin\theta + |k|z\cos\theta - \omega t)}\,\hat{\mathbf{y}}, \end{aligned} \tag{29}$$

where $R = \sqrt{(x-x')^2 + (y-y')^2 + z^2}$. From this we can calculate the reflected field that is "radiated" by the currents,

$$\mathbf{B}_r = \boldsymbol{\nabla} \times \mathbf{A} = \frac{\partial A_y}{\partial z}\,\hat{\mathbf{x}} = \begin{cases} -\frac{E_0}{c}\,e^{i(ky\sin\theta - kz\cos\theta - \omega t)}\,\hat{\mathbf{x}} & (z<0), \\ \frac{E_0}{c}\,e^{i(ky\sin\theta + kz\cos\theta - \omega t)}\,\hat{\mathbf{x}} = -\mathbf{B}_i & (z>0). \end{cases} \tag{30}$$

The reflected electric field is then,

$$\mathbf{E}_r = \begin{cases} -\hat{\mathbf{k}}_r \times c\mathbf{B}_r = -E_0\,e^{i(ky\sin\theta - kz\cos\theta - \omega t)}(\cos\theta\,\hat{\mathbf{y}} + \sin\theta\,\hat{\mathbf{z}}) & (z<0), \\ -\hat{\mathbf{k}}_i \times c\mathbf{B}_r = -\hat{\mathbf{k}}_i \times -c\mathbf{B}_i = -\mathbf{E}_i & (z>0), \end{cases} \tag{31}$$

[7] I don't actually know how to do the integral in eq. (29), so I worked backwards from eq. (30).

noting that for $z > 0$ the reflected wave vector is actually $\mathbf{k}_i$.

Thus, the reflected wave "radiated" into the region $z > 0$ cancels the incident wave there, and the total fields are zero inside the mirror. And, of course, the reflected wave "radiated" into the region $z < 0$ is the same as the reflected wave found by the boundary-value technique.

# 3 Mirror with Large but Finite Conductivity

To comment on the surface charge and current densities $\sigma$ and $\mathbf{K}$ in terms of electrons, we need a model for metals that includes an awareness of their charge $-e$ and mass $m$. Maxwell's equations, by themselves do not include such an awareness, and they were formulated by Maxwell prior to our present understanding of electrical currents as due to the motion of electrons.

## 3.1 Drude's Model of Electrical Conductivity

We follow Drude [18] in making a simple model of the conductivity $\sigma$ of a metal as due to inelastic collisions at frequency $f = 1/\tau$ of the conduction electrons with the lattice of metallic ions. If the effect of a collision is to reset electron's momentum $m\dot{\mathbf{x}}$ to zero, then for frequencies such that $\omega\tau < 1$ this discrete momentum change can be represented by a velocity-dependent friction that acts continually between collisions, and the equation of motion of an electron in an electric field $\mathbf{E} = \mathbf{E}_0\, e^{-i\omega t}$ is approximately,[8]

$$m\ddot{\mathbf{x}} = -e\mathbf{E} - \frac{m\dot{\mathbf{x}}}{\tau}\,, \tag{32}$$

whose solution is,

$$\mathbf{x} = -\frac{ie\tau\mathbf{E}}{m\omega(1 - i\omega\tau)}\,, \qquad \dot{\mathbf{x}} = -\frac{e\tau\mathbf{E}}{m(1 - i\omega\tau)}\,, \tag{33}$$

Then, the current density $\mathbf{J}$ is given by,

$$\mathbf{J} = -Ne\dot{\mathbf{x}} = \frac{Ne^2\tau}{m(1 - i\omega\tau)}\mathbf{E} = \sigma\mathbf{E} \tag{34}$$

where $N$ ($\approx 9 \times 10^{28}/\mathrm{m}^3$ for copper) is the (volume) number density of conduction electrons.[9] The frequency-dependent metallic conductivity $\sigma$ has the form,

$$\sigma = \frac{Ne^2\tau}{m(1 - i\omega\tau)} = \frac{\sigma_0}{1 - i\omega\tau} = \frac{\epsilon_0\omega_p^2\tau}{1 - i\omega\tau}\,, \tag{35}$$

[8]We do not include Lorentz' radiation reaction force (17) in the equation of motion (32) because the conduction electrons do not emit any net radiation. However, if we did include the radiation reaction force $-\omega^2\tau_0 m\dot{\mathbf{x}}$, the effective damping constant $1/\tau + \omega^2\tau_0$ would differ from $1/\tau$ by only a part per million at optical frequencies (and much less than this at rf frequencies). This result tells us that radiation of energy by the conduction electrons is negligible (and, of course, is zero in the limit of a perfect conductor, as discussed in sec. 2.6).

