Radiation in the Near Zone of a Small Loop Antenna

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1 Problem

The electromagnetic fields far from any antenna can be conveniently described as the sum of the radiation fields of a series of oscillating point multipoles, of which the leading term is a dipole in many cases of practical interest. The form of the fields associated with the n^{th} multipole is independent of the details of the physical layout of the antenna (other than that the layout determines the magnitudes of the multipole moments). However, close to the antenna the electromagnetic fields include quasistatic components as well as radiation terms. A well-known argument due to Hertz [1, 2] gives the fields in the near and far zone of an ideal point dipole. In this and companion notes [3, 4] we explore examples in which analytic expressions can be given for the near and far zone fields of antennas of finite dimensions.

Here, the task is to describe the electromagnetic fields, and the Poynting vector [5], of an oscillating current loop of radius a when $a \ll \lambda = 2\pi c/\omega$, where c is the speed of light and ω is the angular frequency of oscillation of the current in the loop. The current is assumed to be independent of the azimuth around the loop.[1]

2 Solution

The qualitative character of the solution is readily anticipated. Within one wavelength of the current loop, the magnetic field looks essentially the same as the dipole field pattern due to a steady current loop, multiplied by $\cos \omega t$. The electric field is azimuthal everywhere, and relatively weak close to the loop, where it is roughly that induced by the oscillating dipole magnetic field according to Faraday's law. Far from the loop, the electric and magnetic fields are transverse to the radial direction, and have the form of ideal magnetic dipole radiation. In the far zone, the Poynting vector is essentially radial, with the well-known $\sin^2\theta$ dependence on the polar angle θ. The lines of the Poynting vector emanate from the current loop. On average, there is no Poynting flux inwards from the loop; the radiated energy flows, on average, outwards from the loop and the lines of Poynting flux are almost purely radial once they are more than a few wavelengths from the antenna.

A real current loop would be fed at some point on its circumference by a pair of leads, perhaps in the form of a coaxial cable. This breaks the perfect azimuthal symmetry of the problem, such that Poynting flux would emanate from the feed point, passing across the space inside the loop all points along the loop (as sketched in the original article of Poynting [5] for a DC current loop), and from there radiating outwards as sketched above.

The assumption of a uniform oscillating current in the loop is not consistent with the loop being a perfect conductor, which cannot sustain electric fields parallel to its surface.

[1]This problem was first solved by FitzGerald [6, 7], prior to the invention of the Poynting vector.

We now give as much of an analytic solution as possible. This solution is inspired by sec. 12.13 of [8], which uses the method of Hertz vectors and scalars, which are reviewed in the Appendix. See also [9]. Early work on this theme is by Pocklington [10] and Rayleigh [11].

We will work in a spherical coordinate system (r, θ, ϕ) whose origin is at the center of the loop and whose z axis coincides with that of the loop. The solution will be based on the use of potentials for the electromagnetic fields. The charge density is effectively zero in this problem, so the scalar potential V may be ignored. The currents in the loop are azimuthal, so the vector potential $\mathbf{A}$ will only have an azimuthal component, and this in independent of the azimuth,

$$\mathbf{A}(\mathbf{r}, t) = A_\phi(r, \theta, t)\hat{\boldsymbol{\phi}}. \tag{1}$$

Furthermore, we restrict our attention to waves of angular frequency ω, and write the vector potential as,

$$\mathbf{A}(\mathbf{r}, t) = A_\phi(r, \theta)e^{-i\omega t}\hat{\boldsymbol{\phi}}. \tag{2}$$

Outside the current loop the vector potential obeys the free-space wave equation,

$$\nabla^2 \mathbf{A} = \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{\omega^2}{c^2}\mathbf{A} = -k^2 \mathbf{A}, \tag{3}$$

where we have introduced the wave number $k = \omega/c$. The time-independent form of eq. (3) is called the Helmholtz equation.

We can also note at the outset that the assumption of uniform current around the ring implies that the wavelength of the radiation is large compared to the radius a (*i.e.*, $ka \ll 1$). Otherwise, the amplitude of the wave would vary from place to place around the ring, and the current would take on spatial modulations.

