The Radiation-Reaction Force and the Radiation Resistance of Small Antennas

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1 Problem

Discussions of the radiation-reaction force are typically in the context of a single, free (*i.e.*, not bound in an atom) electric charge that accelerates due to an external force, where part of the external force is identified as the means of transmission to the charge of the power that is radiated. This identification is accomplished via an invocation of Newton's 3rd law, and so the force in question is called a reaction force.

In technological practice, the conduction electrons in metals is the most common example of "free" charges, and electromagnetic radiation of "free" charges is most often realized using antennas. The latter are conducting structures with time-dependent (typically, oscillating) currents whose geometry permits constructive interference among the radiation of the individual accelerating charges.[1]

In antenna theory, the requirement that the source of the oscillating currents provide the radiated power is summarized by assigning a radiation resistance $R_{\rm rad}$ to the antenna, as if the antenna were a two-terminal device with a characteristic impedance,

$$Z = R_{\rm Ohmic} + R_{\rm rad} + iX_{\rm antenna}, \tag{1}$$

where $R_{\rm Ohmic}$ is the ordinary resistance of the conductors of the antenna as measured at its terminals, and $X_{\rm antenna}$ is the (capacitive and inductive) reactance of the antenna. That is, if an oscillating voltage $V = V_0 e^{-i\omega t}$ of angular frequency ω were applied across the terminals of the antenna (by an appropriate transmission line from the power source), the resulting current (at the terminals) would be $I = I_0 e^{-i\omega t} = V/Z$. The total time-average power "dissipated" by the antenna is then,

$$P_{\rm total} = \frac{1}{2} Re(VI^\star) = \frac{1}{2} Re(II^\star Z) = \frac{I_0^2}{2}\left(R_{\rm Ohmic} + R_{\rm rad}\right) = P_{\rm Ohmic} + P_{\rm rad}. \tag{2}$$

Thus, the radiation resistance of an antenna is given in terms of its time-average radiated power and the peak current at its terminals by,

$$P_{\rm rad} = \frac{I_0^2}{2} R_{\rm rad}. \tag{3}$$

Discuss the relation between the radiation-reaction force on single charges and the radiation resistance of an antenna, using as examples a small, linear, center-fed dipole antenna of length L $\lambda = 2\pi c/\omega$, where c is the speed of light, and also a small circular loop antenna of

[1] Time-independent currents that flow in loops consist of accelerating charges whose radiation fields are totally canceled by destructive interference. See prob. 9 of [1] for further discussion of this point.

circumference $L \ll \lambda$. Note that the small loop emits magnetic-dipole radiation, so that the radiation-reaction forces in this case are not the same as those based on the usual assumption of electric-dipole radiation.

In this problem you may take advantage of the fact that the radiation from a small antenna is well described by an approximation that ignores its conductivity and simply supposes a geometric pattern of the oscillating current which is completely in phase with the drive voltage.

2 Solution

The apparent success of the solutions given below for small antennas is surprising in view of the "radiation paradox" [2, 3],[2] that for good conductors the electric field can have little/no component parallel to the surface of the conductors, and consequently the Poynting vector [5] can have little/no component perpendicular to the conductors. Thus, in the usual accounting of flow of electromagnetic energy, no energy leaves the conductors normal to their surfaces (although some energy enters the conductors normally to sustain the Ohmic losses within). In the Maxwellian view, an antenna is merely a special kind of wave guide that shapes the angular distribution of power that is transmitted from a source (which must include some material other than good conductors) into the far zone.

2.1 The Radiation-Reaction Force on a Single Charge

When an external force $\mathbf{F}_{\rm ext}$ is applied to a mass m, that mass accelerates according to Newton's 2nd law,

$$\mathbf{F}_{\rm ext} = m\mathbf{a} = m\dot{\mathbf{v}}, \tag{4}$$

where c is the speed of light in vacuum, and we assume that the magnitude of the velocity $\mathbf{v}$ obeys $v \ll c$ throughout this problem. As a result, the kinetic energy $mv^2/2$ of the particle changes, and we have,

$$\mathbf{F}_{\rm ext} \cdot \mathbf{v} = m\mathbf{v} \cdot \dot{\mathbf{v}} = \frac{d}{dt}\left(\frac{mv^2}{2}\right). \tag{5}$$

That is, the power supplied by the external force goes into increasing the kinetic energy of the mass.