[9]At very high frequencies all atomic electrons participate in the current, and $N$ is total number density of electrons ($\approx 1.2 \times 10^{30}/\mathrm{m}^3$ for copper).

with,

$$\sigma_0 = \frac{Ne^2\tau}{m}, \qquad \text{and} \qquad \omega_p = \sqrt{\frac{Ne^2}{\epsilon_0 m}}, \tag{36}$$

where $\sigma_0$ ($\approx 6 \times 10^7$ mho/m for copper) is the DC conductivity, and $\omega_p$ ($\approx 10^{16}$ s$^{-1}$ for copper) is the plasma frequency. There are three frequency regimes of interest in Drude's model, $\omega \ll 1/\tau$, $1/\tau \lesssim \omega < \omega_p$, and $\omega > \omega_p$. For copper, the characteristic collision time is $\tau = \sigma_0 m/Ne^2 \approx 2 \times 10^{-14}$ s. Thus, for radio frequencies ($\omega \approx 10^9$ s$^{-1}$, say, for which the wavelength is $\lambda = 2\pi c/\omega \approx 2$ m), $\omega\tau \ll 1$ and the Drude-model conductivity is well approximated by its real, DC value. For optical frequencies ($\omega \approx 4 \times 10^{15}$ s$^{-1}$), $\omega\tau > 1$ and Drude's model predicts that the conductivity is essentially pure imaginary, $\sigma_{\rm optical} \approx -i\epsilon_0\omega_p^2/\omega$. Drude's classical electron model of electrical conductivity is less accurate at optical than rf frequencies, and we must turn to a quantum model for better understanding of metallic conductivity in the optical regime. See, for example, secs. 86-87 of [19]. Drude's model is again rather accurate when $\omega \gg \omega_p$, but as we shall see in sec. 3.2, conductors are essentially transparent in this limit.

One significance of the small imaginary part of the conductivity (35) is that it accounts for the power associated with changes in the time-varying kinetic energy of the conduction electrons.[10] The imaginary part of the conductivity leads to a term in the current density $\mathbf{J} = \sigma\mathbf{E}$ that is out of phase with the electric field, and hence part of the power $\mathbf{J}\cdot\mathbf{E}$ that is delivered to the current $\mathbf{J}$ causes no time-averaged change in the energy of the system, as expected for the oscillatory kinetic energy of the conduction electrons.

In greater detail, if we write the electric field at some point inside the conductor as $\mathbf{E}_c\, e^{-i\omega t}$, then the physical electric field is,

$$\mathbf{E} = Re(\mathbf{E}_c\, e^{-i\omega t}) = Re(\mathbf{E}_c)\cos\omega t + Im(\mathbf{E}_c)\sin\omega t, \tag{37}$$

and the physical current density is,

$$\begin{aligned}\mathbf{J} &= Re(\sigma\mathbf{E}) = Re\left[\sigma_0 \frac{1+i\omega\tau}{1+\omega^2\tau^2}\mathbf{E}_c\, e^{-i\omega t}\right] \qquad (38)\\ &= \frac{Ne^2\tau}{m(1+\omega^2\tau^2)}\{[Re(\mathbf{E}_c)\cos\omega t + Im(\mathbf{E}_c)\sin\omega t] - \omega\tau[Re(\mathbf{E}_c)\sin\omega t - Im(\mathbf{E}_c)\cos\omega t]\}.\end{aligned}$$

Then, the physical density of power delivered to the current density is,

$$\begin{aligned}\mathbf{J}\cdot\mathbf{E} &= \frac{\sigma_0}{1+\omega^2\tau^2}\{[Re(\mathbf{E}_c)\cos\omega t + Im(\mathbf{E}_c)\sin\omega t]^2 \qquad (39)\\ &\quad - \frac{Ne^2\omega\tau^2}{m(1+\omega^2\tau^2)}[Re(\mathbf{E}_c)\sin\omega t - Im(\mathbf{E}_c)\cos\omega t]\cdot[Re(\mathbf{E}_c)\cos\omega t + Im(\mathbf{E}_c)\sin\omega t].\end{aligned}$$

The first term of eq. (39) is, of course, the power dissipated by Joule heating. We relate the second term to the time rate of change of the kinetic energy of the conduction electrons,

$$\frac{d}{dt}u_{\rm KE} = \frac{d}{dt}\left(\frac{Nmv^2}{2}\right) = Nm\mathbf{v}\cdot\mathbf{a}, \tag{40}$$

[10]The velocity of a conduction electron has the form $\mathbf{v}_{\rm random} + \mathbf{v}_{\rm drift}$ where $v_{\rm random} \gg v_{\rm drift}$. The total kinetic energy of these electron is $\sum m(\mathbf{v}_{\rm random} + \mathbf{v}_{\rm drift})^2/2 = \sum mv_{\rm random}^2/2 + \sum mv_{\rm drift}^2/2$. The oscillatory part of the kinetic energy is $\sum mv_{\rm drift}^2/2$, the density of which we call $u_{\rm KE}$.

by noting that the velocity of the conduction electrons is $\mathbf{v} = -\mathbf{J}/Ne$, so that their acceleration $\mathbf{a} = d\mathbf{v}/dt$ is,

$$\mathbf{a} = \frac{e\omega\tau}{m(1+\omega^2\tau^2)}\{[Re(\mathbf{E}_c)\sin\omega t - Im(\mathbf{E}_c)\cos\omega t] + \omega\tau[Re(\mathbf{E}_c)\cos\omega t + Im(\mathbf{E}_c)\sin\omega t]\}. \tag{41}$$

Thus, for $\omega\tau \ll 1$, eqs. (38) and (40)-(41) show that the second term of eq. (39) is the time rate of change of the (drift) kinetic energy of the conduction electrons (plus terms of order $\omega^2\tau^2$).[11]

Although Drude's model gives only an approximate understanding of conductors at optical frequencies, it does predict that in this regime the power dissipated by Joule heating is small compared to the power that changes the kinetic energy of the conduction electrons, and so provides some insight as to a microscopic view of very good conductors in which quasi-free electrons are the charge carriers.