2.1 Series Expansion of the Vector Potential

We need the prescription for the Laplacian operator as applied to a vector (which is much less straightforward in curvilinear coordinates than in rectangular coordinates). From p. 116 of [12] we find that for the vector potential (1),

$$\nabla^2[A_\phi(r, \theta)\hat{\boldsymbol{\phi}}] = \hat{\boldsymbol{\phi}}\left(\nabla^2 A_\phi - \frac{A_\phi}{r^2 \sin^2\theta}\right). \tag{4}$$

Writing out the Laplacian as applied to the scalar A_ϕ, our Helmholtz equation (3) becomes,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_\phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial A_\phi}{\partial \theta}\right) - \frac{A_\phi}{r^2 \sin^2\theta} + k^2 A_\phi = 0. \tag{5}$$

Anticipating the appearance of spherical Bessel functions in A_ϕ, we seek a solution that is a sum of terms of the form,

$$A_\phi = \frac{R(r)}{\sqrt{r}}\Theta(\theta). \tag{6}$$

We insert the trial solution (6) into eq. (5), multiply by r^2 and divide by A_ϕ to find,

$$\frac{r^2}{R^2}\frac{d^2 R}{dr^2} + \frac{r}{R}\frac{dR}{dr} - \frac{1}{4} + k^2 r^2 + \frac{1}{\Theta}\frac{d^2}{d(\cos\theta)^2}[(1-\cos^2\theta)\Theta] - \frac{1}{1-\cos^2\theta} = 0. \tag{7}$$

As is usual for separation-of-variables techniques in spherical coordinates, we introduce a separation constant $\pm n(n+1)$ to obtain the radial and polar equations,

$$\frac{d^2 R_n}{dr^2} + \frac{1}{r}\frac{dR_n}{dr} + \left[k^2 - \frac{(n+\frac{1}{2})^2}{r^2}\right] R_n = 0, \tag{8}$$

$$\frac{d^2}{d(\cos\theta)^2}[(1-\cos^2\theta)\Theta_n] + \left[n(n+1) - \frac{1}{1-\cos^2\theta}\right]\Theta_n = 0. \tag{9}$$

The solution to polar eq. (9) is the associated Legendre function,

$$\Theta_n = P_n^1(\cos\theta). \tag{10}$$

(Since the vector potential must be finite on the z axis, we exclude the solution Q_n^1.) The three lowest-order P_n^1 are,

$$P_1^1(\cos\theta) = \sin\theta, \quad P_2^1(\cos\theta) = 3\cos\theta\sin\theta, \quad P_3^1(\cos\theta) = \frac{3}{2}(5\cos^2\theta - 1)\sin\theta. \tag{11}$$

The present problem is symmetric about the plane of the ring, $\theta = \pi/2$, so that only odd-n Legendre functions can contribute to the solution.

Solutions to the radial equation (9) are the so-called spherical Bessel functions of order n, which are related to ordinary Bessel functions of order $n + \frac{1}{2}$ (see, for example, secs. 5.31 and 5.37 of [8], sec. 10.1 of [13], sec. 9.6 of [16]). At large r, we expect the vector potential to consist of spherical waves of the form $e^{i(kr-\omega t)}/r$. This suggests that we use the spherical Bessel function of the third kind,

$$h_n^{(1)}(kr) = j_n(kr) + iy_n(kr), \tag{12}$$

where j_n and y_n are the spherical Bessel functions of the first and second kind, as the asymptotic is behavior of $h_n^{(1)}$ is,

$$h_n^{(1)}(kr \gg 1)) \to (-i)^{n+1}\frac{e^{ikr}}{kr}. \tag{13}$$

The three lowest-order $h_n^{(1)}$ are,

$$h_0^{(1)}(kr) = -i\frac{e^{ikr}}{kr}, \quad h_1^{(1)}(kr) = -\frac{e^{ikr}}{kr}\left(1 + \frac{i}{kr}\right), \quad h_2^{(1)}(kr) = i\frac{e^{ikr}}{kr}\left(1 + \frac{3i}{kr} - \frac{3}{(kr)^2}\right). \tag{14}$$

Inside the current ring, where $r < a$, it is not appropriate to consider spherical traveling waves. Rather, only standing waves can exist here, with a finite value of the vector potential at the origin. This suggests that we use the $j_n(kr)$ in this region. The three lowest-order j_n are,

$$j_0(kr) = \frac{\sin(kr)}{kr}, \quad j_1(kr) = \frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr}, \quad j_2(kr) = \sin(kr)\left(\frac{3}{(kr)^3} - \frac{1}{kr}\right) - \frac{3\cos(kr)}{(kr)^2}. \tag{15}$$

Altogether, our expansion for the vector potential of the current ring of radius a is,

$$A_\phi(r < a, \theta, t) = i \sum_{\text{odd } n} A_n h_n^{(1)}(ka) j_n(kr) P_n^1(\cos\theta) e^{-i\omega t}, \tag{16}$$

$$A_\phi(r > a, \theta, t) = i \sum_{\text{odd } n} A_n j_n(ka) h_n^{(1)}(kr) P_n^1(\cos\theta) e^{-i\omega t}, \tag{17}$$

where we have inserted the factors i, $h_n^{(1)}(ka)$ and $j_n(ka)$ so that the remaining Fourier coefficients A_n will be real, and the same for $r < a$ and $r > a$.