If, however, the mass m carries electric charge e, then the accelerated charge radiates power at the rate,

$$P_{\rm rad} = \frac{2e^2a^2}{3c^3}, \tag{6}$$

according to the Larmor formula [6]. Assuming that the mass has no internal energy,[3] the radiated power must come from the source that provides the external force. Thus, we are

[2]For further discussion of the radiation paradox, see [4].

[3]This assumption is not valid in general, because the (near) electromagnetic field of the charge stores energy. For periodic motion, the time average of this stored energy is constant, and the radiated energy must come from elsewhere. However, the interpretation of the radiation reaction for nonperiodic motion, beginning with Schott [7], is that part of the radiated energy comes from external sources, and part comes from rearrangement of the electromagnetic energy associated with the charge.

led to modify eq. (5) in the case of radiation to be,

$$\mathbf{F}_{\rm ext} \cdot \mathbf{v} = \frac{d}{dt}\left(\frac{mv^2}{2}\right) + P_{\rm rad}. \tag{7}$$

To determine the partitioning of power from the source between radiation and change in the charge's kinetic energy, we follow Planck [8] in supposing that the radiation exerts a reaction force $\mathbf{F}_{\rm react}$ back on the charge, so that eq. (4) should also be modified,[4]

$$\mathbf{F}_{\rm ext} + \mathbf{F}_{\rm react} = m\dot{\mathbf{v}}. \tag{8}$$

Projecting eq. (8) onto the velocity $\mathbf{v}$ and comparing with eq. (7), we infer that,

$$\mathbf{F}_{\rm react} \cdot \mathbf{v} = -P_{\rm rad}. \tag{9}$$

Equation (9) suggests that we might write $\mathbf{F}_{\rm react} = -P_{\rm rad}\hat{\mathbf{v}}/v$, but this has unacceptable behavior at small velocities (and violates the principle of Galilean invariance that underlies Newton's equations of motion).[5] Planck proposed that we evade this difficulty by integrating eq. (9) over the period $T = 2\pi/\omega$ in the case of periodic motion, as in the present problem,

$$\begin{aligned}\int_0^T \mathbf{F}_{\rm react} \cdot \mathbf{v}\; dt &= -\int_0^T P_{\rm rad}\, dt = -\frac{2e^2}{3c^3}\int_0^T \dot{\mathbf{v}} \cdot \dot{\mathbf{v}}\, dt = \frac{2e^2}{3c^3}\int_0^T \ddot{\mathbf{v}} \cdot \mathbf{v}\, dt - \frac{2e^2}{3c^3}\mathbf{v} \cdot \dot{\mathbf{v}}\Big|_0^T \\ &= \frac{2e^2}{3c^3}\int_0^T \ddot{\mathbf{v}} \cdot \mathbf{v}\, dt. \end{aligned} \tag{10}$$

It then appears plausible to identify the radiation-reaction force as,

$$\mathbf{F}_{\rm react} = \frac{2e^2\ddot{\mathbf{v}}}{3c^3} \qquad (v \ll c). \tag{11}$$

While the result (11) does not obey eq. (9), its average over a period of the motion does. So, we conclude with Planck that eq. (11) is valid on average, and therefore has good utility in the case of periodic motion.[6]

Whether the form (11) is valid at all times, rather than only on average, and whether it applies to nonperiodic motion and in particular to the case of uniform acceleration, has been the subject of extensive debate, some of which is reviewed by the author in [16].

2.2 Small Linear Antenna

We now consider a small, center-fed, linear antenna of length L that extends along the z-axis from $-L/2$ to $L/2$, for which electric-dipole radiation is dominant.

[4]Planck appears to have been inspired in part by Poincaré's commentary that Hertz' observation and analysis of the generation of electromagnetic waves [11, 12, 13] implies that the apparatus should be subject to *amortissement* = damping of the currents and energy due to the emission of radiation.

