“Hidden” Momentum in a Compressed Rod?

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1 Problem

The term “hidden” momentum was popularized by Shockley [1] in considerations of an electromechanical example, and essentially all subsequent use of this term has been for such examples, where one considers the system to consist of matter plus electromagnetic fields.

A definition of “hidden” momentum has been proposed [2] (see also [3]) which can be applied to mechanical systems as well, where a subsystem has a specified volume and can interact with the rest of the system via contact forces and/or transfer of mass/energy across its surface (which can be in motion),

$$\mathbf{P}_{\rm hidden} \equiv \mathbf{P} - M\mathbf{v}_{\rm cm} - \oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm})\,(\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area} = -\int \frac{f^0}{c}(\mathbf{x} - \mathbf{x}_{\rm cm})\,d{\rm Vol}, \quad (1)$$

where $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total “mass”, U is its total energy, c is the speed of light in vacuum, $\mathbf{x}_{\rm cm}$ is its center of mass/energy, $\mathbf{v}_{\rm cm} = d\mathbf{x}_{\rm cm}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its “mass” density, u is its energy density, $\mathbf{v}_b$ is the velocity (field) of its boundary, and,

$$f^\mu = \frac{\partial T^{\mu\nu}}{\partial x^\nu}\,, \quad (2)$$

is the 4-force density due to the subsystem, with $T^{\mu\nu}$ being the stress-energy-momentum 4-tensor of the subsystem.

Does a compressed rod[1] contain “hidden” momentum according to the above definition in its rest frame, and/or in a frame where the rod has velocity v along its axis?

2 Solution

This solution includes interpretations of notes by D. Vanzella, private communications, July 5, 2012 and Feb. 10, 2020.

2.1 Analysis in the Rest Frame of the Rod

The axis of the rod is along the $x^\star$ axis in its rest frame (the * frame), and extends from $x^\star = 0$ to $L^\star$. The cross sectional area of the rod is A.

The momentum $\mathbf{P}^\star$ of the rod is zero, and the velocity $\mathbf{V}^\star$ of its center of mass is also zero. The momentum density $\mathbf{p}^\star$ is zero, and the velocity $\mathbf{v}_b^\star$ of the boundary between the

[1]The rod is part of a larger system that applies the compressive forces to the rod. The larger system can be considered as “isolated”.

rod and its surrounding system is zero. Hence, the "hidden" momentum $\mathbf{P}^\star_{\rm hidden}$ is zero according to the first form of eq. (1).

The stress-energy-momentum tensor of the rod in its rest frame has the form,

$$T^{\star\mu\nu} = \left(\begin{array}{c|ccc} \rho^\star c^2 & & \mathbf{0} & \\ \hline & \sigma^\star & 0 & 0 \\ \mathbf{0} & 0 & 0 & 0 \\ & 0 & 0 & 0 \end{array}\right), \tag{3}$$

where the effective mass density $\rho^\star$ includes the contribution of the elastic potential energy.[2]

Since the stress tensor (3) is constant, its derivative 4-vector f^μ is zero, and the "hidden" momentum of the rod in the rest frame is also zero, according to the second form of eq. (1).

2.2 Analysis in a Frame where the Rod has Velocity $v\,\hat{\mathbf{x}}$

To transform the rest-frame stress tensor (3) into the frame in which that rod has velocity v along the x-axis, we note that the Lorentz-transformation matrix has the form,

$$\mathsf{L}^{\mu\nu}(x) = \left(\begin{array}{c|ccc} \gamma & \gamma v/c & 0 & 0 \\ \hline \gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), \tag{4}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$,

Hence, the stress-energy-momentum tensor in frame where the rod has velocity v is,

$$T^{\mu\nu} = (\mathsf{L}\mathsf{T}^\star\mathsf{L})^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma^2(\rho^\star c^2 + v^2\sigma^\star/c^2) & \gamma^2 v(\rho^\star c^2 + \sigma^\star)/c & 0 & 0 \\ \hline \gamma^2 v(\rho^\star c^2 + \sigma^\star)/c & \gamma^2(\rho^\star v^2 + \sigma^\star) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{5}$$