2.2 Fields Close to the Ring ($r \approx a$)

As remarked earlier, the assumption of a uniform current distribution around the ring is only consistent with large wavelengths, *i.e.*, $ka \ll 1$. Hence, inside the ring we may approximate the spherical Bessel functions by their values for small arguments,

$$j_n(kr \ll 1) \to \frac{(kr)^n}{(2n+1)!!}, \qquad h_n^{(1)}(ka \ll 1) \approx i y_n(ka \ll 1) \to -i \frac{(2n-1)!!}{(ka)^{n+1}}, \tag{18}$$

where $(2n+1)!! = 1 \cdot 3 \cdot 5 \cdots (2n+1)$. The vector potential is therefore,

$$\begin{aligned} A_\phi(r < a, \theta, t) &= Re\left\{ \frac{1}{ka} \sum_{\text{odd } n} \frac{A_n}{2n+1} \left(\frac{r}{a}\right)^n P_n^1(\cos\theta) e^{-i\omega t} \right\} \\ &= \frac{1}{ka} \sum_{\text{odd } n} \frac{A_n}{2n+1} \left(\frac{r}{a}\right)^n P_n^1(\cos\theta) \cos\omega t. \end{aligned} \tag{19}$$

As expected, this is a standing wave, being $\cos\omega t$ times the vector potential of a steady current in the ring. Indeed, the latter is (from, for example, sec. 7.13 of [8] or sec. 5.5 of [16]),

$$\begin{aligned} A_\phi(r < a, \theta) &= \frac{2\pi I}{c} \sum_n \left(\frac{r}{a}\right)^n \frac{P_n^1(0)}{n(n+1)} P_n^1(\cos\theta) \\ &= -\frac{2\pi I}{c} \sum_{\text{odd } n} (-1)^{\frac{n+1}{2}} \frac{1 \cdot 3 \cdot 5 \cdots (n-2)}{2 \cdot 4 \cdot 6 \cdots (n+1)} \left(\frac{r}{a}\right)^n P_n^1(\cos\theta). \end{aligned} \tag{20}$$

This determines the Fourier coefficients A_n. Thus the series expansion of the time-dependent vector potential of a ring of radius a that carries a current $I\cos\omega t$ is,

$$A_\phi(r < a, \theta, t) = -\frac{2\pi I}{c} \sum_{\text{odd } n} (-1)^{\frac{n+1}{2}} \frac{1 \cdot 3 \cdot 5 \cdots (n-2)}{2 \cdot 4 \cdot 6 \cdots (n+1)} \left(\frac{r}{a}\right)^n P_n^1(\cos\theta) \cos\omega t, \tag{21}$$

$$A_\phi(r > a, \theta, t) = \frac{2\pi i I ka}{c} \sum_{\text{odd } n} (-1)^{\frac{n+1}{2}} \frac{1 \cdot 3 \cdot 5 \cdots (n-2)(2n+1)}{2 \cdot 4 \cdot 6 \cdots (n+1)} j_n(ka) h_n^{(1)}(kr) P_n^1(\cos\theta) e^{-i\omega t}. \tag{22}$$

For radii only slightly larger than a, the vector potential is still largely that of a standing wave $\cos\omega t$ multiplied by the vector potential of a steady current,

$$A_\phi(\lambda \gtrsim r > a, \theta, t) \approx -\frac{2\pi I}{c} \sum_{\text{odd } n} (-1)^{\frac{n+1}{2}} \frac{1 \cdot 3 \cdot 5 \cdots (n-2)}{2 \cdot 4 \cdot 6 \cdots (n+1)} \left(\frac{a}{r}\right)^{n+1} P_n^1(\cos\theta) \cos\omega t. \tag{23}$$

The electric and magnetic fields are, of course, obtained from the potentials according to,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{24}$$

