Snowball/Log Rolling down a Snowy Slope

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1 Problem

An ostensibly simple problem is the motion of a snowball (or better, a cylinder/log) that rolls without slipping down a snowy slope, accumulating mass as it moves. A naïve approximation is that the cross section of the ball/log remains circular at all times (which implies that snow moves from the slope to be instantaneously distributed over the entire surface of the rolling object, thereby instantaneously acquiring kinetic energy, momentum and angular momentum).[1] Show, that this (unphysical) assumption leads to different equations of motion via a force/torque analyses about the center or mass of the log and about its line of contact with the slope, as well as different ones based on energy conservation and a Lagrangian.

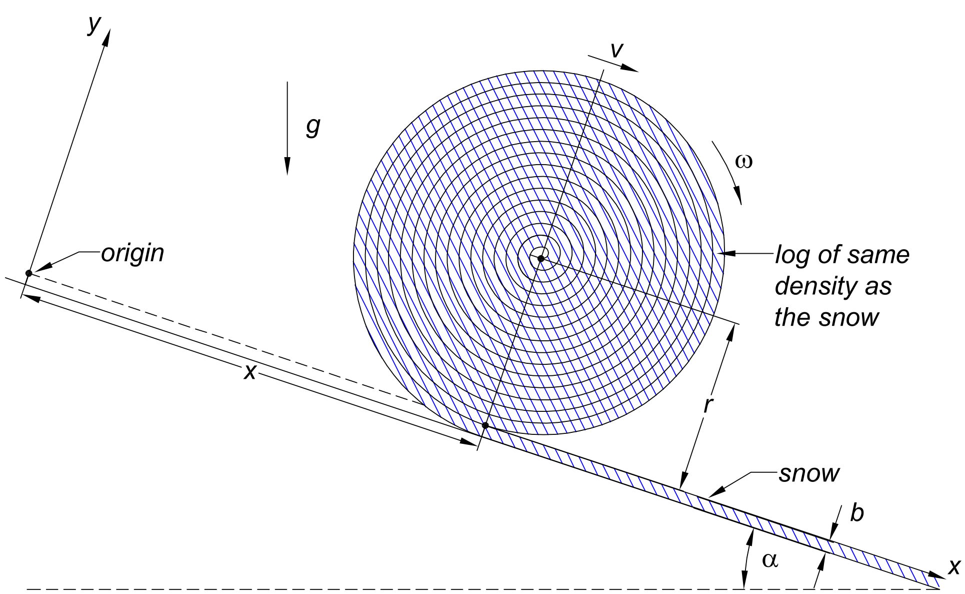
Does the motion have a simple, asymptotic (terminal) character?

Consider also the motion during the first and subsequent full turns of rolling, for which a more accurate analysis can be given.[2]

2 Solution

This problem is a variant on the many examples discussed by the author in [5].

We consider a cylindrical log that rolls down a slope of angle α to the horizontal which is covered by a layer of snow of depth b (normal to the slope), as sketched in the figure below.



[1]This assumption is tacitly made, for example, in [1] and sec. V.D of [2], and was explicit in Art. 203, p. 261, of [3] (which may have been the first discussion of this problem).

[2]Another (simpler) problem in which the analysis can be partitioned into segments of fixed azimuthal rotation is a hexagonal pencil that rolls without slipping on an incline [4].

The rolling log (of mass density ρ, the same as that of the snow) accumulates all of the snow that it encounters, without loss of energy to possible compaction of the snow. As such, the shape of the cylinder is not quite circular, but an appealing approximation is that the cylinder remains circular at all time, with radius $r(t)$.

In this approximation, the mass of the log (of length l) is $m = \pi\rho r^2 l$, and the rate of accumulation of mass is,

$$\frac{dm}{dt} = 2\pi\rho r l \frac{dr}{dt} = \rho b l v = \frac{mbv}{\pi r^2}\,, \qquad \frac{dr}{dt} = \frac{bv}{2\pi r} = \frac{\omega b}{2\pi}\,, \tag{1}$$

where $v = dx/dt$ is the speed of the center of the log down the slope, and $\omega = v/r$ is the angular velocity of the log.

2.1 Force/Torque Analysis

The force (component) F_x parallel to the slope (and uphill) is related to the x-component of the momentum, $p_x = mv$, of the log by,[3]

$$mg\sin\alpha - F_x = \frac{dp_x}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt} = m\frac{dv}{dt} + \frac{mbv^2}{\pi r^2}\,, \tag{2}$$

where g is the acceleration due to gravity.

2.1.1 Torque Analysis about the Center of Mass of the Log

In addition, the torque equation with respect to the center of mass of the log is,

$$\begin{aligned} rF_x &= \tau = \frac{dL}{dt} = \frac{d(I\omega)}{dt} = \frac{d(mrv/2)}{dt} = \frac{rv}{2}\frac{dm}{dt} + \frac{mv}{2}\frac{dr}{dt} + \frac{mr}{2}\frac{dv}{dt} \\ &= \frac{mbv^2}{2\pi r} + \frac{mbv^2}{4\pi r} + \frac{mr}{2}\frac{dv}{dt}\,, \end{aligned} \tag{3}$$

in the approximation that the moment of inertia of the (cylindrical) log about its axis is $I = mr^2/2$. Combining eqs. (2) and (3), we obtain an equation of motion,[4]

$$\frac{3m}{2}\frac{dv}{dt} = mg\sin\alpha - \frac{7mbv^2}{4\pi r^2}\,, \qquad a = \frac{dv}{dt} = \frac{2g}{3}\sin\alpha - \frac{7bv^2}{6\pi r^2}\,. \tag{4}$$

[3]The normal force F_y is related by $F_y - mg\cos\theta = dp_y/dt = d(m\,dr/dt)/dt$, but we don't need to pursue this, as F_y exerts no torque about the center of mass of the log (or about its line of contact with the slope).

[4]On p. 262 of [3], Loney found the equation of motion (4), and noted that the accumulation of snow over distance x increases the radius of the log from r_0 to r according to $\pi r^2 = \pi r_0^2 + b\,x$. On changing variables, $u = v^2 = \dot{x}^2$, then $2a = 2\ddot{x} = \dot{u}/\dot{x} = du/dx = (4g/3)\sin\alpha - 7b\,u/3(\pi r_0^2 + b\,x)$. This can be integrated to give $u = \dot{x}^2 = 2g\sin\alpha(\pi r_0^2 + b\,x)/5b + C/(\pi r_0^2 + b\,x)^{7/3}$, whose time derivative tells us that $a = (g/5)\sin\alpha - 7b\,C/3(\pi r_0^2 + b\,x)^{10/3}$, which goes to the constant value $(g/5)\sin\alpha$ for large x.

2.1.2 Torque Analysis about the Line of Contact of the Log and Slope

We could also consider the torque equation with respect to the line of contact of the log with the slope, which line is instantaneously at rest.

$$\begin{aligned} rmg \sin\alpha &= \tau_C = \frac{dL_C}{dt} = \frac{d(I_C\,\omega)}{dt} = \frac{d(3mrv/2)}{dt} = \frac{3rv}{2}\frac{dm}{dt} + \frac{3mv}{2}\frac{dr}{dt} + \frac{3mr}{2}\frac{dv}{dt} \\ &= \frac{3mbv^2}{2\pi r} + \frac{3mbv^2}{4\pi r} + \frac{3mr}{2}\frac{dv}{dt}\,, \end{aligned} \qquad (5)$$

noting that the moment of inertia about the point of contact is $I_C = I + mr^2 = 3mr^2/2$. The resulting equation of motion is (without need to consider the force at the line of contact),

$$a = \frac{dv}{dt} = \frac{2g}{3}\sin\alpha - \frac{3bv^2}{2\pi r^2}\,. \qquad (6)$$

2.1.3 Comments

The two equations of motion, (4) and (6), differ, which alerts us to the possibility that the preceding analysis is not sufficiently accurate.

In the limit of no snow on the slope, $b \to 0$, both torque analyses yield the well known result (reviewed in the Appendix) that the acceleration of a solid cylinder which rolls without slip down a slope of angle α is $\frac{2}{3}g\sin\alpha$. And, if the correction to the acceleration in case of a slope with thickness b is proportional to that thickness, but independent of g, then dimensional analysis tells us that the correction is proportional to bv^2/r^2. The task of a successful analysis of the motion is to identify the numerical coefficient of this term, which the torque analyses apparently fail to do in a convincing manner.

2.2 Energy Analysis

A different analysis can be based on the approximation that no energy is dissipated by the accumulation of snow on the rolling log, or by air resistance. Then, the mechanical energy, $E = T + V$ is constant.