In this frame, the rod has (Lorentz-contracted) length $L = L^\star/\gamma$, while its cross-sectional area is still A. The mass M of the rod is its mass density $\rho = T^{00}/c^2 = \gamma^2(\rho^\star + v^2\sigma^\star/c^4)$ times its volume AL,

$$M = \gamma^2 AL(\rho^\star + v^2\sigma^\star/c^4). \tag{6}$$

The center of mass of the rod has velocity $\mathbf{v}_{\rm cm} = v\,\mathbf{x}$. Its momentum $\mathbf{P}$ is the momentum density $p_x = T^{0x}/c = \gamma^2 v(\rho^\star + \sigma^\star/c^2)$ times its volume AL,

$$\mathbf{P} = \gamma^2 vAL(\rho^\star + \sigma^\star/c^2)\,\hat{\mathbf{x}}. \tag{7}$$

[2]The stress $\sigma^\star$ is positive for compression of the rod, and negative if it were under tension.

The velocity of the boundaries of the rod with the rest of the system is $v_b\,\hat{\mathbf{x}} = v\,\hat{\mathbf{x}}$. The term $\mathbf{p} - \rho\mathbf{v}_b$ is,

$$\mathbf{p} - \rho\mathbf{v}_b = \left[\gamma^2 v\left(\rho^\star + \frac{\sigma^\star}{c^2}\right) - \gamma^2 v\left(\rho^\star + \frac{v^2\sigma^\star}{c^4}\right)\right]\hat{\mathbf{x}} = \frac{v\sigma^\star}{c^2}\,\hat{\mathbf{x}}. \tag{8}$$

The "hidden" momentum according to the first form of eq. (1) is,

$$\begin{aligned}\mathbf{P}_{\rm hidden} &= \gamma^2 vAL\left(\rho^\star + \frac{\sigma^\star}{c^2}\right)\hat{\mathbf{x}} - \gamma^2 vAL\left(\rho^\star + \frac{v^2\sigma^\star}{c^4}\right)\mathbf{x} - AL\frac{v\sigma^\star}{c^2}\,\hat{\mathbf{x}} \\ &= \left(\frac{vAL\sigma^\star}{c^2} - AL\frac{v\sigma^\star}{c^2}\right)\hat{\mathbf{x}} = 0. \end{aligned}\tag{9}$$

Thus, while $\mathbf{P} - M\mathbf{v}_{\rm cm} = (ALv\sigma^\star/c^2)\,\hat{\mathbf{x}}$ is nonzero, the "hidden" momentum is zero according to the first form of eq. (1).

To use the second form of eq. (1), we note that T^{00} and T^{0x} are constant within rod, whose boundaries in x are $x = vt$ and $vt + L$, while $T^{0y} = 0 = T^{0z}$ everywhere. Further, $T^{00}(x,t) = T^{00}(x - vt)$, such that,

$$f^0 = \partial_0 T^{00} + \partial_i T^{0i} = \frac{\partial T^{00}}{\partial ct} + \frac{\partial T^{0x}}{\partial x} = -\frac{v}{c}\frac{\partial T^{00}}{\partial x} + \frac{\partial T^{0x}}{\partial x}\,, \tag{10}$$

It suffices to complete the calculation for $t = 0$, as the result should be independent of time. Then, f^0 is nonzero only at/near the boundaries $x = 0$ and $x = L = L^\star/\gamma$.