Since the scalar potential V vanishes in this problem, the electric field has only a ϕ component, whose value inside the ring is,

$$E_\phi(r < a, \theta, t) \approx \frac{2\pi I k}{c} \sum_{\text{odd } n} (-1)^{\frac{n+1}{2}} \frac{1 \cdot 3 \cdot 5 \cdots (n-2)}{2 \cdot 4 \cdot 6 \cdots (n+1)} \left(\frac{r}{a}\right)^n P_n^1(\cos\theta) \sin\omega t. \tag{25}$$

Using the fact that,

$$\frac{d[\sin\theta P_n^1(\cos\theta)]}{d\cos\theta} = n(n+1)P_n(\cos\theta), \tag{26}$$

the magnetic field inside the ring has radial and polar components given by,

$$B_r(r < a, \theta, t) = \frac{2\pi I}{ac} \sum_{\text{odd } n} (-1)^{\frac{n+1}{2}} \frac{1 \cdot 3 \cdot 5 \cdots (n)}{2 \cdot 4 \cdot 6 \cdots (n-1)} \left(\frac{r}{a}\right)^{n-1} P_n(\cos\theta) \cos\omega t, \tag{27}$$

$$B_\theta(r < a, \theta, t) = -\frac{2\pi I}{ac} \sum_{\text{odd } n} (-1)^{\frac{n+1}{2}} \frac{1 \cdot 3 \cdot 5 \cdots (n-2)}{2 \cdot 4 \cdot 6 \cdots (n-1)} \left(\frac{r}{a}\right)^{n-1} P_n^1(\cos\theta) \cos\omega t. \tag{28}$$

Inside the ring, the electric field is much smaller than the magnetic field, $E/B \approx ka \ll 1$, as the former is induced by the time variation of the quasistatic magnetic field, whose period $2\pi/\omega$ is small compared to the transit time a/c of light across the ring.

2.3 Radiation in the Near Zone of the Ring

The flow of electromagnetic energy is described by the Poynting vector,

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B}. \tag{29}$$

Using eqs. (25) and (27)-(28), we see that $\mathbf{S}$ has both radial and polar components inside the ring ($r < a$). However, the time average $\langle \mathbf{S} \rangle$ of the Poynting vector vanishes here, since $S \propto \sin\omega t \cos\omega t$. This is to be expected from the standing wave character of the fields inside the ring. There is no (net) radiation from the ring into the region $r < a$.

Just outside the ring, the approximation (23) is an excellent description of the vector potential. However, if we calculate the electric and magnetic fields and the Poynting vector from this approximation, we will find only standing waves and no net radiation.

To discuss radiation in the near zone ($a < r \lesssim \lambda$), we must use eq. (22) for the vector potential. Because $ka \ll 1$, we can use approximation (18) for $j_n(ka)$ to write,

$$A_\phi(r > a, \theta, t) = \frac{2\pi i I k a}{c} \sum_{\text{odd } n} \left(-\frac{1}{2}\right)^{\frac{n+1}{2}} \frac{(n-2)!!}{(\frac{n+1}{2})!!(2n-1)!!} (ka)^n h_n^{(1)}(kr) P_n^1(\cos\theta) e^{-i\omega t}. \tag{30}$$

If we identify the radiation part of this vector potential as those terms that fall off as $1/r$, we will be led to keep only the $n = 1$ term of the series (30), and hence to the radiation fields of a point dipole, as discussed in sec. 2.4. To describe the radiation very close to the ring ($r \approx a$), which will be strong only for $\theta \approx 90°$, we must keep many terms in the series, which converges slowly.

Thus I am disappointed to conclude that the approach taken here is not very good for characterizing radiation in the near zone of a small loop antenna.

2.4 Fields Far from the Ring ($r \gg \lambda \gg a$)

For large r and $ka \ll 1$, the factor $j_n(ka)h_n^{(1)}(kr)$ in the expansion (22) for the vector potential outside the ring can be well approximated by eqs. (13) and (18) as,

$$j_n(ka \ll 1)h_n^{(1)}(kr \gg 1) \to (-i)^{n+1}\frac{e^{ikr}}{kr}\frac{(ka)^n}{(2n+1)!!}\,. \tag{31}$$