## 3.2 Plane Electromagnetic Waves inside a Conductor

The fourth Maxwell equation inside a conductor is, in Drude's model,

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} = \mu_0\sigma\mathbf{E} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\,. \tag{42}$$

Taking the time derivative, using the third Maxwell equation to replace $\mathbf{B}$ by $\mathbf{E}$ and noting that inside a conductor the charge density is negligible, we obtain the wave equation for the electric field,

$$\nabla^2\mathbf{E} = \mu_0\sigma\frac{\partial\mathbf{E}}{\partial t} + \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}\,. \tag{43}$$

For a plane wave of the form $\mathbf{E}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, eqs. (35) and (43) lead to the dispersion relation in a metal,

$$k^2 = \frac{i\omega\mu_0\sigma_0}{1-i\omega\tau} + \frac{\omega^2}{c^2} = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2(1+i/\omega\tau)}\right). \tag{44}$$

At very high frequencies, $\omega \gg \omega_p$, we have that $k \approx \omega/c$ and the conductor is effectively transparent. We will not consider this regime further.

At (radio) frequencies where $\omega\tau \ll 1$, and for "good" conductors, defined as those for which $|\sigma| \gg \epsilon_0\omega$, we have that $k^2 \approx i\mu_0\sigma_0\omega$ and,

$$k_{\rm rf} \approx \frac{1+i}{\delta}\,, \qquad \text{where} \qquad \delta = \sqrt{\frac{2}{\mu_0\sigma_0\omega}}\,, \tag{45}$$

is the skin depth. For $\omega \approx 10^9$ s$^{-1}$, $\delta \approx 5\times10^{-6}$ m, which is small compared to the wavelength $\lambda \approx 2$ m. Radio-frequency waves inside conductors are damped traveling waves.

[11] An implication is that the (drift) kinetic energy of conduction electrons is part of the "electromagnetic field" energy. In AC circuits with negligible capacitance, this field energy is largely "magnetic".

While Maxwell did call the "magnetic" field energy a "kinetic" energy, he did not consider that electric currents involve moving electric charges. See, for example, [20].

At (optical) frequencies where $1/\tau < \omega < \omega_p$, we have that $k^2 \approx -(\omega_p^2/c^2)(1 - i/\omega\tau)$, and,

$$k_{\rm optical} \approx i\frac{\omega_p}{c}\left(1 - \frac{i}{2\omega\tau}\right) = \frac{1}{2\omega\tau\delta'} + \frac{i}{\delta'}\,, \tag{46}$$

where $\delta' = c/\omega_p$ ($\approx 3 \times 10^{-8}$ m for copper) is the effective skin depth (independent of frequency), which is still small compared to a wavelength.

Thus, for $\omega < \omega_p$ the wave number $k$ inside a good conductor always has an imaginary part, and the waves are attenuated as they propagate. In particular, if the waves inside the conductor are excited externally, there can be no propagation of waves inside the conductor towards its surface.

A consequence of the small skin depth is that when a wave propagates into the conductor, which occupies the region $z > 0$, it dies away rapidly with $z$ while its variation in $x$ or $y$ still has periodicity of order the wavelength $\lambda$. The gradient operator then becomes $\boldsymbol{\nabla} \approx \hat{\mathbf{z}}\frac{\partial}{\partial z}$, so that Faraday's law (at angular frequency $\omega$) can be written as,

$$\mathbf{B}_{\rm cond} \approx -i\frac{\hat{\mathbf{z}}}{\omega} \times \frac{\partial \mathbf{E}_{\rm cond}}{\partial z}\,, \tag{47}$$

which implies that the magnetic field has no significant $z$-component (no normal component) inside the conductor. Since the plane waves are still transverse inside the conductor ($\boldsymbol{\nabla}\cdot\mathbf{B} = 0$), the propagation vector can only be in the $z$ direction. That is, the magnetic field inside a conductor must have the form,[12]

$$\mathbf{B}_{\rm cond} \approx \mathbf{B}_\parallel\, e^{i(kz-\omega t)} = \begin{cases} \mathbf{B}_\parallel\, e^{-z/\delta} e^{i(z/\delta - \omega t)} & \text{(rf)}, \\ \mathbf{B}_\parallel\, e^{-z/\delta'} e^{i(z/2\omega\tau\delta' - \omega t)} & \text{(optical)}. \end{cases} \tag{48}$$

Since the tangential component of the magnetic field is continuous across the surface of the conductor, $\mathbf{B}_\parallel$ equals the tangential component of the magnetic field just outside the surface.