The kinetic energy T is related by,

$$T = \frac{m}{2}\left[v^2 + \left(\frac{dr}{dt}\right)^2\right] + \frac{I\omega^2}{2} = \frac{3mv^2}{4} + \frac{m}{2}\left(\frac{dr}{dt}\right)^2 = \frac{3\pi\rho l r^2 v^2}{4} + \frac{\rho l b^2 v^2}{8\pi}\,. \qquad (7)$$

For the potential energy, we suppose that the log started from rest with a radius r_0 and mass $m_0 = \pi\rho r_0^2 l$ and rolled distance x down the slope to its present position. During this time, it accumulated snow of volume blx, such that the present radius r and mass m are related by,

$$\pi r^2 = \pi r_0^2 + bx, \qquad \frac{\partial r}{\partial x} = \frac{b}{2\pi r}\,, \qquad m = m_0 + \rho b l x, \qquad (8)$$

in the approximation that the log is always circular. Then, relative to the origin, the initial potential energy V_0', and the present potential energy V', of the log plus accumulated snow are,

$$V_0' = m_0 g r_0 \cos\alpha - \rho b l x g \frac{x \sin\alpha + b\cos\alpha}{2}\,, \qquad V' = mg(r\cos\alpha - x\sin\alpha). \tag{9}$$

Redefining the initial potential to be zero, the present potential energy V is,

$$\begin{aligned} V &= mg(r\cos\alpha - x\sin\alpha) + \rho b l x g \frac{x\sin\alpha + b\cos\alpha}{2} - m_0 g r_0 \cos\alpha \\ &= \rho l g \frac{(2\pi r^3 + b^2 x)\cos\alpha + (bx^2 - 2\pi r^2 x)\sin\alpha}{2} - m_0 g r_0 \cos\alpha. \end{aligned} \tag{10}$$

In the approximation of conservation of mechanical energy $E = T + V$, we have that,

$$0 = \frac{3\pi\rho l r^2 v^2}{4} + \frac{\rho l b^2 v^2}{8\pi} + \rho l g \frac{(2\pi r^3 + b^2 x)\cos\alpha + (bx^2 - 2\pi r^2 x)\sin\alpha}{2} - m_0 g r_0 \cos\alpha. \tag{11}$$

Taking the time derivative of the energy (and dividing by ρl), we obtain the equation of motion,

$$0 = \frac{3\pi r^2 v}{2}\frac{dv}{dt} + \frac{3bv^3}{4} + \frac{b^2 v}{4\pi}\frac{dv}{dt} + \frac{gv\cos\alpha}{2}(3br + b^2) + g\sin\alpha(bvx - \pi r^2 v - bvx), \tag{12}$$

$$\frac{3\pi r^2}{2}\frac{dv}{dt}\left(1 + \frac{b^2}{6\pi^2 r^2}\right) = \pi r^2 g \sin\alpha - \frac{3bv^2}{4} - \frac{g\cos\alpha}{2}(3br + b^2), \tag{13}$$

$$\frac{dv}{dt}\left(1 + \frac{b^2}{6\pi^2 r^2}\right) = \frac{2g}{3}\sin\alpha - \frac{bv^2}{2\pi r^2} - \frac{g\cos\alpha}{3\pi r^2}(3br + b^2). \tag{14}$$

If we neglect the small terms in b^2, the equation of motion is,

$$\frac{dv}{dt} = \frac{2g}{3}\sin\alpha - \frac{bv^2}{2\pi r^2} - \frac{gb}{\pi r}\cos\alpha, \tag{15}$$

which disagrees with both eqs. (4) and (6), except in the limit of no snow, $b \to 0$. Note the appearance in eq. (15) of the term proportional to gb/r, which did not arise in the torque analyses.

2.3 Lagrangian Method

For completeness, we recall that the equation of motion can also be deduced from the Lagrangian, $\mathcal{L} = T - V$, although strictly this method is for the motion of a rigid body. Here, we take x as the single, independent coordinate, and note that $\dot{x} = v$. Then, recalling eqs. (1) and (8),

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{\partial\mathcal{L}}{\partial x}\,, \tag{16}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{\partial T}{\partial v} = \frac{3\pi\rho l r^2 v}{2} + \frac{\rho l b^2 v}{4\pi}\,, \tag{17}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{3\rho l \pi r^2}{2}\frac{dv}{dt}\left(1 + \frac{b^2}{6\pi^2 r^2}\right) + \frac{3\rho l b v^2}{2}\,, \tag{18}$$

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\rho l g}{2}[(3rb + b^2)\cos\alpha + (2bx - 2\pi r^2 - 2bx)\sin\alpha], \tag{19}$$

$$\frac{3\pi r^2}{2}\frac{dv}{dt}\left(1 + \frac{b^2}{6\pi^2 r^2}\right) = \pi r^2 g \sin\alpha - \frac{3bv^2}{2} - \frac{g\cos\alpha}{2}(3rb + b^2), \tag{20}$$

$$\frac{dv}{dt}\left(1 + \frac{b^2}{6\pi^2 r^2}\right) = \frac{2g}{3}\sin\alpha - \frac{bv^2}{\pi r^2} - \frac{g\cos\alpha}{3\pi r^2}(3rb + b^2). \tag{21}$$

The equation of motion (21) differs slightly from eq. (14), as well as from eqs. (4) and (6), although all four equations agree in the limit that $b \to 0$.

It appears that the approximation of the log as circular at all times as it rolls down a snowy slope does not lead to a consistent equation of motion.

2.4 Terminal Acceleration of the Rolling Log on the Snowy Slope

It is claimed in [1] that although the acceleration of the rolling log is not constant, it approaches a constant (terminal) value, such that the motion of the log is eventually similar to that of rolling on a slope without snow.[5]

The equation of motion used in [1] is based on a torque analysis about the center of mass, as in our sec. 2.1.1 above. Because this equation of motion was derived using the unphysical assumption that the ball/log had a circular cross section at all times, the inference of a nonzero terminal acceleration is doubtful.

We now present an approximate analysis (that will support the existence of a terminal acceleration) which avoids use of the assumption that the log has an exactly circular cross section, instead taking it to be only approximately circular.

The total kinetic energy T of the rolling log can be written as,[6]

$$T = \frac{m v_{\rm cm}^2}{2} + \frac{I_{\rm cm}\omega^2}{2}\,, \tag{22}$$

where all of the parameters vary with time. We define $r_{\rm cm}$ as the distance from the center of mass of the log to the line of contact of the log (which rolls without slipping) on the snowy slope. The mass and moment of inertia of the snow-covered log are approximately related by,

$$m \approx \pi\rho r_{\rm cm}^2 l, \qquad \text{and} \qquad I_{\rm cm} \approx \frac{m r_{\rm cm}^2}{2}\,, \tag{23}$$

where ρ is the mass density of the log, and l is its length. In addition, we approximate the center-of-mass velocity $v_{\rm cm}$ as,

$$v_{\rm cm} \approx \omega r_{\rm cm}, \tag{24}$$

[5]This result was obtained in an analysis that (tacitly) assumed conservation of mechanical energy. In practice, energy is not conserved in the rolling process, which is subject to various forms of energy dissipation that, in general, lead to a terminal velocity (zero terminal acceleration) of the motion. Analysis of these processes is beyond the scope of this note (and of [1]).

[6]This form does not hold in the approximation that the log is always circular in cross section, as this requires instantaneous motion of snow from the line of contact to the entire surface of the log.

which holds exactly only if vectors $\mathbf{v}_{\rm cm}$ and $\mathbf{r}_{\rm cm}$ are perpendicular. Then, the kinetic energy is approximately,

$$T \approx \frac{3mv_{\rm cm}^2}{4}\,, \tag{25}$$

and its time derivative is approximately,

$$\dot T \approx \frac{3\dot m v_{\rm cm}^2}{4} + \frac{3mv_{\rm cm}\dot v_{\rm cm}}{2}\,. \tag{26}$$

The rolling log accumulates mass from the snowy slope at rate,

$$\dot m = \rho b l v_{\rm contact} \approx \rho b l v_{\rm cm} \approx \frac{mbv_{\rm cm}}{\pi r_{\rm cm}^2} \qquad \big(\text{and} \qquad \pi r_{\rm cm}^2 \approx \pi r_0^2 + b x_{\rm cm}\big)\,, \tag{27}$$

noting that the velocity of the line of contact of the log with the slope is approximately the same as the velocity of the center of mass of the log. We can also relate the rate of change $\dot m$ of mass of the log to the rate of change $\dot r_{\rm cm}$ of its radius as approximately,

$$\dot m = 2\pi\rho l r_{\rm cm}\dot r_{\rm cm}, \tag{28}$$

which together with eq. (27) implies that,

$$\dot r_{\rm cm} \approx \frac{bv_{\rm cm}}{2\pi r_{\rm cm}}\,. \tag{29}$$

As the log rolls down the slope its gravitational potential energy V decreases at a rate approximately given by,

$$\dot V \approx -mgv_{\rm cm}\sin\alpha, \tag{30}$$

where we have neglected the small rate of change of the potential energy associated with the changing mass of the log.