Hence, only the $n = 1$ term is significant for large r, and eq. (22) becomes,

$$A_\phi(r \gg a, \theta, t) = \frac{\pi a^2 Iik}{c}\sin\theta\frac{e^{i(kr-i\omega t)}}{r} = ikm\sin\theta\frac{e^{i(kr-i\omega t)}}{r}\,, \tag{32}$$

where the magnetic moment of the current loop is given by,

$$m = \frac{\pi a^2 I}{c}\,. \tag{33}$$

The asymptotic form (32) is identical to the far-zone vector potential of an idealized point magnetic dipole (see, for example, sec. 9.3 of [16]), as is to be expected. The asymptotic electric and magnetic field (the far-zone radiation fields) are, as usual,

$$E_\phi = -\frac{\partial A_\phi}{\partial ct} = -k^2 m\sin\theta\frac{e^{i(kr-i\omega t)}}{r}\,, \qquad B_\theta = -\frac{1}{r}\frac{\partial(rA_\phi)}{\partial r} = k^2 m\sin\theta\frac{e^{i(kr-i\omega t)}}{r}\,. \tag{34}$$

The time average Poynting vector is purely radial,

$$\langle \mathbf{S} \rangle = \frac{c\hat{\mathbf{r}}}{8\pi}Re(E_\phi B_\theta^\star) = \frac{ck^4m^2}{8\pi r^2}\sin^2\theta. \tag{35}$$

The time-average angular distribution of radiated power is,

$$\frac{dP}{d\Omega} = r^2\langle \mathbf{S}\cdot\hat{\mathbf{r}}\rangle = \frac{ck^4m^2}{8\pi}\sin^2\theta. \tag{36}$$

The time-average radiated power is,

$$P = \int\frac{dP}{d\Omega}d\Omega = \frac{ck^4m^2}{3} = \frac{16\pi^6a^4I^2}{3c\lambda^4} = \frac{1}{2}R_{\rm rad}I^2, \tag{37}$$

where the radiation resistance is,

$$R_{\rm rad} = \frac{32\pi^6a^4}{3c\lambda^4} = 307,646\frac{a^4}{\lambda^4}\Omega, \tag{38}$$

recalling that $1/c = 30\ \Omega$.

A Appendix: Hertz Vectors and Scalars

While the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ of many antennas (including a point dipole) can be deduced from considerations of the vector potential $\mathbf{A}$, it is also useful to follow a line of thought due to Hertz in which the scalar and vector potentials, V and $\mathbf{A}$, can be related to another vector, the Hertz vector [1], and hence to the scalar components of the latter.

In empty space the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ can be derived from the scalar and vector potentials V and $\mathbf{A}$ according to,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{39}$$

in Gaussian units. We shall work in the Lorentz gauge where the potentials obey the auxiliary condition,

$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{1}{c}\frac{\partial V}{\partial t}. \tag{40}$$

The potentials then obey the wave equations of the form,

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}, \qquad \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -4\pi\rho, \tag{41}$$

where ρ and $\mathbf{J}$ are the charge and current densities of the sources of the waves. Formal solutions for the (retarded) potentials have been given by Lorenz [14],

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{x}', t' = t - r/c)}{r} d\mathrm{Vol}', \qquad V(\mathbf{x}, t) = \int \frac{\rho(\mathbf{x}', t' = t - r/c)}{r} d\mathrm{Vol}', \tag{42}$$

where $r = |\mathbf{x} - \mathbf{x}'|$.

The present problem involves a loop antenna in which we may suppose that there is nowhere any accumulation of charge, so that $\rho = 0$ and hence the scalar potential V is zero as well. In this case, the charge conservation simplifies to,

$$\boldsymbol{\nabla} \cdot \mathbf{J} = 0, \tag{43}$$

and the Lorentz gauge condition (40) tells us that,

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0, \tag{44}$$

so that the vector potential can be written as the curl of another vector, which we will call $\mathbf{Z}_{\mathrm{M}}$, the magnetic Hertz vector,[2]

$$\mathbf{A} = \boldsymbol{\nabla} \times \mathbf{Z}_{\mathrm{M}} \qquad V = 0. \tag{45}$$

While 3-space vectors have, in general, 3 independent components, the divergence condition (44) implies that in this problem the potentials $\mathbf{A}$ and $\mathbf{Z}_{\mathrm{M}}$ actually have only two independent scalar components.

[2]When the charge density ρ has a nontrivial time dependence (as in the case of linear dipole antennas [3]), one can introduce an appropriate electric Hertz vector $\mathbf{Z}_{\mathrm{E}}$ as well. In this case, $V = -\boldsymbol{\nabla} \cdot \mathbf{Z}_{\mathrm{E}}$, $\mathbf{A} = \partial \mathbf{Z}_{\mathrm{E}}/\partial ct$. See, for example, [15].

