The Tennis Racquet Theorem

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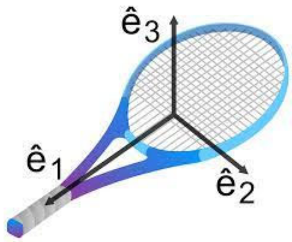
1 Problem

In 1834, Poinsot [1] considered the torque-free motion of a rigid body whose principal moments of inertia are $I_1 < I_2 < I_3$, and deduced what is now known as the tennis-racquet theorem, that motion with the angular velocity $\boldsymbol{\omega}$ initially near principal (body) axis 2 is "unstable".[1] This disconcerting behavior is the subject of many YouTube videos, such as
https://www.youtube.com/watch?v=1VPfZ_XzisU https://www.youtube.com/watch?v=L2o9eBl_Gzw



Examine the special case where the kinetic energy has the form $T = L^2/2I_2$, and $\mathbf{L}$ is the angular momentum about the center of mass. Use expressions for T and $\mathbf{L}$ to show that,

$$\omega_1^2 = \frac{I_3 - I_2}{I_3 - I_1}\frac{L^2 - I_2^2\omega_2^2}{I_1 I_2}\,, \qquad \omega_3^2 = \frac{I_2 - I_1}{I_3 - I_1}\frac{L^2 - I_2^2\omega_2^2}{I_2 I_3}\,, \tag{1}$$

and that Euler's equations lead to,

$$\omega_1 = \omega_{1,\max}\,\mathrm{sech}[k\,\omega_{2,\max}\,(t - t_0)], \tag{2}$$

$$\omega_2 = \omega_{2,\max}\tanh[k\,\omega_{2,\max}\,(t - t_0)], \qquad \omega_{2,\max} = \frac{L}{I_2}\,, \qquad k = \sqrt{\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}}\,, \tag{3}$$

$$\omega_3 = \omega_{3,\max}\,\mathrm{sech}[k\,\omega_{2,\max}\,(t - t_0)]. \tag{4}$$

As $t \to \infty$, $\omega_1,\ \omega_3 \to 0$, while $\omega_2 \to \omega_{2,\max}$, and the final rotation is about axis 2. Thus, for this special case. a kind of stability occurs.

Relate this special case to the general behavior when $T \neq L^2/2I_2$.

2 Solution

In general, for principal (body) axes $\hat{\mathbf{1}}$, $\hat{\mathbf{2}}$ and $\hat{\mathbf{3}}$ we have,

$$T = \frac{I_1\,\omega_1^2}{2} + \frac{I_2\,\omega_2^2}{2} + \frac{I_3\,\omega_3^2}{2}\,, \tag{5}$$

$$\mathbf{L} = I_1\omega_1\,\hat{\mathbf{1}} + I_2\omega_2\,\hat{\mathbf{2}} + I_3\omega_3\,\hat{\mathbf{3}}, \qquad L^2 = I_1^2\,\omega_1^2 + I_2^2\,\omega_2^2 + I_3^2\,\omega_3^2. \tag{6}$$

[1] See, for example, sec. 37 of [2].

For the special case considered here, $2I_2T = I_1I_2\,\omega_1^2 + I_2^2\,\omega_2^2 + I_2I_3\,\omega_3^2 = L^2$. From eq. (6), we have $I_3^2\,\omega_3^2 = L^2 - I_1^2\,\omega_1^2 - I_2^2\,\omega_2^2$, which leads to,

$$2I_2I_3T = I_1I_2I_3\,\omega_1^2 + I_2^2I_2\,\omega_2^2 + I_2(L^2 - I_1^2\,\omega_1^2 - I_2^2\,\omega_2^2) = I_3L^2, \tag{7}$$

$$\omega_1^2 I_1I_2(I_3 - I_2) = (I_3 - I_2)(L^2 - I_2^2\,\omega_2^2), \tag{8}$$

$$\omega_1^2 = \frac{I_3 - I_2}{I_3 - I_1}\frac{L^2 - I_2^2\omega_2^2}{I_1I_2}, \qquad \text{and with } 1 \leftrightarrow 3, \qquad \omega_3^2 = \frac{I_2 - I_1}{I_3 - I_1}\frac{L^2 - I_2^2\omega_2^2}{I_2I_3}. \tag{9}$$

Then, Euler's equation for $\dot\omega_2$ in torque-free motion[2] leads to,

$$\dot\omega_2 = -\frac{I_1 - I_3}{I_2}\omega_1\omega_3 = -\frac{I_1 - I_3}{I_2}\sqrt{\frac{I_3 - I_2}{I_3 - I_1}\frac{I_2 - I_1}{I_3 - I_1}}\frac{L^2 - I_2^2\omega_2^2}{I_2\sqrt{I_1I_3}}$$

$$= \sqrt{\frac{(I_3 - I_2)(I_2 - I_1)}{I_1I_3}}\left(\frac{L^2}{I_2^2} - \omega_2^2\right), \tag{10}$$

$$\frac{d\,\omega_2}{\omega_{2,\max}^2 - \omega_2^2} = k\,dt, \qquad \text{with} \qquad k = \sqrt{\frac{(I_3 - I_2)(I_2 - I_1)}{I_1I_3}}, \qquad \omega_{2,\max} = \frac{L}{I_2}, \tag{11}$$

$$\frac{1}{\omega_{2,\max}}\tanh^{-1}\frac{\omega_2}{\omega_{2,\max}} = k(t - t_0), \qquad \omega_2 = \omega_{2,\max}\tanh[k\,\omega_{2,\max}\,(t - t_0)], \tag{12}$$

using Dwight 140.1, http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf.