In the approximation of conservation of mechanical energy, we have that,

$$\dot T + \dot V = 0 \approx \frac{3\dot m v_{\rm cm}^2}{4} + \frac{3mv_{\rm cm}\dot v_{\rm cm}}{2} - mgv_{\rm cm}\sin\alpha \approx \frac{3mv_{\rm cm}}{2}\left(\dot v_{\rm cm} - \frac{2g\sin\alpha}{3} + \frac{bv_{\rm cm}^2}{2\pi r_{\rm cm}^2}\right). \tag{31}$$

Thus, we obtain an approximate equation of motion,

$$\dot v_{\rm cm} \approx \frac{2g\sin\alpha}{3} - \frac{bv_{\rm cm}^2}{2\pi r_{\rm cm}^2}\,, \tag{32}$$

which is similar to (but not the same as) the four equations of motion previously deduced using the unphysical assumption of a circular cross section of the log at all times. This gives some confidence that the form of these equations of motion has (approximate) physical relevance.

A clever suggestion in [1] is to take the time derivative of the equation of motion (32),

$$\ddot v_{\rm cm} \approx \frac{bv_{\rm cm}\dot r_{\rm cm}}{2\pi r_{\rm cm}^3} - \frac{b\dot v_{\rm cm}}{2\pi r_{\rm cm}^2} \approx \frac{b}{2\pi r_{\rm cm}^2}\left(\frac{bv_{\rm cm}^2}{2\pi r_{\rm cm}^3} - \dot v_{\rm cm}\right), \tag{33}$$

recalling eq. (29), which indicates that the acceleration $\dot{v}_{\rm cm}$ takes on a constant (terminal) value,

$$a_{\rm term} \approx \left. \frac{b v_{\rm cm}^2}{2\pi r_{\rm cm}^3} \right|_{\rm term} . \tag{34}$$

Using this value in the equation of motion (32), we learn that the terminal acceleration is,

$$a_{\rm term} \approx \frac{g \sin\alpha}{3} , \tag{35}$$

which is 1/2 the acceleration of a solid cylinder that rolls without slipping on a slope without snow. That is, the log starts from rest on the snowy slope with acceleration $2g\sin\alpha/3$, but decelerates (while its velocity increases) until the acceleration is only $g\sin\alpha/3$, after which the velocity increases linearly with time.

From eq. (32), we see that $\dot{v}_{\rm cm}$ would be zero if $bv_{\rm cm}^2/2\pi r_{\rm cm}^2 = 2a_{\rm term}$. For large times, we have that $v_{\rm cm} \approx a_{\rm term}t$, $x_{\rm cm} \approx a_{\rm cm}t^2/2$, and from eq. (31), $\pi r_{\rm cm}^2 \approx bx_{\rm cm}$, such that $bv_{\rm cm}^2/2\pi r_{\rm cm}^2 \approx a_{\rm term}$, so $\dot{v}_{\rm cm}$ remains at $a_{\rm term}$ at large times, and never drops to zero (i.e., there is no terminal velocity in the present approximations).

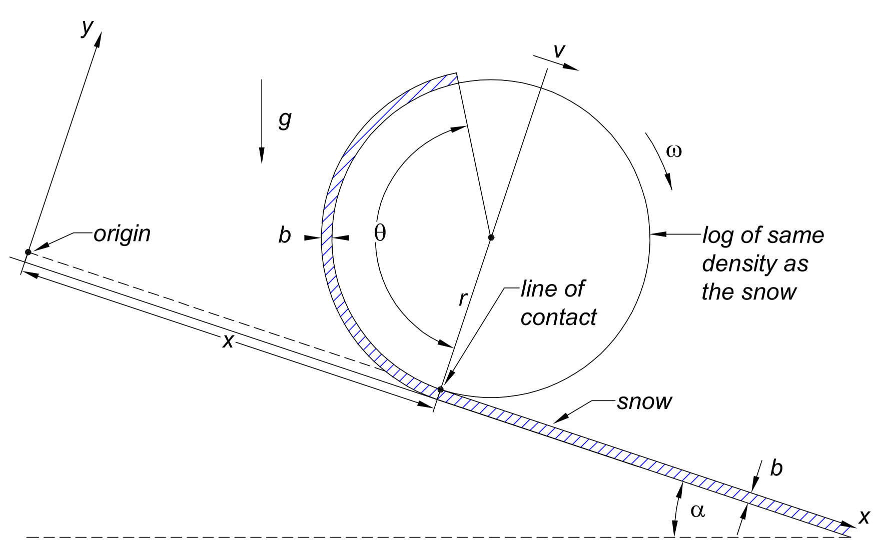
The approximate character of the derivation of the equation of motion (32) does not exclude the existence of an additional small term, such as,

$$-\frac{gb\cos\alpha}{\pi r} , \tag{36}$$

that was found in secs. 2.2-3 above. As the radius r of the rolling log grows with time, the effect of such a term becomes negligible, and the acceleration approaches a constant, terminal value, albeit this approach is slower than in the absence of the term (36).

3 Single Turn of a Rolling Log

We can make a more physical analysis, without the approximation that the rolling log is circular at all times, for the first full turn of rolling, as sketched below.



When the "center" of the log, at (x, r_0), has moved distance x down the slope, an arc of angle $\theta = x/r_0$ of snow has accumulated over a portion of the surface of the log, where r_0 is the initial radius of the (initially circular) log of length l

We make the (unphysical) assumption that the snow within the circular arc of thickness b has angular velocity ω about the line of contact at $(x, 0)$, which implies that the snow at $(x, -b)$ instantaneously takes on velocity ωb. While this is a small (unphysical) effect, it limits the accuracy of the analysis for large times. Of course, at the end of the first full turn of rolling, the "step" in the thickness of the snow on the rolling log encounters the snowy slope, and the analysis is not readily continued.

The initial mass of the log is,[7]

$$m_0 = \pi\rho\, l\, r_0^2, \tag{37}$$

and the mass of accumulated snow is, neglecting terms of order b^2/r_0^2,

$$m_s = \pi\rho\, l[(r_0 + b)^2 - r_0^2]\frac{\theta}{2\pi} \approx \rho\, b\, l\, r_0\, \theta = m_0 \frac{b\,\theta}{\pi r_0}\,, \tag{38}$$

taking the radial thickness of the layer accumulated on the log to be b, and the density ρ of the accumulated snow to be the same as that of the initial log. We also suppose that the thickness of the snow on the slope is b, which implies that the density of the snow on the slope is slightly greater than ρ.

Once the log has accumulated snow, its center of mass is not at the nominal center (x, r_0) of the log. To analyze this, we first compute the center of mass (x_s, y_s) coordinates of the accumulated snow, neglecting terms of order b^2/r_0^2,

$$\begin{aligned} m_s\, x_s &= \int_{r_0}^{r_0+b}\int_0^{\theta} \rho\, l\, r'\, dr' d\theta'\,(x - r'\sin\theta') = m_s x - \rho\, l \frac{(r_0+b)^3 - r_0^3}{3}\,(1 - \cos\theta) \\ &\approx m_s\, x - \rho\, l\, r_0^2\, b\,(1 - \cos\theta) = m_s\, x - m_0 \frac{b}{\pi}\,(1 - \cos\theta)\,, \end{aligned} \tag{39}$$

$$\begin{aligned} m_s\, y_s &= \int_{r_0}^{r_0+b}\int_0^{\theta} \rho l\, r'\, dr' d\theta'\,(r - r'\cos\theta') = m_s r - \rho l \frac{(r_0+b)^3 - r_0^3}{3}\sin\theta \\ &\approx m_s\, r_0 - \rho\, l\, r_0^2\, b\sin\theta = m_s\, r_0 - m_0 \frac{b}{\pi}\sin\theta. \end{aligned} \tag{40}$$