Similarly, the fourth Maxwell equation inside the conductor becomes,

$$\mu_0(\sigma - i\epsilon_0\omega)\mathbf{E}_{\rm cond} \approx \mu_0\sigma\mathbf{E}_{\rm cond} \approx \hat{\mathbf{z}} \times \frac{\partial \mathbf{B}_{\rm cond}}{\partial z}\,, \quad i.e., \quad \mathbf{E}_{\rm cond} \approx \frac{ik}{\mu_0\sigma}\,\hat{\mathbf{z}} \times \mathbf{B}_{\rm cond}, \tag{49}$$

using the form (47) for $\mathbf{B}_{\rm cond}$. In particular,

$$\mathbf{E}_{\rm cond} \approx \begin{cases} \frac{1+i}{\mu_0\sigma_0\delta}\,\hat{\mathbf{z}} \times \mathbf{B}_\parallel\, e^{-z/\delta} e^{i(z/\delta-\omega t)} = (1+i)\frac{\pi\delta}{\lambda}\,\hat{\mathbf{z}} \times c\mathbf{B}_\parallel\, e^{-z/\delta} e^{i(z/\delta-\omega t)} & \text{(rf)}, \\ \frac{i\omega\tau}{\mu_0\sigma_0\delta'}\,\hat{\mathbf{z}} \times \mathbf{B}_\parallel\, e^{-z/\delta'} e^{i(z/2\omega\tau\delta'-\omega t)} = \frac{i\omega}{\omega_p}\,\hat{\mathbf{z}} \times c\mathbf{B}_\parallel\, e^{-z/\delta'} e^{i(z/2\omega\tau\delta'-\omega t)} & \text{(optical)}. \end{cases} \tag{50}$$

Thus, $\mathbf{E}_{\rm cond}$ is small in magnitude compared to $c\mathbf{B}_{\rm cond}$ (and out of phase with it by 45° at radio frequencies and by $\approx 90°$ at optical frequencies).

We can now rewrite eqs. (47) and (49) as,

$$\mathbf{B}_{\rm cond} \approx \frac{k}{\omega}\,\hat{\mathbf{z}} \times \mathbf{E}_{\rm cond} \approx \frac{i\mu_0\sigma}{k}\,\hat{\mathbf{z}} \times \mathbf{E}_{\rm cond}. \tag{51}$$

[12]Propagation of waves in the $-z$-direction inside the conductor is possible in principle, but this requires a large energy source at large $z$, which we exclude as unphysical in the present problem. Mirrors involving thin metallic films will have waves propagating in both directions inside the films. See, for example, [21].

### 3.3 Energetics

Since the electric field is small compared to the magnetic field inside a conductor, the stored electromagnetic energy is largely magnetic. The time-average density $\langle u_{\rm EM}\rangle$ of electromagnetic energy is,

$$\langle u_{\rm EM}\rangle \approx \frac{\langle B_{\rm cond}^2\rangle}{2\mu_0} \approx \frac{B_\parallel^2(z=0)e^{-2z/\Delta}}{4\mu_0}\,, \tag{52}$$

where $\Delta = \delta$ at radio frequencies, and $= \delta'$ at optical frequencies.

The electrical currents $\mathbf{J} = \sigma\mathbf{E}$ carry kinetic energy density $Nmv^2/2$, where $v$ is the average speed of the conduction electrons, which is related by $J = Nev$. Hence, the density of mechanical energy in the conductor is,[13]

$$\langle u_{\rm mech}\rangle \approx \frac{m\,|\sigma|^2\,\langle E_{\rm cond}^2\rangle}{2Ne^2} \approx \frac{m\,|k_z|^2\,\langle B_{\rm cond}^2\rangle}{2\mu_0^2 Ne^2} \approx \frac{\epsilon_0 mc^2}{Ne^2\Delta^2}\,\langle u_{\rm EM}\rangle = \frac{1}{4\pi N r_e \Delta^2}\,\langle u_{\rm EM}\rangle\,, \tag{53}$$

where $r_e = e^2/4\pi\epsilon_0 mc^2 \approx 3\times 10^{-15}$ m is the classical electron radius. At radio frequencies, $4\pi N r_e\delta^2 = (4\pi/\mu_0)(2Nr_e/\sigma_0\omega) \approx 10^7\cdot 2\times 10^{29}\cdot 3\times 10^{-15}/(6\times 10^7\cdot 10^9) \approx 10^5$, so that $\langle u_{\rm mech}\rangle \ll \langle u_{\rm EM}\rangle$, but at optical frequencies the two energy densities are comparable in Drude's model, since $4\pi N r_e\delta'^2 \approx 10\cdot 10^{29}\cdot 3\times 10^{-15}\cdot 10^{-15} \approx 3$.[14]

The conduction currents transfer energy to the lattice ions at the time-average rate per unit volume,

$$\frac{Re(\mathbf{J}\cdot\mathbf{E}_{\rm cond}^\star)}{2} = \frac{Re(\sigma\mathbf{E}_{\rm cond}\cdot\mathbf{E}_{\rm cond}^\star)}{2} = Re(\sigma)\frac{|E_{\rm cond}(z)|^2}{2}\,. \tag{54}$$

According to Drude's model, the conductivity (35) is almost purely imaginary at optical frequencies, but we should not neglect the small real part if we wish to model the partial absorption of the wave by the metal.