Plank's derivation in 1896 was before Larmor [6] published his formula (6), and Planck relied on Hertz' analysis [12] or radiation by an oscillating charge (oscillating electric dipole).

[5]Equation (9) is, however, valid for uniform circular motion.

[6]The force (11) had been found by Lorentz in 1892 [14] during an investigation of the self force of an extended, accelerated charge, with no mention of radiation. See also Note 18 of [15].

2.2.1 Radiation Resistance via a Conventional Argument

The current $I(z,t)$has value $I_0 e^{-i\omega t}$ at the feedpoint $z = 0$, and vanishes at the endpoints of the antenna, $z = \pm L/2$. For $L \ll \lambda$, the spatial variation of the current can only be linear,

$$I(z,t) = I_0 \left(1 - \frac{|z|}{L/2}\right) e^{-i\omega t}. \tag{12}$$

This spatial variation results in an accumulation of charge along the antenna, according to the equation of continuity,

$$\frac{\partial \rho}{\partial t} = -\frac{\partial I}{\partial z} = \mp \frac{I_0}{L/2} e^{-i\omega t}, \tag{13}$$

so that the charge density ρ is uniform over each of the two arms of the antenna, but with opposite signs,

$$\rho(z,t) = \pm \frac{iI_0}{\omega L/2} e^{-i\omega t}. \tag{14}$$

The electric-dipole moment p of this charge distribution is,

$$p(t) = \int_{-L/2}^{L/2} \rho z \; dz = \frac{iI_0 L}{2\omega} e^{-i\omega t}. \tag{15}$$

The time-average radiated power $P_{\rm ave}$ follows from the electric-dipole version of the Larmor formula,

$$P_{\rm ave} = \frac{|\ddot{p}|^2}{3c^3} = \frac{I_0^2 \omega^2 L^2}{12c^3} = \frac{I_0^2}{2} \frac{2\pi^2}{3c} \left(\frac{L}{\lambda}\right)^2 = \frac{I_0^2}{2} R_{\rm rad}, \tag{16}$$

where the radiation resistance of the small dipole antenna is,

$$R_{\rm rad} = \frac{2\pi^2}{3c} \left(\frac{L}{\lambda}\right)^2 = 197 \left(\frac{L}{\lambda}\right)^2 \Omega. \tag{17}$$

2.2.2 Radiation Resistance via the Radiation-Reaction Force

The conduction electrons in the antenna have average velocity,

$$v(z,t) = v_0(z) e^{-i\omega t} = \frac{I(z,t)}{neA_0} = \frac{I_0}{neA_0} \left(1 - \frac{|z|}{L/2}\right) e^{-i\omega t}, \tag{18}$$

at position z and time t, where n is the number density of the conduction electrons and A_0 is the cross sectional area of the antenna.

The radiation-reaction force (11) on a conduction electron at position z is,

$$F_{\rm react} = \frac{2e^2 \ddot{v}(z)}{3c^3} = -\frac{2e^2 \omega^2 v_0(z)}{3c^3} e^{-i\omega t}. \tag{19}$$

The time-average power delivered to a single electron by the radiation-reaction force (19) is,

$$-\frac{1}{2} Re(F_{\rm react} v^\star) = \frac{e^2 \omega^2 v_0^2(z)}{3c^3}. \tag{20}$$

This power is delivered coherently to all of the conduction electrons in the antenna, so the total power is the square of the sum of the amplitudes of the power per electron. This amplitude is the square root of eq. (20), namely $e\omega v_0(z)/\sqrt{3c^3}$. The number of conduction electrons per unit length along the antenna is nA_0, so the total power delivered by the radiation-reaction force is,

$$
\begin{aligned}
P_{\rm react} &= \left[\int_{-L/2}^{L/2} \frac{e\omega v_0(z)}{\sqrt{3c^3}} nA_0\ dz\right]^2 = \left[\frac{\omega I_0}{\sqrt{3c^3}}\int_{-L/2}^{L/2}\left(1-\frac{|z|}{L/2}\right)\ dz\right]^2 = \left[\frac{\omega I_0 L}{2\sqrt{3c^3}}\right]^2 \\
&= \frac{I_0^2}{2}\frac{2\pi^2}{3c}\left(\frac{L}{\lambda}\right)^2 = \frac{I_0^2}{2}R_{\rm rad} = P_{\rm rad}, \qquad (21)
\end{aligned}
$$

as found in eq. (16).[7]