The center of mass (cm) coordinates of the rolling log are then related by, to order b/r_0,

$$(m_0 + m_s)x_{\rm cm} = m_0 x + m_s x_s = (m_0 + m_s)x - m_0 \frac{b}{\pi}\,(1 - \cos\theta)\,, \tag{41}$$

$$(m_0 + m_s)y_{\rm cm} = m_0\, r_0 + m_s\, y_s = (m_0 + m_s)r_0 - m_0 \frac{b}{\pi}\sin\theta\,, \tag{42}$$

and, noting from eq. (38) that $m_0/(m_0 + m_s) \approx 1 - b\,\theta/\pi r_0$, the cm coordinates are,

$$x_{\rm cm} = x - \frac{b}{\pi}\,(1 - \cos\theta)\,, \qquad y_{\rm cm} = r_0 - \frac{b}{\pi}\sin\theta, \tag{43}$$

[7]If the initial log had very low mass, it would not roll down the slope, as the snow must be lifted as it sticks to the log.

$$\dot{x}_{\rm cm} = r_0\,\dot\theta - \frac{b\,\dot\theta\sin\theta}{\pi}\,, \qquad \dot{y}_{\rm cm} = -\frac{b\,\dot\theta\cos\theta}{\pi}\,, \tag{44}$$

$$\ddot{x}_{\rm cm} = r_0\,\ddot\theta - \frac{b}{\pi}\left(\ddot\theta\sin\theta + \dot\theta^2\cos\theta\right), \qquad \ddot{y}_{\rm cm} = -\frac{b}{\pi}\left(\ddot\theta\cos\theta - \dot\theta^2\sin\theta\right). \tag{45}$$

The velocity of the nominal center of the log is again $v = \dot x$, and the angular velocity of the log is again $\omega = \dot\theta = v/r_0 = \dot x/r_0$, while now the nominal radius r_0 is constant during the first turn of rolling.

The angular momentum with respect to the center of mass is,

$$\mathbf{L}_{\rm cm} = \int dm\,(\mathbf{x} - \mathbf{x}_{\rm cm}) \times (\mathbf{v} - \mathbf{v}_{\rm cm}) = \int dm\,(\mathbf{x} - \mathbf{x}_{\rm cm}) \times \mathbf{v}, \tag{46}$$

since $\int dm\,(\mathbf{x}-\mathbf{x}_{\rm cm}) = 0$. To carry out the integral, we note that for a mass element centered on (r', θ', z') in cylindrical coordinates about the axis of the log, its rectangular coordinates (x', y, z') are,

$$x' = x - r'\sin\theta', \quad y' = r - r'\cos\theta', \quad \dot{x}' = \dot x - r'\,\dot\theta'\cos\theta = (r - r'\cos\theta')\dot\theta, \quad \dot{y}' = r'\,\dot\theta\sin\theta', \tag{47}$$

considering that as the log rolls, r' of a mass element stays constant while its angular velocity $\dot\theta'$ is that of the rigid log, $\dot\theta' = \dot\theta$. Then, the angular momentum (46) about the center of mass has only a z-component,

$$\begin{aligned}
\mathbf{L}_{\rm cm} &= \left\{\int_0^{r_0}\int_0^{2\pi} \rho\,l\,r'\,dr'\,d\theta' + \int_{r_0}^{r_0+b}\int_0^{\theta} \rho\,l\,r'\,dr'\,d\theta'\right\}(\mathbf{x}' - \mathbf{x}_{\rm cm}) \times \mathbf{v}' \\
&= \left\{\int_0^{r_0}\int_0^{2\pi} \rho\,l\,r'\,dr'\,d\theta' + \int_{r_0}^{r_0+b}\int_0^{\theta} \rho\,l\,r'\,dr'\,d\theta'\right\} \\
&\left\{\hat{\mathbf{x}}\left[-r_0\sin\theta' + \frac{b}{\pi}(1-\cos\theta)\right] + \hat{\mathbf{y}}\left[-r_0\cos\theta' + \frac{b}{\pi}\sin\theta\right]\right\} \times [\hat{\mathbf{x}}\,(r_0 - r'\cos\theta') + \hat{\mathbf{y}}\,r'\sin\theta']\,\dot\theta \\
&= -\rho\,l\,\dot\theta\,\hat{\mathbf{z}}\left\{\int_0^{r_0}\int_0^{2\pi} r'\,dr'\,d\theta' + \int_{r_0}^{r_0+b}\int_0^{\theta} r'\,dr'\,d\theta'\right\} \\
&\quad \left\{r_0\,r'(1-\cos\theta') + \frac{b\,r_0}{\pi}\sin\theta + \frac{b\,r'}{\pi}[\cos\theta'\sin\theta + \sin\theta'(1-\cos\theta)]\right\} \\
&\approx -\rho\,l\left[\frac{\pi r_0^4}{2} + \frac{b\,\pi r_0^3\sin\theta}{\pi} + \frac{b\,\pi r_0^3}{\pi}(\theta - \sin\theta)\right]\dot\theta\,\hat{\mathbf{z}} = -\left[\frac{m_0\,r_0^2}{2} + \frac{m_0\,b\,r_0\,\theta}{\pi}\right]\dot\theta\,\hat{\mathbf{z}},
\end{aligned} \tag{48}$$

to order b/r_0. This is the same as the angular momentum about the axis of the cylinder, to order b/r_0.

3.1 Torque Analysis

A torque analysis about the center of mass would include the torque due to the contact force $\mathbf{F}_C$ on the line of contact of the initial log with the snowy slope. Instead, we perform a torque analysis in the lab frame about the point $\mathbf{x}_C = (x, 0, 0)$ of contact of the rolling log with the snowy slope, as the torque $\boldsymbol{\tau}_C$ does not involve the contact force $\mathbf{F}_C$. However, the rolling log as considered in this section does not have a symmetry axis, in contrast to the assumption of sec. 2 above, which complicates the torque analysis, as reviewed in [6].

In particular, while the torque analysis involves the time derivative $d\mathbf{L}_C/dt$, this does not equal $\partial\mathbf{L}_C/\partial t$, such that one should not first compute the angular momentum $\mathbf{L}_C$ about point C and then take its (partial) time derivative. Furthermore, there are two possible meanings of the angular momentum about point C, which can be termed the absolute angular momentum,

$$\mathbf{L}_C = \int dm\,(\mathbf{x} - \mathbf{x}_C) \times \mathbf{v}, \tag{49}$$

and the relative angular momentum,

$$\mathbf{L}'_C = \int dm\,(\mathbf{x} - \mathbf{x}_C) \times (\mathbf{v} - \mathbf{v}_C). \tag{50}$$

And, there are at least three possible interpretations of point C, as fixed in the lab frame, as fixed in the rolling log, and as the moving point of contact along the snowy slope. As such, there are (at least) five variants of torque analyses based on the point C of contact [6].

Here, we consider use of the absolute angular momentum $\mathbf{L}_C$ about the point of contact, taking this to be that point C in the rolling log that happens to be the point of contact at the time of interest. In this convention, the point C of contact is instantaneously at rest, $\mathbf{v}_C = 0$, but it has nonzero acceleration perpendicular to the snowy slope. Then, according to eq. (13) of [6], the torque equation of motion is,

$$\frac{d\mathbf{L}_C}{dt} = \boldsymbol{\tau}_C = \frac{d\mathbf{L}_{\rm cm}}{dt} + (\mathbf{x}_{\rm cm} - \mathbf{x}_C) \times m\,\mathbf{a}_{\rm cm}. \tag{51}$$

The torque $\boldsymbol{\tau}_C$ about the line of contact is due to the force of gravity, $(m_0 + m_s)\,\mathbf{g} = (m_0 + m_s)(\hat{\mathbf{x}}\,g\sin\alpha - \hat{\mathbf{y}}\,g\cos\alpha)$, which acts at the center of mass, whose position relative to the line of contact is $\hat{\mathbf{x}}\,(x_{\rm cm} - x) + \hat{\mathbf{y}}\,y_{\rm cm}$,

$$\begin{aligned}
\boldsymbol{\tau}_C &= [\hat{\mathbf{x}}\,(x_{\rm cm} - x) + \hat{\mathbf{y}}\,y_{\rm cm}] \times (m_0 + m_s)(\hat{\mathbf{x}}\,g\sin\alpha - \hat{\mathbf{y}}\,g\cos\alpha) \\
&= (m_0 + m_s)\,g\,[(x_{\rm cm} - x)(-\cos\alpha) - y_{\rm cm}\sin\alpha]\,\hat{\mathbf{z}} \\
&\approx m_0\,g\left(1 + \frac{b\,\theta}{\pi r_0}\right)\left[\frac{b}{\pi}(1 - \cos\theta)\cos\alpha - \left(r_0 - \frac{b}{\pi}\sin\theta\right)\sin\alpha\right]\hat{\mathbf{z}} \\
&\approx -m_0\,g\left[\left(r_0 + \frac{b\,\theta}{\pi} - \frac{b}{\pi}\sin\theta\right)\sin\alpha - \frac{b}{\pi}(1 - \cos\theta)\cos\alpha\right]\hat{\mathbf{z}}.
\end{aligned} \tag{52}$$