Since $|E_{\rm cond}(z)|^2 \propto e^{-2\,Im(k)z}$, the total rate of energy absorbed, per unit area, at larger $z$ is,

$$\frac{Re(\sigma)\,|E_{\rm cond}(z)|^2}{4\,Im(k)}\,. \tag{55}$$

The flow $\mathbf{S}$ of electromagnetic energy inside conductor is described by the Poynting vector, whose time average is, recalling eqs. (49) and (51),

$$\begin{aligned}\langle\mathbf{S}\rangle_{\rm cond} = \frac{Re(\mathbf{E}_{\rm cond}\times\mathbf{B}_{\rm cond}^\star)}{2\mu_0} &= Re\left(\frac{-i\sigma^\star}{k^\star}\right)\frac{|E_{\rm cond}|^2}{2}\,\hat{\mathbf{z}} \\ &= Re\left(\frac{-2ik}{\mu_0\sigma}\right)\frac{|B_{\rm cond}|^2}{4\mu_0}\,\hat{\mathbf{z}} = Re\left(\frac{-2ik}{\mu_0\sigma}\right)\langle u_{\rm EM}\rangle\,\hat{\mathbf{z}}. \end{aligned} \tag{56}$$

At radio frequencies, $k = (1+i)/\delta$, while $\sigma = \sigma_0$ is real, so $Re(-i\sigma^\star/k^\star) = Re(\sigma)/2\,Im(k)$, and energy transported to unit area at coordinate $z$ equals the rate (55) of absorption

[13] An argument which avoids detailed discussion of the currents and fields inside the conductor is that surface currents of magnitude $K = J\Delta = Nev\Delta = B_{0c}/\mu_0$ exist in a layer of thickness $\Delta$. The density of mechanical energy of the electrons of mass $m$ is $u_{\rm mech} = Nmv^2/2 = mB_{0c}^2/2\mu_0^2Ne^2\Delta^2 = (B_{0c}^2/2\mu_0)(1/N\Delta^2)(\epsilon_0 mc^2/e^2) = u_{\rm EM}/4\pi N r_e\Delta^2$.

[14] See [20] for a discussion of the ratio of mechanical kinetic energy to magnetic energy in examples with DC currents.

of energy at all larger $z$. At optical frequencies, $\sigma \approx Re(\sigma)(1 + i\omega\tau)$ from eq. (35) and $k \approx i\, Im(k)(1 - i/2\omega\tau)$ from eq. (46), so again $Re(-i\sigma^\star/k^\star) = Re(\sigma)/2\, Im(k)$.

The second line of eq. (56) permits us to identify the speed of the flow of electromagnetic energy inside the conductor as,

$$v_{\rm energy} = Re\left(\frac{-2ik}{\mu_0\sigma}\right) = \begin{cases} \frac{2\pi\delta}{\lambda}\, c & \text{(rf)}, \\ \frac{1}{\omega_p\tau}\, c & \text{(optical)}, \end{cases} \tag{57}$$

which is small compared to the speed of light in both frequency regimes.

The phase and group velocities, $v_p$ and $v_g$, of the waves inside the conductor are,

$$\begin{aligned} v_p = \frac{\omega}{Re(k)} \quad &= \begin{cases} \frac{2\pi\delta}{\lambda}\, c & \text{(rf)}, \\ 2\omega\tau\frac{\omega}{\omega_p}\, c & \text{(optical)}, \end{cases} \\ v_g = \frac{d\omega}{d[Re(k)]} = \frac{1}{d[Re(k)]/d\omega} \quad &= \begin{cases} \infty & \text{(rf)}, \\ -v_p & \text{(optical)}. \end{cases} \end{aligned} \tag{58}$$

We see that the formal group velocity does not have the significance of the velocity of energy flow inside conductors.

## 3.4 Detailed Solution of the Boundary-Value Problem

For completeness, we consider metallic reflection as a boundary-value problem, restricting the frequency to the realm where $\omega\tau \ll 1$, such that the conductivity is purely real ($\sigma = \sigma_0 \gg \epsilon_0\omega$).