2.3 The Radiation-Reaction Force for Magnetic-Dipole Radiation

We shall consider the case of a small loop antenna (sec. 2.4), in which all electric-multipole radiation vanishes and magnetic-dipole radiation dominates. We restrict our discussion to the case that the charges move in a circle of fixed radius r. Then, the contribution of a charge e with azimuthal velocity,

$$\mathbf{v} = v_0 e^{-i\omega t}\ \boldsymbol{\phi} \qquad (24)$$

to the magnetic moment is $\boldsymbol{\mu} = \sum e\mathbf{r}\times\mathbf{v}/2c$ is,[8]

$$\boldsymbol{\mu}_e = \frac{e}{2c}\mathbf{r}\times\mathbf{v}. \qquad (25)$$

For a magnetic dipole created by oscillating conduction currents in a loop, the azimuthal acceleration at angular frequency ω is much larger than the centripetal acceleration v^2/r.

[7] Recall the "antenna formula" for radiation by a known, time-harmonic current distribution $\mathbf{J}(\mathbf{x})e^{-i\omega t}$ (see, for example, eq. (14-53) of [17]),

$$
\begin{aligned}
\frac{dP_{\rm rad}}{d\Omega} &= \frac{\omega^2}{8\pi c^3}\left|\int \mathbf{J}(\mathbf{x})\times\hat{\mathbf{k}}\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, d\mathrm{Vol}\right|^2 = \frac{3}{8\pi}\left|\int \frac{ne\omega\mathbf{v}(\mathbf{x})}{\sqrt{3c^3}}\times\hat{\mathbf{k}}\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, d\mathrm{Vol}\right|^2 \\
&= \frac{3}{8\pi}\left|\int n\sqrt{-\frac{1}{2}Re(F_{\rm react}v^\star)}\times\hat{\mathbf{k}}\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, d\mathrm{Vol}\right|^2 \approx \frac{3\sin^2\theta}{8\pi}\left|\int n\sqrt{-\frac{1}{2}Re(F_{\rm rad}v^\star)}\, d\mathrm{Vol}\right|^2, \qquad (22)
\end{aligned}
$$

where the approximation holds for small antennas (of size $L \ll \lambda$). Thus, for small antennas the total radiated power is,

$$P_{\rm rad} = \int \frac{dP_{\rm rad}}{d\Omega}\, d\Omega = \left|\int n\sqrt{-\frac{1}{2}Re(F_{\rm rad}v^\star)}\, d\mathrm{Vol}\right|^2, \qquad (23)$$

which is the method used in eq. (21).

For antennas that are not small compared to a wavelength the present technique does not work. Returning to the derivation of the radiation reaction force in sec. 2.1, we note that for an extended distribution of charges, the integration by parts of eq. (10) cannot be carried out, and the radiation reaction force on individual charges cannot readily be identified. In such cases the total radiation reaction force can still be deduced as the negative of the rate of radiation of momentum [22].

[8] We neglect the tiny contribution of the intrinsic magnetic moments of the conduction electrons to the magnetic moment of the current loop.

So, in taking time derivatives we have to a very good approximation,

$$\dot{\mathbf{v}} = -i\omega\mathbf{v}. \tag{26}$$

Thus,

$$\dot{\boldsymbol{\mu}}_e = \frac{e}{2c}\mathbf{r}\times\dot{\mathbf{v}} = -i\frac{e\omega}{2c}\mathbf{r}\times\mathbf{v}, \tag{27}$$

and,

$$\ddot{\boldsymbol{\mu}}_e = -i\frac{e\omega}{2c}\mathbf{r}\times\dot{\mathbf{v}} = -\frac{e\omega^2}{2c}\mathbf{r}\times\mathbf{v} = \frac{e}{2c}\mathbf{r}\times\ddot{\mathbf{v}}. \tag{28}$$