From eq. (48), we have,

$$\frac{d\mathbf{L}_{\rm cm}}{dt} = -\left[\frac{m_0\,r_0^2}{2} + \frac{m_0\,b\,r_0\,\theta}{\pi}\right]\ddot{\theta}\,\hat{\mathbf{z}} - \frac{m_0\,b\,r_0}{\pi}\,\dot{\theta}^2\,\hat{\mathbf{z}}. \tag{53}$$

and also,

$$
\begin{aligned}
(\mathbf{x}_{\rm cm} - \mathbf{x}_C) \times m\, \mathbf{a}_{\rm cm} &= \left[-\hat{\mathbf{x}}\, \frac{b}{\pi}(1-\cos\theta) + \hat{\mathbf{y}} \left(r_0 - \frac{b}{\pi}\sin\theta \right) \right] \\
&\times m_0 \left(1 + \frac{b\,\theta}{\pi r_0} \right) \left\{ \hat{\mathbf{x}} \left[r_0\, \ddot\theta - \frac{b}{\pi} \left(\ddot\theta \sin\theta + \dot\theta^2 \cos\theta \right) \right] - \hat{\mathbf{y}} \left[\frac{b}{\pi} \left(\ddot\theta\cos\theta - \dot\theta^2 \sin\theta \right) \right] \right\} \\
&\approx m_0 \left\{ -\left[r_0^2 \ddot\theta - \frac{b\, r_0}{\pi} \left(\ddot\theta\sin\theta + \dot\theta^2\cos\theta \right) \right] + \frac{b\, r_0\, \ddot\theta \sin\theta}{\pi} - \frac{b\, r_0\, \theta\, \ddot\theta}{\pi} \right\} \hat{\mathbf{z}} \\
&= -m_0 \left[r_0^2 \ddot\theta + \frac{b\, r}{\pi} \left(\theta\, \ddot\theta - 2\ddot\theta\sin\theta - \dot\theta^2 \cos\theta \right) \right] \hat{\mathbf{z}}. \qquad (54)
\end{aligned}
$$

With eqs. (52)-(54) in eq. (51) we have the torque equation of motion,

$$
\begin{aligned}
-\frac{dL_C}{dt} &= \left[\frac{m_0\, r_0^2}{2} + \frac{m_0\, b\, r_0\, \theta}{\pi} \right] \ddot\theta + \frac{m_0\, b\, r_0}{\pi}\, \dot\theta^2 + m_0 \left[r_0^2 \ddot\theta + \frac{b\, r_0}{\pi} \left(\theta\, \ddot\theta - 2\ddot\theta \sin\theta - \dot\theta^2 \cos\theta \right) \right] \\
&= m_0 \left[\frac{3 r_0^2}{2} + \frac{b\, r_0}{\pi}(2\theta - 2\sin\theta) \right] \ddot\theta + \frac{m_0\, b\, r_0}{\pi}(1-\cos\theta)\, \dot\theta^2 \\
&= -\tau_C \approx m_0\, g \left[\left(r_0 + \frac{b\,\theta}{\pi} - \frac{b}{\pi}\sin\theta \right) \sin\alpha - \frac{b}{\pi}\,(1-\cos\theta)\cos\alpha \right]. \qquad (55)
\end{aligned}
$$

For later comparisons we note that (to order b/r_0) the moment of inertia I_C of the log (plus accumulated snow) about the line of contact is,

$$
\begin{aligned}
I_C &= \frac{3 m_0\, r_0^2}{2} + \int_{r_0}^{r_0+b} \int_0^\theta \rho\, l\, r'\, dr'\, d\theta'\, (r_0^2 + r'^2 - 2 r_0\, r' \cos\theta') \\
&= \frac{3 m_0\, r_0^2}{2} + \rho\, l\, \theta \left(r_0^2 \frac{(r_0+b)^2 - r_0^2}{2} + \frac{(r_0+b)^4 - r_0^4}{4} \right) - 2\rho\, l\, r_0 \frac{(r_0+b)^3 - r_0^3}{3} \sin\theta \\
&\approx \frac{3 m_0\, r_0^2}{2} + \frac{2 m_0\, b\, r_0}{\pi}\theta - \frac{2 m_0\, b\, r_0}{\pi}\sin\theta. \qquad (56)
\end{aligned}
$$

It is noteworthy that although $L_C = -I_C\, \dot\theta$, the torque equation (55) can be written as,

$$
-\frac{dL_C}{dt} = I_C\, \ddot\theta + \frac{\dot\theta}{2} \frac{dI_C}{dt} = -\tau_C, \qquad (57)
$$

rather than $d(I_C\, \dot\theta)/dt = -\tau_C$, as first deduced by Loney (1909), Art. 214, p. 287 of [3].

We also make the substitutions $v = r_0\, \dot\theta$ and $dv/dt = r_0\, \ddot\theta$ to write the torque equation (55) as,

$$
I_C \frac{dv}{dt} + \frac{m_0\, b\, v^2}{\pi}(1-\cos\theta) \approx m\, g\, r_0 \left[\left(r_0 + \frac{b\,\theta}{\pi} - \frac{b}{\pi}\sin\theta \right) \sin\alpha - \frac{b}{\pi}\,(1-\cos\theta)\cos\alpha \right]. \qquad (58)
$$

From this equation, which holds only for the first turn of the rolling motion, it is not very apparent that the acceleration dv/dt approaches a constant value. See also sec. 3.4 below.

3.2 Energy Analysis

The rolling log plus accumulated snow is instantaneously rotating with angular velocity ω about the line of contact, so its kinetic energy is,[8]

$$T = \frac{I_C\,\omega^2}{2} = \frac{I_C\,v^2}{2r_0^2}\,, \qquad \frac{dT}{dt} = \frac{I_C v}{r^2}\frac{dv}{dt} + \frac{v^2}{2r_0^2}\frac{dI_C}{dt} = \frac{I_C\,v}{r_0^2}\frac{dv}{dt} + \frac{m_0\,b\,v^3}{\pi r_0^2}(1-\cos\theta). \qquad (59)$$

Relative to the origin, the initial gravitational potential energy V_0', and the present potential energy V', of the log plus accumulated snow are,

$$V_0' = m_0\,g\,r_0\cos\alpha - m_s\,g\frac{x\sin\alpha + b\cos\alpha}{2}\,, \qquad (60)$$

$$V' = m_0\,g(r\cos\alpha - x\sin\alpha) + m_s\,g(-x_s\sin\alpha + y_s\cos\alpha). \qquad (61)$$

Redefining the initial potential to be zero, the present potential energy V is, to order b/r_0,

$$\begin{aligned}
V &= -m_0\,g\,x\sin\alpha + m_s\,g(-x_s\sin\alpha + y_s\cos\alpha) + m_s\,g\frac{x\sin\alpha + b\cos\alpha}{2} \\
&\approx -m_0\,g\,x\sin\alpha - \left[m_s\,x - m_0\frac{b}{\pi}(1-\cos\theta)\right] g\sin\alpha \\
&\quad + \left[m_s\,r_0 - m_0\frac{b}{\pi}\sin\theta\right] g\cos\alpha + m_s\,g\frac{x\sin\alpha + b\cos\alpha}{2} \\
&\approx -m_0\,g\,r_0\theta\sin\alpha - \left[\frac{m_0\,b\theta^2}{2\pi} - m_0\frac{b}{\pi}(1-\cos\theta)\right] g\sin\alpha \\
&\quad + \left[m_0\frac{b\,\theta}{\pi} - m_0\frac{b}{\pi}\sin\theta\right] g\cos\alpha. \qquad (62)\\
-\frac{dV}{dt} &= \frac{v}{r_0}m_0\,g\,r_0\sin\alpha + \frac{v}{r_0}m_0\left[\frac{\theta}{\pi} - \frac{b}{\pi}\sin\theta\right] g\sin\alpha - \frac{v}{r_0}m_0\,g\frac{b}{\pi}(1-\cos\theta)\cos\alpha. \qquad (63)
\end{aligned}$$

Assuming that mechanical energy is conserved during the rolling, $dT/dt = -dV/dt$, and we arrive at the equation of motion,

$$I_C\frac{dv}{dt} = m_0\,g\,r_0\left[\left(r_0 + \frac{b\,\theta}{\pi} - \frac{b}{\pi}\sin\theta\right)\sin\alpha - \frac{b}{\pi}(1-\cos\theta)\cos\alpha\right] - \frac{m_0\,b\,v^2(1-\cos\theta)}{\pi}\,, \qquad (64)$$

in agreement with eq. (58).