### 3.4.1 Polarization Parallel to the Plane of Incidence

The magnetic field has only an $x$-component when the electric field is polarized in the plane of incidence, again taking this be the $y$-$z$ plane. We anticipate that the angle of reflection equals the angle of incidence, so the magnetic field in vacuum ($z < 0$) has only the component,

$$cB_x = -E_0\, e^{i(k_0 y\sin\theta + k_0 z\cos\theta - \omega t)} + E_{0r}\, e^{i(k_0 y\sin\theta - k_0 z\cos\theta - \omega t)}, \tag{59}$$

where $k_0 = \omega/c$, and the amplitude of the reflected wave is not necessarily minus that of the incident wave. The magnetic field inside the conductor ($z > 0$) has only the component,

$$B_{cx} = B_{0c}\, e^{i(k_y y + k_z z - \omega t)} \tag{60}$$

where $k_y^2 + k_z^2 = k^2$. For finite conductivity, $B_x$ is continuous at the surface $z = 0$, which implies that,

$$k_y = k_0\sin\theta, \qquad k_z^2 = k^2 - k_0^2\sin^2\theta, \tag{61}$$

and that,

$$cB_{0c} = E_{0r} - E_0. \tag{62}$$

For large conductivity $\sigma_0$, $|k| \gg k_0$ so that $k_z$ is independent of the angle of incidence $\theta$,

$$k_z \approx k \approx \frac{1+i}{\delta}, \qquad k_z^2 \approx \frac{2i}{\delta^2} = i\mu_0\sigma_0 k_0 c. \tag{63}$$

Recalling sec. 2.1.2, the electric field in vacuum has the form,

$$\begin{aligned}
\mathbf{E} &= (E_0\, e^{ik_0 z\cos\theta} + E_{0r}\, e^{-ik_0 z\cos\theta})\, e^{i(k_0 y\sin\theta - \omega t)} \cos\theta\, \hat{\mathbf{y}} \\
&\quad -(E_0\, e^{ik_0 z\cos\theta} - E_{0r}\, e^{-ik_0 z\cos\theta})\, e^{i(k_0 y\sin\theta - \omega t)} \sin\theta\, \hat{\mathbf{z}}.
\end{aligned} \tag{64}$$

The fourth Maxwell equation inside the conductor tells us that,

$$\begin{aligned}
\mu_0(\sigma_0 - i\epsilon_0\omega)\mathbf{E}_c \approx \mu_0\sigma_0\mathbf{E}_c &= \boldsymbol{\nabla}\times\mathbf{B}_c = \frac{\partial B_{cx}}{\partial z}\,\hat{\mathbf{y}} - \frac{\partial B_{cx}}{\partial y}\,\hat{\mathbf{z}} \\
&= ik_z B_{0c}\, e^{i(k_0 y\sin\theta + k_z z - \omega t)} \left(\hat{\mathbf{y}} - \frac{k_0}{k_z}\sin\theta\,\hat{\mathbf{z}}\right).
\end{aligned} \tag{65}$$

Continuity of the tangential component of the electric field at the surface of the conductor implies that,

$$(E_0 + E_{0r})\cos\theta = \frac{ik_z}{\mu_0(\sigma_0 - i\epsilon_0\omega)} B_{0c} \approx \frac{ik_z}{\mu_0\sigma_0} B_{0c} = -\frac{k_0}{k_z} c B_{0c}, \tag{66}$$

where the approximation holds for “good” conductors.

Combining eqs. (62) and (66), we find that,

$$E_{0r} \approx -E_0\left(1 - \frac{2k_0}{k_z\cos\theta}\right). \tag{67}$$

At most angles of incidence, $E_{0r} \approx -E_0$, as holds in the limit of perfect conductivity, but at grazing incidence ($\theta \approx 90°$), the reflected electric field can depart considerably from the case on an ideal mirror.

The electric field in vacuum is,

$$\begin{aligned}
\mathbf{E} &\approx 2E_0\left(i\sin(k_0 z\cos\theta) + \frac{k_0}{k_z\cos\theta}\, e^{-ik_0 z\cos\theta}\right) e^{i(k_0 y\sin\theta - \omega t)} \cos\theta\,\hat{\mathbf{y}} \\
&\quad -2E_0\left(\cos(k_0 z\cos\theta) - \frac{k_0}{k_z\cos\theta}\, e^{-ik_0 z\cos\theta}\right) e^{i(k_0 y\sin\theta - \omega t)} \sin\theta\,\hat{\mathbf{z}}.
\end{aligned} \tag{68}$$

The electric field in vacuum next to the surface of the conductor now includes a small tangential component (as needed for the Poynting vector to have a component normal to the conductor to supply the energy lost to Joule heating),

$$\begin{aligned}
\mathbf{E}(z=0) &= \frac{2k_0 E_0}{k_z} e^{i(k_0 y\sin\theta - \omega t)}\,\hat{\mathbf{y}}. \\
&\quad -2E_0\left(1 - \frac{k_0}{k_z\cos\theta}\right) e^{i(k_0 y\sin\theta - \omega t)} \sin\theta\,\hat{\mathbf{z}}.
\end{aligned} \tag{69}$$

The ratio $W$ of the tangential to normal components of the electric field in vacuum at the surface of the conductor is,

$$W \approx -\frac{k_0}{k_z \sin\theta} = -\frac{\pi(1-i)\,\delta}{\lambda \sin\theta}\,, \tag{70}$$

which ratio is sometimes called the wave tilt [21, 22].

