Seppala's Paradox

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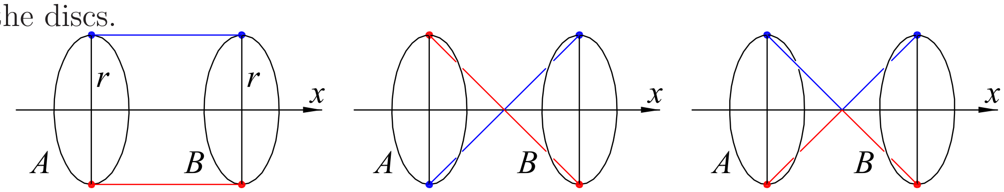
1 Problem

A special-relativity puzzler recently posed by David Seppala (private communication) concerns two discs, A and B of radius r, with a common axis, x.

The discs can rotate about this axis, but only when driven by a motor that insures a constant angular velocity.

As sketched on the left below, the two discs are at rest (not rotating) at time $t < 0$ in the inertial rest frame of the centers of the discs. The two discs are connected by two elastic bands, as shown below in blue and red, fixed to points at the ends of diameters of the discs. For $t < 0$, these two diameters are parallel to each other, and the bands are parallel to the line of centers of the discs.

inertial rest frame of the centers of the discs



$t = 0$, discs A and B at rest; $t = T/2$, disc A 1/2 turn, disc B at rest; $t = T$, disc A 1 turn, disc B at rest

At time $t \geq 0$ disc A is driven to rotate at constant angular velocity with period T (in the inertial rest frame of the centers of the discs), while disc B remains at rest.

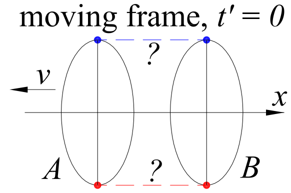
After 1/2 turn of disc A, at time $t = T/2$, the two elastic bands touch at the midpoint of the line of centers of the discs, as shown in the middle figure above.

Thereafter, the two bands remain in contact at this midpoint, and become increasingly twisted/intertwined about this point. However, between the midpoint and disc B, the bands remain, for all later times, in the configuration they have at time $t = T/2$. The configuration at time $t = T$, when disc A has made one full turn, is shown in the right figure above.

We now consider an inertial frame, the $'$ frame, that moves in the x direction with respect to the inertial rest frame of the discs. In particular we suppose that the parameters of this example are such that at time $t' = 0$ in the $'$ frame, a clock at the center of disc A (in the inertial rest frame) would read $t_A = 0$, while a clock at the center of disc B would read $t_B = T$. Then, in the $'$ frame, both discs are at rest, which suggests that the elastic bands lie along straight lines parallel to the axis of the discs, as shown in the figure below.

If so, how/when do observers in the $'$ frame consider that the bands become intertwined?

moving frame, $t' = 0$



$t_A = 0$, $t_B = T$
only disc A rotates

2 Solution

This example is a twisty variant of the "pole in the barn paradox".[1]

It is helpful to consider first the simpler case that the two discs rotate in phase, starting at $t = 0$ in the rest frame of their centers.

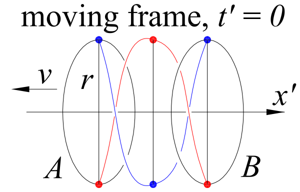
2.1 The Two Discs Rotate in Phase

When the two discs rotate in phase, the elastic bands remain unstretched, and parallel to the axis of the discs, at all times in the rest frame of the centers of the discs.

Hence, in a frame that moves parallel to the (x) axis of the discs, points on the bands are always at radius r from the axis.

In the $'$ frame at time $t' = 0$, the time in the rest frame at disc B is $t = T$, when this disc has rotated one full turn.

moving frame, $t' = 0$



$t_A = 0,\ \ t = T/2,\ \ t_B = T$

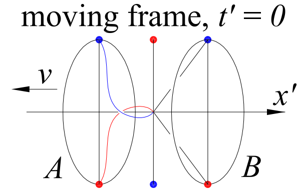
discs rotate in phase

Points in the plane perpendicular to the axis and midway between the two discs are at time $t = T/2$ in the rest frame when at time $t' = 0$ in the $'$ (moving) frame. This means that the midpoints of the two bands have each rotated by 180° at time $t' = 0$. That is, the bands appear to be helices of radius r in the moving frame, as sketched above.

2.2 Only Disc A Rotates

As noted in sec. 1 above, when only disc A rotates, the portions of the bands between the midpoint and disc B remain in the configuration they had at time $t = T/2$ for all later times. In particular, this applies at time $t' = 0$ in the moving frame, when the midpoint is at time $t = T/2$ (in the rest frame) and disc B is at time $t = T$. The portions of the bands between disc A and the midpoint at time $t' = 0$ have times between $t = 0$ and $t = T/2$, when these portions are in motion due to the rotation of disc A. As such, the paths of these portions of the bands appear in the moving frame as spirals, from their end points on disc A at time $t = 0$ into the midpoint at time $t = T/2$, as sketched in the figure below.

moving frame, $t' = 0$



$t_A = 0,\ \ t = T/2,\ \ t_B = T$

only disc A rotates

[1]See, for example, https://en.wikipedia.org/wiki/Ladder_paradox

At time $t' = -\gamma T = -T/\sqrt{1 - v^2/c^2}$ in the moving frame, discs A and B are at times $t_A = -T$ and $t_B = 0$ in the rest frame. The figure at the bottom of p. 1 above holds at this time (and all earlier times). For $t' > -\gamma T$, we have that $t_B > 0$, so the bands have departed from their initial, parallel configuration, at least near disc B. For $-\gamma T < t' < 0$ the configuration of the bands in the moving frame is complex, evolving from that in the bottom figure on p. 1 to that in the lower figure on p. 2.