In the far zone of an antenna the electromagnetic fields are transverse to the vector $\mathbf{r}$ from the center of the antenna. In the near zone the fields are more complicated, but we may hope that one of $\mathbf{E}$ or $\mathbf{B}$ remains transverse to $\mathbf{r}$ for antennas of sufficient symmetry. That is, we are led to seek fields that can be characterized as transverse electric (TE) or transverse magnetic (TM).

This suggests that we take one of the two independent components of the Hertz vector $\mathbf{Z}_{\rm M}$ to be radial, say $\mathbf{r}Z_{\rm TE}$, and the other to be transverse, say $\mathbf{r}\times\boldsymbol{\nabla}Z_{\rm TM} = \boldsymbol{\nabla}\times\mathbf{r}Z_{\rm TM}$. That is,

$$\mathbf{Z}_{\rm M} = \mathbf{r}Z_{\rm TE} + \mathbf{r}\times\boldsymbol{\nabla}Z_{\rm TM}. \tag{46}$$

Then,

$$\begin{aligned} \mathbf{A} &= \boldsymbol{\nabla}\times\mathbf{Z}_{\rm M} = \boldsymbol{\nabla}\times\mathbf{r}Z_{\rm TE} + \boldsymbol{\nabla}\times(\mathbf{r}\times\boldsymbol{\nabla}Z_{\rm TM}) \\ &= \boldsymbol{\nabla}Z_{\rm TE}\times\mathbf{r} + \mathbf{r}\nabla^2 Z_{\rm TM} - (\mathbf{r}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}Z_{\rm TM} - 2\boldsymbol{\nabla}Z_{\rm TM} \\ &= \boldsymbol{\nabla}Z_{\rm TE}\times\mathbf{r} + \mathbf{r}\nabla^2 Z_{\rm TM} - \boldsymbol{\nabla}(\mathbf{r}\cdot\boldsymbol{\nabla}Z_{\rm TM} - Z_{\rm TM}), \end{aligned} \tag{47}$$

using the identity that,

$$\boldsymbol{\nabla}(\mathbf{r}\cdot\boldsymbol{\nabla}Z_{\rm TM}) = (\mathbf{r}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}Z_{\rm TM} + \boldsymbol{\nabla}Z_{\rm TM}. \tag{48}$$

Thus, if $Z_{\rm TM} = 0$, then $\mathbf{A} = \boldsymbol{\nabla}Z_{\rm TE}\times\mathbf{r}$ is transverse, and hence the electric field $\mathbf{E} = -\partial\mathbf{A}/\partial ct$ is transverse also. On the other hand, if $Z_{\rm TE} = 0$, then $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} = \boldsymbol{\nabla}(\nabla^2 Z_{\rm TM})\times\mathbf{r}$ is transverse. Hence, the subscripts TE and TM in the decomposition (46) indeed label scalar superpotentials that lead to transverse electric and transverse magnetic fields, respectively.

In view of the definition (46) of the magnetic Hertz vector and the wave equation (41) for the vector potential, we have,

$$\nabla^2\mathbf{A} = \nabla^2(\boldsymbol{\nabla}\times\mathbf{Z}_{\rm M}) = \boldsymbol{\nabla}\times\nabla^2\mathbf{Z}_{\rm M} = \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \frac{4\pi}{c}\mathbf{J} = \boldsymbol{\nabla}\times\frac{1}{c^2}\frac{\partial^2\mathbf{Z}_M}{\partial t^2} - \frac{4\pi}{c}\mathbf{J} \tag{49}$$

If we write the current density as,

$$\mathbf{J} = c\boldsymbol{\nabla}\times\mathbf{M}, \tag{50}$$

in terms of a magnetization density $\mathbf{M}$, the magnetic Hertz vector satisfies the wave equation,

$$\nabla^2\mathbf{Z}_{\rm M} - \frac{1}{c^2}\frac{\partial^2\mathbf{Z}_{\rm M}}{\partial t^2} = -4\pi\mathbf{M} \tag{51}$$

(although strictly speaking, the proper wave equation is the curl of eq. (51)). This justifies the alternative terminology that the magnetic Hertz vector is a polarization potential, with the formal solution,

$$\mathbf{Z}_{\rm M}(\mathbf{x},t) = \int \frac{\mathbf{M}(\mathbf{x}', t' = t - r/c)}{r} d{\rm Vol}'. \tag{52}$$