From eqs. (62) and (67) we now have that,

$$cB_{0c} \approx -2E_0 \left(1 + \frac{k_0}{k_z \cos\theta}\right) \approx -2E_0. \tag{71}$$

The fields inside the conductor are then, recalling eqs. (60) and (65),

$$\mathbf{E}_c \approx \frac{2k_0 E_0}{k_z}\, e^{-z/\delta} e^{i(k_0 y \sin\theta + z/\delta - \omega t)} \left(\hat{\mathbf{y}} - \frac{k_0}{k_z}\sin\theta\,\hat{\mathbf{z}}\right), \tag{72}$$

$$c\mathbf{B}_c \approx -2E_0\, e^{-z/\delta}\, e^{i(k_0 y \sin\theta + z/\delta - \omega t)}\,\hat{\mathbf{x}}. \tag{73}$$

The damped, transmitted wave makes a small angle $\theta_c$ to the $z$-axis,

$$\theta_c \approx k_0 \delta \sin\theta = \frac{2\pi\delta}{\lambda}\sin\theta. \tag{74}$$

The phase velocity of the wave inside the conductor is,

$$\mathbf{v}_p = \frac{2\pi\delta}{\lambda} c \left(\hat{\mathbf{z}} + \frac{2\pi}{\lambda}\sin\theta\,\hat{\mathbf{y}}\right), \tag{75}$$

whose magnitude $v_p \approx (2\pi\delta/\lambda)c$ is very small compared to the speed of light in vacuum.

The time-average Poynting vector in the conductor is entirely in the $+z$-direction,

$$\langle \mathbf{S} \rangle = \frac{c\,\epsilon_0}{2} Re(\mathbf{E}_c \times c\mathbf{B}_c^\star) \approx \frac{2\pi\delta}{\lambda} c\,\epsilon_0 E_0^2\, e^{-2z/\delta}\,\hat{\mathbf{z}}. \tag{76}$$

#### 3.4.2 Polarization Perpendicular to the Plane of Incidence

For completeness, we also consider the case where the electric field is polarized in the $x$-direction (*i.e.*, perpendicular to the plane of incidence).

We anticipate that the angle of reflection equals the angle of incidence, so the electric field in vacuum ($z < 0$) has only the component,

$$E_x = E_0\, e^{i(k_0 y \sin\theta + k_0 z \cos\theta - \omega t)} + E_{0r}\, e^{i(k_0 y \sin\theta - k_0 z \cos\theta - \omega t)}. \tag{77}$$

The electric field inside the conductor ($z > 0$) has only the component,

$$E_{cx} = E_{0c}\, e^{i(k_y y + k_z z - \omega t)} \tag{78}$$

where $k_y^2 + k_z^2 = k^2$. For finite conductivity, $E_x$ is continuous at the surface $z = 0$, which implies that,

$$k_y = k_0 \sin\theta, \qquad k_z^2 = k^2 - k_0^2 \sin^2\theta \approx k^2. \tag{79}$$

and that,

$$E_{0c} = E_0 + E_{0r}. \tag{80}$$

Recalling sec. 2.1.1, the magnetic field in vacuum has the form,

$$\begin{aligned} c\mathbf{B} &= (E_0\, e^{ik_0 z\cos\theta} - E_{0r}\, e^{-ik_0 z\cos\theta})\, e^{i(k_0 y\sin\theta - \omega t)} \cos\theta\, \hat{\mathbf{y}} \\ &\quad -(E_0\, e^{ik_0 z\cos\theta} + E_{0r}\, e^{-ik_0 z\cos\theta})\, e^{i(k_0 y\sin\theta - \omega t)} \sin\theta\, \hat{\mathbf{z}}. \end{aligned} \tag{81}$$

Faraday's law inside the conductor tells us that,

$$\begin{aligned} -i\omega\mathbf{B}_c = -ik_0 c\mathbf{B}_c &= \boldsymbol{\nabla}\times\mathbf{E}_c = \frac{\partial E_{cx}}{\partial z}\,\hat{\mathbf{y}} - \frac{\partial E_{cx}}{\partial y}\,\hat{\mathbf{z}} \\ &= ik_z E_{0c}\, e^{i(k_0 y\sin\theta + k_z z - \omega t)} \left(\hat{\mathbf{y}} - \frac{k_0}{k_z}\sin\theta\,\hat{\mathbf{z}}\right). \end{aligned} \tag{82}$$

Continuity of the tangential component of the magnetic field at the surface of the conductor implies that,

$$(E_0 - E_{0r})\cos\theta = -\frac{k_z}{k_0} E_{0c}. \tag{83}$$

Combining eqs. (80) and (83), we find that,

$$E_{0r} \approx -E_0 \left(1 - \frac{2k_0\cos\theta}{k_z}\right). \tag{84}$$

At all angles of incidence, $E_{0r} \approx -E_0$, as holds in the limit of perfect conductivity, so this case is simpler than that of polarization parallel to the plane of incidence.