3.3 Lagrangian Analysis

We now take the independent coordinate to be θ, with $\dot\theta = \omega = v/r_0$. Then,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\omega} = \frac{\partial\mathcal{L}}{\partial\theta}\,, \qquad (65)$$

[8]We could not use eq. (59) when we assumed that the rolling log is always circular, as this implies that it is not a rigid body, but has instantaneous motion of snow over its entire surface.

$$\frac{\partial \mathcal{L}}{\partial \omega} = \frac{\partial T}{\partial \omega} = I_C\,\omega = \frac{I_C v}{r}\,, \quad (66)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \omega} = \frac{I_C}{r_0}\frac{dv}{dt} + \frac{2m_0\,b\,v^2}{\pi r}(1-\cos\theta), \quad (67)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{m_0\,b\,v^2}{\pi r}(1-\cos\theta) + m_0\,g\sin\alpha\left(r_0 + \frac{b}{\pi} - \frac{b}{\pi}\sin\theta\right) - m_0\,g\cos\alpha\frac{b}{\pi}\,(1-\cos\theta)\,, \quad (68)$$

$$I_C\frac{dv}{dt} = m_0\,g\,r_0\left[\left(r_0 + \frac{b}{\pi} - \frac{b}{\pi}\sin\theta\right)\sin\alpha - \frac{b}{\pi}\,(1-\cos\theta)\cos\alpha\right] - \frac{m_0\,b\,v^2}{\pi}(1-\cos\theta), \quad (69)$$

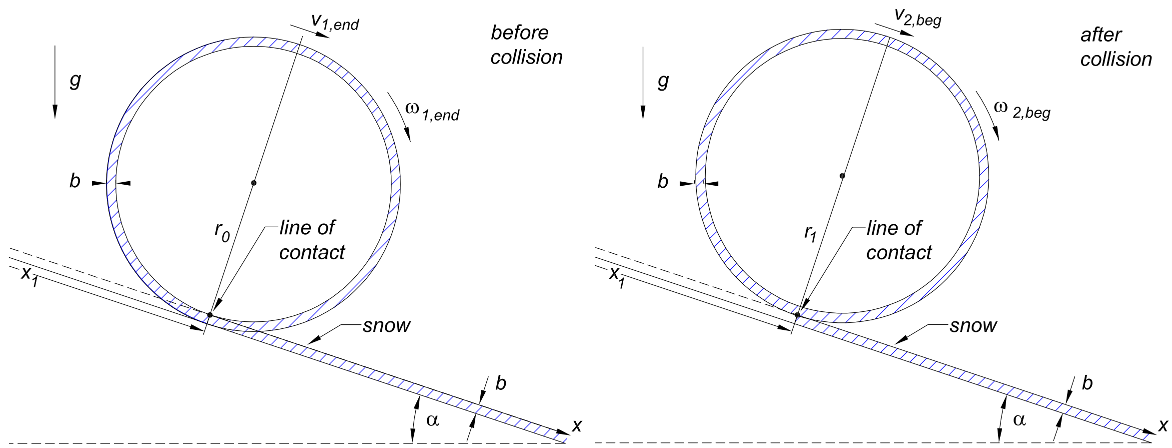
which agrees with eqs. (58) and (64).

3.4 Motion during the Second (and Later) Full Turn

At (or slightly before) the end of the first full turn of rolling, the new layer of snow on the log encounters the layer of snow still on the slope, and a kind of collision occurs. Here, we make the idealization that just after this collision the snow log is circular with radius and mass given by,

$$y_1 = r_1 = r_0 + b, \qquad m_1 = 2\pi\rho\,l\,r_1^2 = m_0\left(1 + \frac{(r_0+b)^2 - r_0^2}{r_0^2}\right) \approx m_0\left(1 + \frac{2b}{r_0}\right), \qquad (70)$$

and that its center is at $(x_1, y_1) = (2\pi r_0, r_0 + b) = (2\pi r_0, r_1)$. The configurations of the snowy log just before and after the collision are illustrated below.



After the collision, the snowy log rolls and accumulates more snow in the manner of its behavior during the first full turn, except that the initial velocity v_1 is not zero, and the roles of r_0 and m_0 during the first full turn are played by r_1 and m_1 during the second full turn. Again, assuming that energy is conserved during the rolling motion of the second full turn, the equation of motion follows from eq. (64) (or eq. (58) or (69)) as,

$$I_{C_1}\frac{dv}{dt} = m_1\,g\,r_1\left[\left(r_1 + \frac{b\,\theta}{\pi} - \frac{b}{\pi}\sin\theta\right)\sin\alpha - \frac{b}{\pi}\,(1-\cos\theta)\cos\alpha\right] - \frac{m_1\,b\,v^2(1-\cos\theta)}{\pi}\,, (71)$$

where,

$$I_{C_1} \approx \frac{3m_1\,r_1^2}{2} + \frac{2m_1\,b\,r_1}{\pi}\theta - \frac{2m_1\,b\,r_1}{\pi}\sin\theta, \qquad (72)$$

to order b/r_1.

If we accept the above model of the collision at the end of a full turn, we can extrapolate that during the $n^{\rm th}$ full turn, the equation of motion is,

$$I_{C_n}\frac{dv}{dt} = m_n\, g\, r_n \left[\left(r_n + \frac{b\,\theta}{\pi} - \frac{b}{\pi}\sin\theta\right)\sin\alpha - \frac{b}{\pi}\,(1-\cos\theta)\cos\alpha\right] - \frac{m_n\, b\, v^2(1-\cos\theta)}{\pi}\,, \tag{73}$$

where,

$$y_n = r_n = r_{n-1} + b = r_0 + nb, \qquad x_n = x_{n-1} + 2\pi r_{n-1} = 2\pi n r_0 + n(n-1)\pi b, \tag{74}$$

$$m_n = \pi\rho\, l\, r_n^2 = m_0\left(1 + \frac{nb}{r_0}\right)^2, \tag{75}$$

and,

$$I_{C_n} \approx \frac{3m_n\, r_n^2}{2} + \frac{2m_n\, b\, r_n}{\pi}\theta - \frac{2m_n\, b\, r_n}{\pi}\sin\theta, \tag{76}$$

to order b/r_n.

For large n, $I_{C_n} \to 3m_n\, r_n^2/2$, and the equation of motion (73) becomes,

$$\frac{dv}{dt} \approx \frac{2g\sin\alpha}{3} - \frac{2v^2(1-\cos\theta)}{3\pi n^2 b}\,, \tag{77}$$

Now, by conservation of energy, the velocity during the $n^{\rm th}$ turn is roughly given by,

$$\frac{3m_n\, v^2}{2} \approx \frac{m_n\, g\, x_n\sin\alpha}{2}\,, \qquad v^2 \approx \frac{\pi g\, n^2 b\sin\alpha}{3}\,, \tag{78}$$

for large n, such that,

$$\frac{dv}{dt} \approx \frac{2g\sin\alpha}{3} - \frac{2g\sin\alpha(1-\cos\theta)}{9} \approx \frac{4g\sin\alpha}{9}\,, \tag{79}$$

where the last form is the average acceleration. However, the term $1-\cos\theta$ in eq. (77) oscillates between 0 and 2, and indicates that the snowy log never achieves a steady acceleration, but rather oscillates about the value (79).[9,10]

For completeness, we note that the velocity $v_{n,\rm beg}$ and $v_{n,\rm end}$ of the center of the log at the beginning and end of the $n^{\rm th}$ full turn follows from the assumption of conservation of energy except during the collision at the end of a turn (and that there is no "hopping").

At the beginning of the $n^{\rm th}$ turn, the snowy log is circular, with mass m_{n-1}, radius r_{n-1}, angular velocity,

$$\dot\theta_{n,\rm beg} = \frac{v_{n,\rm beg}}{r_{n-1}}\,, \tag{80}$$

[9]The results of sec. 2 above, which suggest that the acceleration of the snowy log takes on a terminal velocity, is an artifact of the assumption that the log is perfectly circular in cross section at all times. In reality, the accumulation of snow is not azimuthally symmetric, and therefore the acceleration of the log is never steady.