The power of the magnetic-dipole radiation emitted by a single charge with oscillatory azimuthal velocity (24) is given by the magnetic version of the Larmor formula,[9]

$$P_{\rm M1} = \frac{2\left|\ddot{\boldsymbol{\mu}}_e\right|^2}{3c^3} = \frac{e^2r^2\left|\ddot{\mathbf{v}}\right|^2}{6c^5} = \frac{e^2r^2\omega^4\left|\mathbf{v}\right|^2}{6c^5}\,. \tag{29}$$

According to eq. (9), we identify this expression with the product $-\mathbf{F}_{\rm react,M1}\cdot\mathbf{v}$, where $\mathbf{F}_{\rm react,M1}$ is the radiation-reaction force appropriate to magnetic-dipole radiation. Thus, we find,

$$\mathbf{F}_{\rm react,M1} = -\frac{e^2r^2\omega^4}{6c^5}\mathbf{v}. \tag{30}$$

An alternative derivation that parallels eq. (10) is,

$$\begin{aligned}\int_0^T \mathbf{F}_{\rm react,M1}\cdot\mathbf{v}\;dt &= -\int_0^T P_{\rm M1}\;dt = -\frac{e^2r^2}{6c^5}\int_0^T \ddot{\mathbf{v}}\cdot\ddot{\mathbf{v}}\;dt = \frac{e^2r^2}{6c^5}\int_0^T \dddot{\mathbf{v}}\cdot\dot{\mathbf{v}}\;dt - \frac{2e^2}{3c^3}\ddot{\mathbf{v}}\cdot\dot{\mathbf{v}}\bigg|_0^T \\ &= -\frac{e^2r^2}{6c^5}\int_0^T \ddddot{\mathbf{v}}\cdot\mathbf{v}\;dt + \frac{2e^2}{3c^3}\dddot{\mathbf{v}}\cdot\mathbf{v}\bigg|_0^T = -\frac{e^2r^2}{6c^5}\int_0^T \ddddot{\mathbf{v}}\cdot\mathbf{v}\;dt,\end{aligned} \tag{31}$$

from which we infer that,

$$\mathbf{F}_{\rm react,M1} = -\frac{e^2r^2}{6c^5}\ddddot{\mathbf{v}} = -\frac{e^2r^2\omega^4}{6c^5}\mathbf{v}. \tag{32}$$

The form (32) has also been deduced by Itoh via an expansion of the Liénard-Wiechert potentials to 4$^{\rm th}$ order [19].[10]

A single charge not at the origin has nonzero electric and magnetic moments of all orders. If the charge is accelerated, it emits electric and magnetic multipole radiation of all orders, of which electric-dipole radiation is typically the strongest. The above discussion indicates that the radiation-reaction force associated with higher-order multipole radiation depends on higher-order derivatives of the charge's velocity. In extremes cases, the higher-order derivatives could be larger than $\ddot{\mathbf{v}}$*, such that a higher-order radiation-reaction force is the largest. This reminds us that Lorentz' derivation [14, 15] of the self force on an accelerated charge, which yielded the form (11), this result is only approximate.*

[9]See, for example, sec. 71 of [18].

[10]Comments on radiation damping for magnetic dipoles also appear in [20, 21].

2.4 Small Loop Antenna

We now consider a circular loop of circumference $L \ll \lambda = 2\pi c/\omega$ that carries a current $I_0 e^{-i\omega t}$, which is assumed to be independent of azimuth around the loop. Then, the equation of continuity for electrical charge density ρ tells us that $\partial\rho/\partial t = -\partial I/\partial l = 0$, where l is the spatial coordinate along the loop. Thus, the electric-multipole moments of the loop have no time dependence, and the loop emits no electric-multipole radiation. The leading magnetic-multipole moment of the loop is, of course, its magnetic-dipole moment,

$$\mu(t) = \frac{IA}{c} = \frac{I_0 L^2}{4\pi c} e^{-i\omega t} \tag{33}$$

(in Gaussian units). The time-averaged radiated power $P_{\rm rad}$ associated with the oscillating magnetic dipole (33) follows from the magnetic version of the Larmor formula,[11]

$$P_{\rm rad} = \frac{|\ddot{\mu}|^2}{3c^3} = \frac{\omega^4 I_0^2 L^4}{48\pi^2 c^5} = \frac{I_0^2}{2}\frac{2\pi^2}{3c}\left(\frac{L}{\lambda}\right)^4 = \frac{I_0^2}{2} R_{\rm rad}, \tag{34}$$

where the radiation resistance is,

$$R_{\rm rad} = \frac{2\pi^2}{3c}\left(\frac{L}{\lambda}\right)^4 = 197\left(\frac{L}{\lambda}\right)^4\ \Omega, \tag{35}$$

recalling that $1/c = 30\ \Omega$.