$$\begin{aligned}\mathbf{P}_{\rm hidden}(t=0) &= -\int \frac{f^0}{c}(\mathbf{x} - \mathbf{x}_{\rm cm})\,d{\rm Vol} = -\frac{A}{c}\int_0^L dx\,\left(-\frac{v}{c}\frac{\partial T^{00}}{\partial x} + \frac{\partial T^{0x}}{\partial x}\right)\left(x - \frac{L}{2}\right)\hat{\mathbf{x}} \\ &= \frac{Av}{c^2}\left[xT^{00}\big|_0^L - \int_0^L dx\,T^{00} - \frac{L}{2}[T^{00}(L) - T^{00}(0)]\right]\hat{\mathbf{x}} \\ &\quad -\frac{A}{c}\left[xT^{0x}\big|_0^L - \int_0^L dx\,T^{0x} - \frac{L}{2}[T^{0x}(L) - T^{0x}(0)]\right]\hat{\mathbf{x}} = 0, \end{aligned}\tag{11}$$

in agreement with eq. (9), taking T^{00} and T^{0x} to have values their nonzero, constant values within the interval $0 \le x \le L$. and zero outside this.[3]

[3]An analysis of eq. (11) more in the style of Vanzella's note of July 5, 2012, which invokes Heaviside step functions Θ and Dirac delta functions δ, considers that in the frame where the rod has velocity v, the nonzero components of $T^{0\mu}$ can be written as,

$$T^{00} = \gamma^2(\rho^\star c^2 + v^2\sigma^\star/c^2)[\Theta(x - vt) - \Theta(x - L - vt)], \tag{12}$$
$$T^{0x} = \gamma^2 v(\rho^\star c^2 + \sigma^\star)/c\,[\Theta(x - vt) - \Theta(x - L - vt)], \tag{13}$$

where $\Theta(x) = 1$ for $x > 0$ and $= 0$ for $x < 0$. Then,

$$\frac{\partial T^{00}}{\partial ct} = -\gamma^2(\rho^\star c^2 + v^2\sigma^\star/c^2)\frac{v}{c}[\delta(x - vt) - \delta(x - L - vt)], \tag{14}$$
$$\frac{\partial T^{0x}}{\partial x} = \gamma^2 v(\rho^\star c^2 + \sigma^\star)/c[\delta(x - vt) - \delta(x - L - vt)], \tag{15}$$
$$P_{{\rm hidden},x}(t=0) = -\int \frac{f^0}{c}(x - x_{\rm cm})\,d{\rm Vol} = -\frac{A}{c}\int_0^L dx\,\left(\frac{\partial T^{00}}{\partial ct} + \frac{\partial T^{0x}}{\partial x}\right)(x - x_{\rm cm})$$

Thus, according to the calculations (9) and (11), there is no "hidden" momentum in the "all-mechanical" example of a compressed rod in either its rest frame or the frame in which the rod has velocity $v\,\hat{\mathbf{x}}$.

Part of the momentum (7) in a frame in which the rod is moving is associated with its internal stress, but this is not a "hidden" momentum according to definition (1).[4]

As noted in sec. VI of [5], "hidden" momentum is associated with (sub)systems that have internal motion when "at rest", which is not the case for a compressed rod.

2.2.1 Stress in the Surrounding Mechanical Press

The rod is compressed by a surrounding mechanical press, such as that shown below.

In the rest frame of the system, the stress in the plates of the press in contact with the

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\begin{aligned}
&= \frac{A}{c}\int_0^L dx\,\gamma^2(\rho^\star c^2 + v^2\sigma^\star/c^2)\frac{v}{c}[\delta(x)-\delta(x-L)](x-L/2) \\
&\quad -\frac{A}{c}\int_0^L dx\,\gamma^2 v(\rho^\star c^2+\sigma^\star)/c[\delta(x)-\delta(x-L)](x-L/2) \\
&= -\frac{ALv}{c^2}\gamma^2(\rho^\star c^2 + v^2\sigma^\star/c^2) + \frac{ALv}{c^2}\gamma^2(\rho^\star c^2+\sigma^\star) = \frac{ALv\sigma^\star}{c^2}\,. \qquad (16)
\end{aligned}
$$

Note that $(ALv\sigma^\star/c^2)\,\hat{\mathbf{x}}$ is the same as $\mathbf{P} - M\mathbf{v}_{\rm cm}$ since $\mathbf{v}_{\rm cm} = v\,\hat{\mathbf{x}}$, so if the boundary integral in the first form of eq. (1) were ignored, then the two forms of that expression would both lead to the same, nonzero "hidden" momentum in the moving, compressed rod.