The magnetization density $\mathbf{M}$ has the same formal relation to the current density $\mathbf{J} = c\boldsymbol{\nabla}\times\mathbf{M}$ as does the Hertz vector $\mathbf{Z}_M$ to the vector potential $\mathbf{A} = \boldsymbol{\nabla}\times\mathbf{Z}_M$. Hence, following eq. (46), we can write the magnetization as,

$$\mathbf{M} = \mathbf{r}\psi_{\rm TE} + \mathbf{r}\times\boldsymbol{\nabla}\psi_{\rm TM}, \tag{53}$$

and the current density as,

$$\mathbf{J} = c\boldsymbol{\nabla}\times\mathbf{M} = c\boldsymbol{\nabla}\times\mathbf{r}\psi_{\rm TE} + c\boldsymbol{\nabla}\times(\mathbf{r}\times\boldsymbol{\nabla}\psi_{\rm TM}), \tag{54}$$

in terms of two scalar source fields $\psi_{\rm TE}$ and $\psi_{\rm TM}$.

We also seek the wave equations for the scalar superpotentials $Z_{\rm TE}$ and $Z_{\rm TM}$, for which we need to evaluate $\nabla^2(\mathbf{r}Z_{\rm TE})$ and $\nabla^2(\mathbf{r}\times\boldsymbol{\nabla}Z_{\rm TM.})$. This is more easily done in rectangular coordinates than in spherical coordinates. We write the ith component of $\nabla^2(\mathbf{r}Z_{\rm TE})$ as,

$$\begin{aligned}\nabla^2(x_i Z_{\rm TE}) &= \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}(x_i Z_{\rm TE}) = \frac{\partial}{\partial x_j}\left(\delta_{ij}Z_{\rm TE} + x_i\frac{\partial Z_{\rm TE}}{\partial x_j}\right) = 2\delta_{ij}\frac{\partial Z_{\rm TE}}{\partial x_j} + x_i\frac{\partial^2 Z_{\rm TE}}{\partial x_j\partial x_j} \\ &= x_i\nabla^2 Z_{\rm TE} + 2\frac{\partial Z_{\rm TE}}{\partial x_i},\end{aligned} \tag{55}$$

using the summation convention over repeated indices. That is,

$$\nabla^2(\mathbf{r}Z_{\rm TE}) = \mathbf{r}\nabla^2 Z_{\rm TE} + 2\boldsymbol{\nabla}Z_{\rm TE}. \tag{56}$$

Similarly, the ith component of $\nabla^2(\mathbf{r}\times\boldsymbol{\nabla}Z_{\rm TM.})$ is,

$$\begin{aligned}\nabla^2\left(\epsilon_{ijk}x_j\frac{\partial Z_{\rm TM}}{\partial x_k}\right) &= \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_l}\left(\epsilon_{ijk}x_j\frac{\partial Z_{\rm TM}}{\partial x_k}\right) = \frac{\partial}{\partial x_l}\epsilon_{ijk}\left(\delta_{jl}\frac{\partial Z_{\rm TM}}{\partial x_k} + x_j\frac{\partial^2 Z_{\rm TM}}{\partial x_k\partial x_l}\right) \\ &= 2\epsilon_{ijk}\delta_{jl}\frac{\partial^2 Z_{\rm TM}}{\partial x_k\partial x_l} + \epsilon_{ijk}x_j\frac{\partial^3 Z_{\rm TM}}{\partial x_k\partial x_l\partial x_l} = 2\epsilon_{ijk}\frac{\partial^2 Z_{\rm TM}}{\partial x_k\partial x_j} + \epsilon_{ijk}x_j\frac{\partial\nabla^2 Z_{\rm TM}}{\partial x_k} \\ &= \epsilon_{ijk}x_j\frac{\partial\nabla^2 Z_{\rm TM}}{\partial x_k}\end{aligned} \tag{57}$$

Hence,

$$\nabla^2(\mathbf{r}\times\boldsymbol{\nabla}Z_{\rm TM.}) = \mathbf{r}\times\boldsymbol{\nabla}\nabla^2 Z_{\rm TM}. \tag{58}$$