We now verify that the total power associated with the magnetic-dipole radiation-reaction force (30) on all moving charges in the loop antenna equals the radiated power (34).

First, we recall that if the charge carriers (electrons) in the loop antenna have charge e, average velocity $\mathbf{v} = v_0 e^{-i\omega t}\hat{\boldsymbol{\phi}}$ (at time t) and number density n, then the current density $\mathbf{J}$ obeys,

$$\mathbf{J} = ne\mathbf{v}. \tag{36}$$

If the current I in the loop antenna is confined to a region of area A_0 in the cross section of the conductor, then $I = JA_0$. Thus, the average velocity is,

$$v = \frac{I}{neA_0} = \frac{I_0}{neA_0} e^{-i\omega t} = v_0 e^{-i\omega t}. \tag{37}$$

For example, in copper, the number density n of conduction electrons is about $10^{23}/\mathrm{cm}^3$. A current of $I_0 = 1$ Amp corresponds to about $6 \times 10^{18} e/\mathrm{s}$. Then, for a cross sectional area of $A_0 = 1\ \mathrm{mm}^2 = 0.01\ \mathrm{cm}^2$, the magnitude of the velocity of the charges is,

$$v_0 = \frac{I_0}{neA_0} \approx \frac{6 \times 10^{18} e/\mathrm{s}}{(10^{23} e/\mathrm{cm}^3)(0.01\ \mathrm{cm}^2)} = 6 \times 10^{-3}\ \mathrm{cm/s}. \tag{38}$$

The peak radiation-reaction force (30) due to the magnetic-dipole radiation of a single electron with velocity (37) is,

$$F_{\rm react,M1} = \frac{e^2 r^2 \omega^4 v_0}{6c^5} = \frac{2\pi^2 e^2 L^2 v_0}{3c\lambda^4}. \tag{39}$$

[11] See, for example, sec. 71 of [18].

The time-average power delivered to a single electron by this force is $F_{\rm react,M1} v_0/2$. The total number of conduction electrons involved producing the radiation from the small loop is $N = nA_0L$. Because these electrons act coherently, the total time-average power provided by the radiation-reaction force is $N^2 v_0/2$ times the peak force (39) on a single electron. That is,

$$P_{\rm react} = \frac{N^2 F_{\rm react,M1} v_0}{2} = \frac{\pi^2 n^2 e^2 A_0^2 L^4 v_0^2}{3c\lambda^4} = \frac{I_0^2}{2} \frac{2\pi^2}{3c} \left(\frac{L}{\lambda}\right)^4 = \frac{I_0^2}{2} R_{\rm rad} = P_{\rm rad}, \qquad (40)$$

as found in eq. (34).

References

[1] K.T. McDonald, *Ph501 Problem Set 8*, http://kirkmcd.princeton.edu/examples/ph501set8.pdf

[2] H.H. Macdonald, *Radiation from Conductors*, Proc. London Math. Soc. **1**, 459 (1904), http://kirkmcd.princeton.edu/examples/EM/macdonald_plms_1_459_04.pdf

[3] See footnote 18 of S.A. Schelkunoff, *Theory of Antennas of Arbitrary Shape and Size*, Proc. I.R.E. **29**, 493 (1941), http://kirkmcd.princeton.edu/examples/schelkunoff_procire_29_493_41.pdf

[4] K.T. McDonald, *Currents in a Center-Fed Linear Dipole Antenna* (June 27, 2007), http://kirkmcd.princeton.edu/examples/transmitter.pdf