This author finds the delta functions in the expressions (14)-(15) for the 4-force density f^μ very unappealing physically, and so prefers the analysis in the main text that avoids them, with the implication that there is zero "hidden" momentum in the present example.

[4] In sec. 2 of [4], the term $(Av\sigma^\star/c^2)\,\hat{\mathbf{x}} = \mathbf{P} - M\mathbf{v}_{\rm cm}$ of our eq. (9) was called a "hidden" momentum.

rod is essentially the same as that in the rod, but in general the mass density is different,

$$T^{\star\mu\nu}_{\rm press}(\text{next to rod}) = \left(\begin{array}{c|ccc} \rho^\star_{\rm press} c^2 & & \mathbf{0} & \\ \hline & \sigma^\star & 0 & 0 \\ \mathbf{0} & 0 & 0 & 0 \\ & 0 & 0 & 0 \end{array} \right). \qquad (17)$$

In the frame in which the rod and press have velocity $v\,\mathbf{x}$, the stress tensor next to the rod is,

$$T^{\mu\nu}_{\rm press}(\text{next to rod}) = \left(\begin{array}{c|ccc} \gamma^2(\rho^\star_{\rm press} c^2 + v^2\sigma^\star/c^2) & \gamma^2 v(\rho^\star_{\rm press} c^2 + \sigma^\star)/c & 0 & 0 \\ \hline \gamma^2 v(\rho^\star_{\rm press} c^2 + \sigma^\star)/c & \gamma^2(\rho^\star_{\rm press} v^2 + \sigma^\star) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right). \qquad (18)$$

Comparing with the stress tensor (5), we see that the stress T^{xx} is not continuous across the boundary between the rod and the press.

Perhaps we should infer that mechanical stress analysis in not very meaningful in frames where objects have velocities comparable to the speed of light.

However, even if we restrict the discussion to $v \ll c$, setting $\gamma \approx 1$, and ignore terms in v^2/c^2, the analysis of "hidden" momentum given above remains essentially the same.

2.2.2 A Relativistic Effect

Written Sept. 12, 2020, following a comment by D. Vanzella.

Suppose the compressive forces on the rod are removed simultaneously in the lab frame. Then, they are not removed simultaneously in the initial rest frame of the compressed rod.

Indeed, in the initial rest from of the rod, the force on the "leading" end is removed before that on the "trailing" end. This gives a "forward kick" to the rod, and its velocity becomes nonzero (with respect to this frame).

Meanwhile, an observer in the lab frame also notices this effect as an increase in the velocity of the rod with respect to the lab frame. But, it is inappropriate for the lab-frame observer to say that this change is a result of "hidden" momentum being converted to "overt" momentum, since it is just an effect an "overt" force that is unbalanced for a short time due to the relativity of simultaneity.[5]

References

[1] W. Shockley and R.P. James, *"Try Simplest Cases" Discovery of "Hidden Momentum" Forces on "Magnetic Currents"*, Phys. Rev. Lett. **18**, 876 (1967), http://kirkmcd.princeton.edu/examples/EM/shockley_prl_18_876_67.pdf

[5]While "hidden" momentum can be called a relativistic effect, not all relativistic effects involve "hidden" momentum.

[2] D. Vanzella, Private communication, (June 29, 2012).

[3] K.T. McDonald, *On the Definition of "Hidden" Momentum* (July 9, 2012), http://kirkmcd.princeton.edu/examples/hiddendef.pdf

[4] K. Szymański, *On the momentum of mechanical plane waves*, Physica B **403**, 2296 (2008), http://kirkmcd.princeton.edu/examples/GR/szymanski_physica_b403_2296_08.pdf

[5] D. Babson *et al.*, *Hidden momentum, field momentum, and electromagnetic impulse*, Am. J. Phys. **77**, 826 (2009), http://kirkmcd.princeton.edu/examples/EM/babson_ajp_77_826_09.pdf