Then, from eq. (9),

$$\omega_1^2 = \frac{I_3 - I_2}{I_3 - I_1}\frac{I_2}{I_1}(\omega_{2,\max}^2 - \omega_2^2) = \frac{I_3 - I_2}{I_3 - I_1}\frac{I_2}{I_1}\omega_{2,\max}^2\,\mathrm{sech}^2[k\,\omega_{2,\max}\,(t - t_0)], \tag{13}$$

$$\omega_1 = \omega_{2,\max}\sqrt{\frac{I_2}{I_1}\frac{I_3 - I_2}{I_3 - I_1}}\mathrm{sech}[k\,\omega_{2,\max}\,(t - t_0)] = \omega_{1,\max}\,\mathrm{sech}[k\,\omega_{2,\max}\,(t - t_0)]. \tag{14}$$

And, exchanging indices 1 and 3,

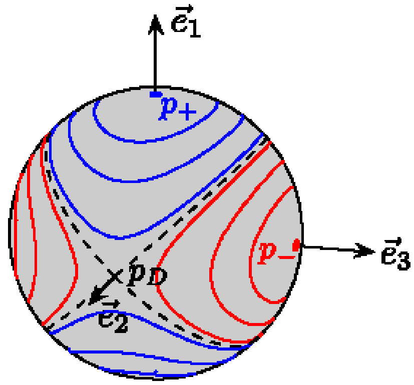
$$\omega_3 = \omega_{2,\max}\sqrt{\frac{I_2}{I_3}\frac{I_2 - I_1}{I_3 - I_1}}\mathrm{sech}[k\,\omega_{2,\max}\,(t - t_0)] = \omega_{3,\max}\,\mathrm{sech}[k\,\omega_{2,\max}\,(t - t_0)]. \tag{15}$$

As $t \to \infty$, $\omega_1,\ \omega_3 \to 0$, while $\omega_2 \to \omega_{2,\max}$, and the final rotation is about axis 2. Thus, for the special case of $T = L^2/2I_2$, a kind of stability occurs.

3 Comments

Poinsot [1] noted that it is useful to consider the path (polhode) of the angular velocity vector $\boldsymbol{\omega}$ on the inertia ellipsoid, which is the surface defined by eq. (5) for constant kinetic energy T in torque-free, rigid-body motion. The figure below (from [3]) illustrates various possible polhodes.

[2] See, for example, eq. (36.4) of [2].



When the angular velocity vector $\boldsymbol{\omega}$ is close to principal axes $\pm\hat{\mathbf{1}}$ or $\pm\hat{\mathbf{3}}$, the polhodes are nearly circular, and an observer readily characterizes the motion of the spinning free body as "stable". But for polhodes far from these axes, the angular velocity $\boldsymbol{\omega}$ passes close to both $\hat{\mathbf{2}}$ and $-\hat{\mathbf{2}}$ during each cycle of the (mathematically stable) motion. To an observer, this behavior seems rather disconcerting, and it is commonly called "unstable".

The special case considered here, $T = L^2/2I_2$, corresponds to motion along the "separating polhodes", shown as dashed curves in the above figure. We have seen that it takes an infinite time for $\boldsymbol{\omega}$ to move from its initial direction to alignment with either $\hat{\mathbf{2}}$ or $-\hat{\mathbf{2}}$, so the motion never completes a full cycle around the separating polhodes. As remarked above, this behavior has a kind of "stability".

We infer from this problem that the cycle time for a trajectory very close to the separating polhodes is very long, approaching infinity in the limit considered here. The long period of such cycles contributes to the impression by the "casual" observer that the motion is "unstable".[3]

References

[1] L. Poinsot, *Théorie Nouvelle de la Rotation des Corps* (Bachlier, 1851),
http://kirkmcd.princeton.edu/examples/mechanics/poinsot_motion_34.pdf
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[2] L.D. Landau and E.M. Lifshitz, *Mechanics*, 3rd ed. (Pergamon, 1976),
http://kirkmcd.princeton.edu/examples/mechanics/landau_mechanics_76.pdf

[3] L. van Damme, P. Mardešic and D, Sugny, *The tennis racket effect in a three-dimensional rigid body*, Physica D **338**, 17 (2017),
http://kirkmcd.princeton.edu/examples/mechanics/vandamme_physica_d338_17_17.pdf

[4] C. Peterson and W. Schwalm, *Euler's rigid rotators, Jacobi elliptic functions, and the Dzhanibekov or tennis racket effect*, Am. J. Phys. **89**, 349 (2021),
http://kirkmcd.princeton.edu/examples/mechanics/peterson_ajp_89_349_21.pdf

[3] A detailed calculation of the cycle times via elliptic functions is given, for example, in [3, 4].