As remarked above, it is the curl of the wave equation (51) for the magnetic Hertz vector (46) which has physical significance. This can now be rewritten as,

$$\begin{aligned}-\frac{4\pi}{c}\mathbf{J} &= -4\pi\boldsymbol{\nabla}\times\mathbf{r}\psi_{\rm TE} - 4\pi\boldsymbol{\nabla}\times(\mathbf{r}\times\boldsymbol{\nabla}\psi_{\rm TM}) \\ &= \boldsymbol{\nabla}\times\left[\mathbf{r}\nabla^2 Z_{\rm TE} + 2\boldsymbol{\nabla}Z_{\rm TE} + \mathbf{r}\times\boldsymbol{\nabla}\nabla^2 Z_{\rm TM} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\mathbf{r}Z_{\rm TE} + \mathbf{r}\times\boldsymbol{\nabla})Z_{\rm TM}\right] \\ &= \boldsymbol{\nabla}\times\mathbf{r}\left[\nabla^2 Z_{\rm TE} - \frac{1}{c^2}\frac{\partial^2 Z_{\rm TE}}{\partial t^2}\right] + \boldsymbol{\nabla}\times\left\{\mathbf{r}\times\boldsymbol{\nabla}\left[\nabla^2 Z_{\rm TM} - \frac{1}{c^2}\frac{\partial^2 Z_{\rm TM}}{\partial t^2}\right]\right\}.\end{aligned} \tag{59}$$

Thus, at some length we deduce that the Hertz scalars $Z_{\rm TE}$ and $Z_{\rm TM}$ also satisfy the wave equations,

$$\nabla^2 Z_{\rm TE} - \frac{1}{c^2}\frac{\partial^2 Z_{\rm TE}}{\partial t^2} = -4\pi\psi_{\rm TE}, \qquad \nabla^2 Z_{\rm TM} - \frac{1}{c^2}\frac{\partial^2 Z_{\rm TM}}{\partial t^2} = -4\pi\psi_{\rm TM}. \tag{60}$$

These wave equations, of course, have the formal, retarded solutions,

$$Z(\mathbf{x},t) = \int\frac{\psi(\mathbf{x}',t'=t-r/c)}{r}d{\rm Vol}', \tag{61}$$

where $r = |\mathbf{x} - \mathbf{x}'|$.

We now restrict our attention to sources, and hence waves, of pure angular frequency ω, writing,

$$\psi(\mathbf{x}, t) = \psi(\mathbf{x})e^{-i\omega t}, \qquad \text{and} \qquad Z(\mathbf{x}, t) = Z(\mathbf{x})e^{-i\omega t} \tag{62}$$

for the scalar source fields $\psi_{\rm TE}$ and $\psi_{\rm TM}$ and for the Hertz scalars $Z_{\rm TE}$ or $Z_{\rm TM}$. The scalar wave equations (60) then reduce to the Helmholtz equation,

$$\nabla^2 Z(\mathbf{x}) + \frac{\omega^2}{c^2} Z = -4\pi\psi(\mathbf{x}), \tag{63}$$

whose formal solution is,

$$Z(\mathbf{x}) = \int \frac{\psi(\mathbf{x}')e^{ikr}}{r} d\text{Vol}'. \tag{64}$$

As discussed above in sec. 2.1, in case the sources have azimuthal symmetry and the region of interest extends over all polar angles θ, a suitable expansion for a Hertz scalar is,

$$Z(r < a, \theta, t) = i \sum_{\text{odd } n} A_n h_n^{(1)}(ka) j_n(kr) P_n^1(\cos\theta) e^{-i\omega t}, \tag{65}$$

$$Z(r > a, \theta, t) = i \sum_{\text{odd } n} A_n j_n(ka) h_n^{(1)}(kr) P_n^1(\cos\theta) e^{-i\omega t}. \tag{66}$$

Multipole expansions of the source fields $\psi_{\rm TE}$ and $\psi_{\rm TM}$ have been given in [17], where they are called ψ and χ, respectively. These are variants on the multipole expansions in term of vector spherical harmonics given, for example, in [16]. The lowest-order moment contributing to $\psi_{\rm TE}$ is the magnetic dipole moment, with respect to whose axis the radiation electric field is well known to the azimuthal (and hence transverse). The lowest-order moment contributing to $\psi_{\rm TM}$ has been called the toroid moment [17], which is nonzero for poloidal (also called meridianal) current distributions as flow on the surface of a torus with no azimuthal component. In this case the radiation magnetic field is azimuthal, and the radiation electric field has the form of that due to an electric dipole, but proportional to k^3 rather than k^2. For an example, see [18].

B Appendix: Use of Toroidal Coordinates (July 10, 2020)

The present problem has a formal solution in toroidal coordinates. See, for example, [19], where sec. 3 discusses (TE) waves with azimuthal electric field.

The example of a toroidal ring with a gap and a DC azimuthal current is discussed in [20].

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