[10]Conceivably, this oscillatory acceleration could be associated with "hopping", in which the snowball briefly leaves the snowy surface once each turn. Compare with the discussion in Appendix A.4 of [7].

moment of inertia about the point C of contact with the snowy slope,

$$I_{C,n,\text{beg}} = \frac{3m_{n-1}\, r_{n-1}^2}{2}\,, \tag{81}$$

kinetic energy,

$$T_{n,\text{beg}} = \frac{I_{C,n,\text{beg}}\, \dot{\theta}_{n,\text{beg}}^2}{2} = \frac{3m_{n-1}\, v_{n,\text{beg}}^2}{4} \tag{82}$$

and potential energy defined to be zero,

$$V_{n,\text{beg}} = 0. \tag{83}$$

At the end of the n^{th} turn, the snowy log is again circular, with mass m_n and radius r_n, while the distance from the point C of contact with the snowy slope is still r_{n-1}, so the various kinematic parameters are now,

$$\dot{\theta}_{n,\text{end}} = \frac{v_{n,\text{end}}}{r_{n-1}}\,, \tag{84}$$

$$I_{C,n,\text{end}} = I_{\text{cm},n,\text{end}} + m_n\, r_{n-1}^2 = \frac{m_n\, r_n^2}{2} + m_n\, r_{n-1}^2 \tag{85}$$

$$T_{n,\text{end}} = \frac{I_{C,n,\text{end}}\, \dot{\theta}_{n,\text{end}}^2}{2} = m_n \left(1 + \frac{r_n^2}{2r_{n-1}^2}\right) \frac{v_{n,\text{end}}^2}{2}\,, \tag{86}$$

$$V_{n,\text{end}} = (m_n - m_{n-1}) g \cos\alpha - \left(m_n - \frac{\rho\, l\, b\, x_n}{2}\right) g\, x_n \sin\alpha. \tag{87}$$

Conservation of energy during the n^{th} turn then tells us that,

$$T_{n,\text{beg}} = T_{n,\text{end}} + V_{n,\text{end}}, \tag{88}$$

which gives a somewhat lengthy expression for $v_{n,\text{end}}^2$ in terms of $v_{n,\text{beg}}^2$ and other, known kinematic parameters.

All that remains is to relate the velocities just before and after the collision at the end of a full turn. For this, we note that to a good approximation, the angular momentum about the point C of contact is conserved during the collision,

$$L_{C,n,\text{end}} = L_{C,n+1,\text{beg}}, \qquad \frac{I_{C,n,\text{end}}\, v_{n,\text{end}}}{r_{n-1}} = \frac{I_{C,n+1,\text{beg}}\, v_{n+1,\text{beg}}}{r_n}\,. \tag{89}$$

A Appendix: Cylinder Rolling on a Snowless Slope

As in the figure on p. 1, the coordinate x of the line of contact of the cylinder with the slope has nonzero acceleration $a = dv/dt = d^2x/dt^2$. This is also the x-coordinate of the center of mass of the rolling log.

Force/torque analyses about points P that depend on the coordinate $x(t)$ of the line of contact can be done either in the lab frame or in an accelerated frame based on point P. In the latter case, the analysis must include the "fictitious" (coordinate) force $-m\, \mathbf{a}_P$ which appears to act on the center of mass of the cylinder in the accelerated frame.

We first note that the moments of inertia for a solid cylinder of mass m and radius r are,

$$I_{\text{cm}} = \frac{mr^2}{2}\,, \qquad I_{\text{contact}} = \frac{3mr^2}{2}\,. \tag{90}$$

Also, as in the figure on p. x = distance along the slope to the point of contact

We consider rolling without slipping, such that the angular velocity ω of the rolling cylinder is related by,

$$\omega = \frac{v}{r}\,. \tag{91}$$

A.1 Energy Analysis

The cylinder is in instantaneous rotation about the point of contact, so the kinetic energy T (in the lab frame) is related by,

$$T = \frac{I_{\text{contact}}\,\omega^2}{2} = \frac{3mr^2\omega^2}{4} = \frac{3mv^2}{4}\,. \tag{92}$$

The gravitational potential energy V is related by,

$$V = -mgx\sin\alpha, \tag{93}$$

taking V to be zero when $x = 0$.

The total mechanical energy $E = T + V$ is constant (neglecting possible dissipative interactions), so,

$$\frac{dE}{dt} = 0 = 3mva/2 - mgv\sin\alpha, \tag{94}$$

and the acceleration a of the cylinder down the slope is,

$$a = \frac{2g}{3}\sin\alpha. \tag{95}$$

A.2 Lagrangian Analysis

The equation of motion can also be deduced from the Lagrangian, $\mathcal{L} = T - V$. Here, we take x as the single, independent coordinate, and note that $\dot{x} = v$. Then,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{\partial\mathcal{L}}{\partial x}\,, \tag{96}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{\partial T}{\partial v} = \frac{3mv}{2}\,, \tag{97}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{3ma}{2}\,, \tag{98}$$

$$\frac{\partial\mathcal{L}}{\partial x} = mg\sin\alpha, \tag{99}$$

$$a = \frac{2g}{3}\sin\alpha. \tag{100}$$

A.3 Torque Analysis about the Line of Contact

A.3.1 Analysis in the Lab Frame

The lab-frame torque equation about the line of contact is simply,

$$\tau_{\rm contact} = rmg\sin\alpha = \frac{d}{dt}L_{\rm contact} = \frac{d}{dt}(I_{\rm contact}\,\omega) = \frac{d}{dt}\left(\frac{3mr^2}{2}\frac{v}{r}\right) = \frac{3mra}{2}\,, \tag{101}$$

and hence,

$$a = \frac{2g}{3}\sin\alpha. \tag{102}$$

A.3.2 Analysis in the Accelerated Frame

When considering a torque analysis about a point $\mathbf{x}_P(t)$ that is accelerated in the lab frame, it is often tacitly assumed that the origin of the accelerated coordinate system at time t is at the point P. However, we do not make this assumption here.

The x-y coordinates of the line of contact are $\mathbf{x}_P = (x, 0)$, so the lab-frame acceleration of the reference point is $\mathbf{a}_P = \mathbf{a}\,\hat{\mathbf{x}}$. In the accelerated frame, the center of mass of the rolling cylinder is at rest, so its acceleration $\mathbf{a}'_{\rm cm}$ in the accelerated frame is zero. That is, we can rewrite the lab-frame relation $F_x + mg\sin\alpha = ma_{\rm cm}$ as,

$$F_x + mg\sin\alpha - ma = ma'_{\rm cm} = 0, \tag{103}$$

where "fictitious" force $-m\,\mathbf{a}_P = -ma\,\hat{\mathbf{x}}$ acts on the center of mass of the cylinder.

A torque analysis about a general point P in the accelerated frame can be written in the form,

$$\boldsymbol{\tau}'_P = \frac{d\mathbf{L}'_P}{dt}\,, \tag{104}$$

where the torque $\boldsymbol{\tau}'_P$ is the sum of the torque about P in the (inertial) lab frame and the "fictitious" torque associated with the "fictitious" force that appears to act on the center of mass of the cylinder,

$$\boldsymbol{\tau}'_P = \boldsymbol{\tau}_P + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times (-m\,\mathbf{a}_P), \tag{105}$$

and the angular momentum $\mathbf{L}'_P$ relative to the moving point P is related lab-frame quantities by (see, for example, sec. 3 of [8], and the Appendix of [9]),

$$\begin{aligned} \mathbf{L}'_P &= \int dm[(\mathbf{x} - \mathbf{x}_P) \times (\mathbf{v} - \mathbf{v}_P)] = \mathbf{L} - \mathbf{x}_P \times m\mathbf{v}_{\rm cm} - (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\mathbf{v}_P \\ &= \mathbf{L}_{\rm cm} + m(\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times (\mathbf{v}_{\rm cm} - \mathbf{v}_P) = \mathbf{L}_P - (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\mathbf{v}_P, \end{aligned} \tag{106}$$

where,

$$\mathbf{L} = \int dm\,\mathbf{x} \times \mathbf{v} = \mathbf{L}_{\rm cm} + \mathbf{x}_{\rm cm} \times m\mathbf{v}_{\rm cm}, \tag{107}$$

is the (lab-frame) angular momentum about the origin, and,

$$\mathbf{L}_P = \int dm\,(\mathbf{x} - \mathbf{x}_P) \times \mathbf{v} = \mathbf{L} - \mathbf{x}_P \times m\mathbf{v}_{\rm cm} = \mathbf{L}_{\rm cm} + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\mathbf{v}_{\rm cm}, \tag{108}$$

is the angular momentum about point P regarding this point as fixed in the lab frame.[11]

For the present case where $P = (x(t), 0)$ is on the line of contact, $\mathbf{v}_P = \mathbf{v}_{\rm cm} = \mathbf{v}$, so,

$$\mathbf{L}'_P = \mathbf{L}_{\rm cm}, \qquad L'_P = L_{\rm cm} = I_{\rm cm}\,\omega = \frac{mr^2}{2}\frac{v}{r} = \frac{mrv}{2}\,, \tag{111}$$

and,

$$\tau'_P = rmg\sin\alpha - rma. \tag{112}$$

The torque equation (104) implies,

$$\tau'_P = rmg\sin\alpha - rma = \frac{dL'_P}{dt} = \frac{mra}{2}\,, \tag{113}$$

and hence,

$$a = \frac{2g}{3}\sin\alpha, \tag{114}$$

as found previously.