[5] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc./ London **175**, 343 (1884), http://kirkmcd.princeton.edu/examples/EM/poynting_pt_175_343_84.pdf

[6] J. Larmor, *On the Theory of the Magnetic Influence on Spectra; and on the Radiation from moving Ions*, Phil. Mag. **44**, 503 (1897), http://kirkmcd.princeton.edu/examples/EM/larmor_pm_44_503_97.pdf

[7] G.A. Schott, *On the Motion of the Lorentz Electron*, Phil. Mag. **29**, 49-69 (1915), http://kirkmcd.princeton.edu/examples/EM/schott_pm_29_49_15.pdf

[8] M. Planck, *Über electrische Schwingungen, welche durch Resonanz erregt und durch Strahlung gedämpft werden*, Ann. d. Phys. **60**, 577 (1897), http://kirkmcd.princeton.edu/examples/EM/planck_ap_60_577_97.pdf

[9] H. Poincaré, *Sur la théorie des oscillations hertziennes*, Compte Rendus Acad. Sci. **113**, 515 (1891), http://kirkmcd.princeton.edu/examples/EM/poincare_cras_113_515_91.pdf

[10] H. Poincaré, *Les Oscillations Électriques* (Paris, 1894), http://kirkmcd.princeton.edu/examples/EM/poincare_osc_elec_94.pdf

[11] H. Hertz, *Ueber electrodynamische Wellen im Luftraume und deren Reflexion*, Ann. d. Phys. **34**, 609 (1889), http://kirkmcd.princeton.edu/examples/EM/hertz_apc_34_609_88.pdf

[12] H. Hertz, *Die Kräfte electrischer Schwingungen, behandelt nach der Maxwell'schen Theorie*, Ann. d. Phys. **36**, 1 (1889),
http://kirkmcd.princeton.edu/examples/EM/hertz_ap_36_1_89.pdf
The Forces of Electrical Oscillations Treated According to Maxwell's Theory, Nature **39**, 402, 450, 547 (1889), http://kirkmcd.princeton.edu/examples/EM/hertz_nature_39_402_89.pdf

[13] H. Hertz *Electric Waves* (Macmillan, 1900),
http://kirkmcd.princeton.edu/examples/EM/hertz_electric_oscillations.pdf

[14] H.A. Lorentz, *La Théorie Électromagnétique de Maxwell et son Application aux Corps Mouvants*, Arch. Neérl. **25**, 363 (1892),
http://kirkmcd.princeton.edu/examples/EM/lorentz_theorie_electromagnetique_92.pdf

[15] H.A. Lorentz, *The Theory of Electrons* (Teubner, Leipzig, 1906), secs. 26 and 37 and note 18. http://kirkmcd.princeton.edu/examples/EM/lorentz_theory_of_electrons_09.pdf

[16] K.T. McDonald, *On the History of the Radiation Reaction* (May 6, 2017),
http://kirkmcd.princeton.edu/examples/selfforce.pdf

[17] W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd ed. (Addison-Wesley, 1962), http://kirkmcd.princeton.edu/examples/EM/panofsky-phillips.pdf

[18] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Pergamon Press, 1975), http://kirkmcd.princeton.edu/examples/EM/landau_ctf_71.pdf

[19] N. Itoh, *Radiation reaction due to magnetic dipole radiation*, Phys. Rev. A **43**, 1002 (1991), http://kirkmcd.princeton.edu/examples/EM/itoh_pra_43_1002_91.pdf

[20] D.R. Stump and G.L. Pollack, *Magnetic dipole oscillations and radiation damping*, Am. J. Phys. **65**, 81 (1997), http://kirkmcd.princeton.edu/examples/EM/stump_ajp_65_81_97.pdf

[21] W. Unruh, *Radiation reaction fields for an accelerated dipole for scalar and electromagnetic radiation*, Phys. Rev. A **59**, 131 (1999),
http://kirkmcd.princeton.edu/examples/EM/unruh_pra_59_131_99.pdf

[22] K.T. McDonald, *The Force on an Antenna Array* (May 1, 1979; updated Oct. 8, 2007),
http://kirkmcd.princeton.edu/examples/antenna_force.pdf