The electromagnetic fields in vacuum are,

$$\mathbf{E} \approx 2E_0 \left(i\sin(k_0 z\cos\theta) + \frac{k_0\cos\theta}{k_z}\, e^{-ik_0 z\cos\theta}\right) e^{i(k_0 y\sin\theta - \omega t)}\,\hat{\mathbf{x}}, \tag{85}$$

$$\begin{aligned} c\mathbf{B} &\approx 2E_0 \left(\cos(k_0 z\cos\theta) - \frac{k_0\cos\theta}{k_z}\, e^{-ik_0 z\cos\theta}\right) e^{i(k_0 y\sin\theta - \omega t)} \cos\theta\,\hat{\mathbf{y}} \\ &\quad -2E_0 \left(i\sin(k_0 z\cos\theta) + \frac{k_0\cos\theta}{k_z}\, e^{-ik_0 z\cos\theta}\right) e^{i(k_0 y\sin\theta - \omega t)} \sin\theta\,\hat{\mathbf{z}}. \end{aligned} \tag{86}$$

The electric field in vacuum next to the surface of the conductor again includes a small tangential component (as needed for the Poynting vector to have a component normal to the conductor to supply the energy lost to Joule heating),

$$\mathbf{E}(z=0) = \frac{2k_0\cos^2\theta\, E_0}{k_z} e^{i(k_0 y\sin\theta - \omega t)}\,\hat{\mathbf{x}}. \tag{87}$$

The ratio $W$ of the normal to tangential components of the magnetic field in vacuum at the surface of the conductor is,

$$W \approx -\frac{k_0\sin\theta}{k_z} = -\frac{\pi(1-i)\,\delta\sin\theta}{\lambda}, \tag{88}$$

which ratio could be called the wave tilt, but apparently is not.

From eqs. (80) and (84) we now have that,

$$E_{0c} \approx \frac{2k_0 \cos\theta E_0}{k_z} \,. \tag{89}$$

The fields inside the conductor are then, recalling eqs. (78) and (82),

$$\mathbf{E}_c \approx \frac{2k_0 \cos\theta E_0}{k_z} \, e^{-z/\delta} e^{i(k_0 y \sin\theta + z/\delta - \omega t)} \, \hat{\mathbf{x}}, \tag{90}$$

$$c\mathbf{B}_c \approx -2\cos\theta E_0 \, e^{-z/\delta} e^{i(k_0 y \sin\theta + z/\delta - \omega t)} \left( \hat{\mathbf{y}} - \frac{k_0}{k_z} \sin\theta \, \hat{\mathbf{z}} \right) . \tag{91}$$

The time-average Poynting vector in the conductor is entirely in the $+z$-direction,

$$\langle \mathbf{S} \rangle = \frac{c\,\epsilon_0}{2} Re(\mathbf{E}_c \times c\mathbf{B}_c^\star) \approx \frac{2\pi\delta}{\lambda} c\,\epsilon_0 \cos^2\theta E_0^2 \, e^{-2z/\delta} \, \hat{\mathbf{z}}, \tag{92}$$

and vanishes as the angle of incidence approaches 90°. This result reinforces the previous impression that the case of polarization perpendicular to the plane of incidence on a conductive mirror is less interesting than the case of parallel polarization.

# A Appendix: Does a Free Electron in a Plane Electromagnetic Wave Experience a Radiation Pressure?

That the answer to the above question is `no` has been reviewed in [23].

However, this issue is somewhat subtle in that if a plane electromagnetic wave overtakes an initially free electron, the latter takes on a constant drift velocity in the direction of propagation of the wave, as first noted by McMillan [24].[15] See also [26], and references therein. During the interval in which the strength of the wave at the position of the charge rises from zero to its steady-state value, the electron experiences a transient force in the direction of propagation that could be characterized as a "radiation pressure".[16]

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[This meeting featured an inspirational address by W. Thomson (later Lord Kelvin) as a memorial to Herschel; among many other topics Thomson speculates on the size of atoms, on the origin of life on Earth as due to primitive organisms arriving in meteorites, and on how the Sun's source of energy cannot be an influx of matter as might, however,


[15] This result is implicit, but not explicit, in the earlier discussion by Landau [25].

[16] When the plane wave passes beyond the electron, the latter receives an impulse that restores its velocity to zero.

explain the small advance of the perihelion of Mercury measured by LeVerrier.]; *On Æthereal Friction*, Brit. Assoc. Reports, 43rd Meeting, Notes and Abstracts, pp. 32-35 (1873), http://kirkmcd.princeton.edu/examples/EM/stewart_bar_43_32_73.pdf
Stewart argued that the radiation resistance felt by a charge moving through blackbody radiation should vanish as the temperature of the bath went to zero, just as he expected the electrical resistance of a conductor to vanish at zero temperature. [The 43rd meeting was also the occasion of a report by Maxwell on the exponential atmosphere as an example of statistical mechanics (pp. 29-32), by Rayleigh on the diffraction limit to the sharpness of spectral lines (p. 39), and perhaps of greatest significance to the attendees, a note by A.H. Allen on the detection of adulteration of tea (p. 62).]

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