A.4 Torque Analysis about the Center of Mass

A.4.1 Analysis in the Lab Frame

The force $\mathbf{F}$ that acts on the cylinder at the line of contact has components related by,

$$F_x + mg\sin\alpha = ma, \qquad F_y - mg = 0, \tag{115}$$

The lab-frame torque equation about the center of mass is,

$$\tau_{\rm cm} = -rF_x = -rma + rmg\sin\alpha = \frac{d}{dt}L_{\rm cm} = \frac{d}{dt}(I_{\rm cm}\omega) = \frac{d}{dt}\frac{mrv}{2} = \frac{mra}{2}\,, \tag{116}$$

$$\frac{3mra}{2} = mgr\sin\alpha\,, \tag{117}$$

and again,

$$a = \frac{2g}{3}\sin\alpha. \tag{118}$$

[11]In general, the torque equation for $\mathbf{L}_P$ of eq. (108) is not $d\mathbf{L}_P/dt = \boldsymbol{\tau}_P$, but rather,

$$\frac{d\mathbf{L}_P}{dt} = \tilde{\boldsymbol{\tau}}_P = \boldsymbol{\tau}_P + m\mathbf{v}_{\rm cm} \times \mathbf{v}_P. \tag{109}$$

as discussed, for example, in sec. 11.8, p. 256 of [10] and sec. 2 of [8]. See also [11].

For the present case where $P = (x(t), 0)$ is on the line of contact, $\mathbf{v}_P = \mathbf{v}_{\rm cm} = \mathbf{v}$, and $\mathbf{x}_P$ is parallel to $\mathbf{v}_{\rm cm}$, so $\tilde{\tau}_P = \tau_P = rmg\sin\alpha$. Then, from eq. (108), $L_P = L_CM + mrv = 3mrv/2$, and the torque equation (109) implies,

$$\frac{dL_P}{dt} = \frac{d}{dt}\frac{3mrv}{2} = \frac{3mra}{2}\tilde{\tau}_P = rmg\sin\alpha, \qquad \text{and hence,} \qquad a = \frac{2g}{3}\sin\alpha, \qquad \text{as found previously.} \tag{110}$$

A.4.2 Analysis in the Accelerated Frame

In the accelerated frame defined by the position of the center of mass of the cylinder at point $P = \mathbf{x}_{\rm cm} = (x, r)$, we have that $\mathbf{x}_P = \mathbf{x}_{\rm cm}$ and $\mathbf{v}_P = \mathbf{v}_{\rm cm}$. Then, according to eqs. (105) and (116),

$$\boldsymbol{\tau}'_P = \boldsymbol{\tau}_P = \boldsymbol{\tau}_{\rm cm}, \qquad \tau'_P = -rma + rmg\sin\alpha, \tag{119}$$

and according to eq. (106),

$$\mathbf{L}'_P = \mathbf{L}_{\rm cm}. \tag{120}$$

The torque equation $\boldsymbol{\tau}'_{\rm cm} = d\mathbf{L}'_{\rm cm}/dt$ in the accelerated frame implies,

$$\tau'_{\rm cm} = -rma + rmg\sin\alpha = \frac{dL'_{\rm cm}}{dt} = \frac{dL_{\rm cm}}{dt} = \frac{mra}{2}\,, \tag{121}$$

and again the equation of motion is,

$$a = \frac{2g}{3}\sin\alpha. \tag{122}$$

A.5 Torque Analysis about the Point $(x, 2r)$

A.5.1 Analysis in the Lab Frame

The moment of inertia I_p about this point is the same, $3mr^2/2$, as about the line of contact.

The force $\mathbf{F}$ that acts on the cylinder a the line of contact has components related by eq. (115), and the lab-frame torque equation about P is,

$$\tau_P = -2rF_x - rmg\sin\alpha = -2mar + mgr\sin\alpha. \tag{123}$$

The lab-frame angular momentum about point P is,

$$\mathbf{L}_P = \mathbf{L}_{\rm cm} + (\mathbf{x}_{\rm cm} - \mathbf{x}_P) \times m\mathbf{v}_{\rm cm}, \qquad L_P = \frac{mrv}{2} - rmv = -\frac{mrv}{2} \tag{124}$$

so the lab-frame torque equation is,

$$\tau_P = -2mar + mgr\sin\alpha = \frac{dL_P}{dt} = -\frac{mra}{2}\,, \tag{125}$$

and again,

$$a = \frac{2g}{3}\sin\alpha. \tag{126}$$

A.5.2 Analysis in the Accelerated Frame

In the accelerated frame defined by the point $P = (x, 2r)$, we have that and $\mathbf{v}_P = \mathbf{v}$. Then, according to eqs. (105) and (123),

$$\tau'_P = \tau_P + rma = -rma + rmg \sin\alpha, \tag{127}$$

and according to eq. (106),

$$\mathbf{L}'_P = \mathbf{L}_{\rm cm}. \tag{128}$$

The torque equation $\boldsymbol{\tau}'_P = d\mathbf{L}'_P/dt$ in the accelerated frame implies,

$$\tau'_P = -rma + rmg \sin\alpha = \frac{dL'_P}{dt} = \frac{dL_{\rm cm}}{dt} = \frac{mra}{2}\,, \tag{129}$$

and again the equation of motion is,

$$a = \frac{2g}{3} \sin\alpha. \tag{130}$$

References

[1] S. Rubin, *A Variable-Mass Snowball Rolling Down a Snowy Slope*, Phys. Teach. **57**, 150 (2019), http://kirkmcd.princeton.edu/examples/mechanics/rubin_pt_57_150_19.pdf

[2] J.E. Hurtado, *Analytical Dynamics of Variable-Mass Systems*, J. Guid. Control Dyn. **41**, 701 (2018), http://kirkmcd.princeton.edu/examples/mechanics/hurtado_jgcd_41_701_18.pdf

[3] S.A. Loney, *An Elementary Treatise on the Dynamics of a Particle and of Rigid Bodies* (Cambridge U. Press, 1909), http://kirkmcd.princeton.edu/examples/mechanics/loney_13.pdf http://kirkmcd.princeton.edu/examples/mechanics/loney_09_pp261-262.pdf

[4] K.T. McDonald, *Hexagonal Pencil Rolling on an Inclined Plane* (Nov. 14, 2006), http://kirkmcd.princeton.edu/examples/pencil.pdf

[5] K.T. McDonald, *Uncoiling on a Horizontal Surface* (June 2, 2018), http://kirkmcd.princeton.edu/examples/uncoil.pdf

[6] K.T. McDonald, *Comments on Torque Analyses* (Apr. 28, 2019), http://kirkmcd.princeton.edu/examples/torque.pdf

[7] K.T. McDonald, *The Rolling Motion of a Half-Full Beer Can* (Nov. 14, 1996), http://kirkmcd.princeton.edu/examples/beercan.pdf

[8] J.H. Jensen, *Rules for rolling as a rotation about the instantaneous point of contact*, Eur. J. Phys. **32**, 389 (2011), http://kirkmcd.princeton.edu/examples/mechanics/jensen_ejp_32_389_11.pdf

[9] K.T. McDonald, *Torque Analyses of a Sliding Ladder* (May 6, 2007), http://kirkmcd.princeton.edu/examples/ladder.pdf

[10] J.M. Knudsen and P.G. Hjorth, *Elements of Newtonian Mechanics*, 3rd ed. (Springer, 2000), http://kirkmcd.princeton.edu/examples/mechanics/knudsen_3ed_00.pdf

[11] J.H. Jensen, *Five ways of deriving the equation of motion for rolling bodies*, Am. J. Phys. **80**, 1073 (2012), http://kirkmcd.princeton.edu/examples/mechanics/jensen_ajp_80_1073_12.pdf