What is the Stiffness of Spacetime?

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A question asked at the end of the 2018 Hamilton Lecture (Princeton), *Exploring the Universe with Gravitational Waves* by Kip Thorne,[1] concerned the Young's modulus of spacetime.[2]

1 A "Classical" Answer

The Young's modulus Y of a elastic medium[3] appears in the stress-strain relation,

$$\frac{F}{A} = Y \frac{\Delta l}{l}\,, \tag{1}$$

such that Y has the dimensions of pressure (and energy density), $m^1 l^{-1} t^{-2}$ (called Pa in the SI system of units).

For a "classical" answer, note that Newton's gravitational constant is $G = 6.67 \times 10^{-11}$ N·m^2·kg^{-2}, with dimensions $m^{-1} l^3 t^{-2}$, and the speed of light in vacuum is $c = 3 \times 10^8$ m/s, with dimensions $l^1 t^{-1}$. Then, we can write (by dimensional analysis),

$$Y_{\rm spacetime} \approx \frac{c^2 f^2}{G} = 4.5 \times 10^{27} f^2 \text{ Pa}, \tag{2}$$

where f ($\propto t^{-1}$) is the frequency of the gravitational waves in Hz.[4]

For example, $f = 100$ Hz is representative of the gravitational waves recently detected by LIGO [4]-[10], so the corresponding Young's modulus is,

$$Y_{\rm spacetime}(f = 100 \text{ Hz}) \approx 4.5 \times 10^{31} \text{ Pa} \approx 10^{20} Y_{\rm steel}, \tag{4}$$

as $Y_{\rm steel} \approx 200$ GPa.

The answer (4) was given by Rainer Weiss at the Hamilton Lecture.[5]

[1] http://www.kaltura.com/tiny/r63ta. The question about the stiffness of spacetime is at 1:22:22.

[2] This question is in the spirit of the æther theories of the 1800's. For example, on p. 126 of [1] (1800), Young wrote: *The rapid transmission of electrical shock, shows that the electrical medium is possessed of an elasticity as great as is necessary to be proposed for the propagation of light. Whether the electric ether is to be considered as the same with the luminous ether, if such a fluid exists, may perhaps at some future time be discovered by experiment.*

The concept of the luminiferous ether was introduced by Huygens (1678), p. 10 of [2] (p. 11 of the English translation).

[3] In the engineering literature, the modulus of elasticity is often denoted as E.

[4] Also note that Einstein's field equations of gravitation [3],

$$G_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) = -\frac{8\pi G}{c^2 f^2}\frac{f^2}{c^2}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) = -\frac{8\pi}{Y}\frac{f^2}{c^2}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right), \tag{3}$$

have the form of a frequency-dependent stress-strain relation, since the stress-energy tensor $T_{\mu\nu}$ has the same dimensions as the Young's modulus Y, and $G_{\mu\nu}$ has dimensions l^{-2}.

[5] According to Prof. Weiss (email, Apr. 15, 2018), the energy densities u of stress waves in steel (see, for

2 A "Quantum" Answer

However, we might seek a "quantum" answer, in which the Young's modulus is independent of frequency.[6] For this, we note that $\hbar = 1.05 \times 10^{-34}$ m^2·kg/s, with dimensions $m^1 l^2 t^{-1}$, so we can write (again by dimensional analysis),

$$Y_{\rm quantum\ gravity} \approx \frac{c^7}{\hbar G^2} = 5 \times 10^{113} \text{ Pa}. \tag{6}$$

We might also suppose that the Young's modulus is roughly the bulk modulus, so the gravitational wave speed is,[7]

$$c \approx \sqrt{\frac{Y}{\rho}}\,, \tag{7}$$

where the mass density ρ of the quantum fluctuations of "empty" space would then be,

$$\rho \approx \frac{Y}{c^2} \approx 5 \times 10^{96}\ \frac{\rm kg}{\rm m^3}\,. \tag{8}$$

This implausibly large mass density is often called the "cosmological-constant problem",[8] and perhaps indicates that our "classical" estimate of the Young's modulus of spacetime is more meaningful than the "quantum" estimate.[9]

A Appendix: Cosmological Sound Waves

This section was inspired by [20].

The Universe has a very small mass density, $\rho \sim \rho_{\rm critical} \approx 10^{-26}$ kg/m^3, that could be associated with sound waves with speed,

$$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{\gamma k T}{m}}\,, \tag{9}$$

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example, p. 495 of [11]) and in plane gravitational waves are related by (denoting the strain by h, and using eq. (107.11) of [12] or eq. (10.3.6) of [13]),

$$u_{\rm steel} = \frac{Y_{\rm steel}\, h^2}{2}\,, \qquad u_{\rm gravity\ wave} = \frac{\pi c^2 f^2 h^2}{8G} = \frac{Y_{\rm spacetime}\, h^2}{2}\,, \qquad \text{with} \qquad Y_{\rm spacetime} = \frac{\pi c^2 f^2}{4G}\,. \tag{5}$$

[6]Discussions of general relativity as a frequency-independent, stress-strain theory are given in, for example, [14, 15, 16, 17, 18].

[7]Both gravitational waves and gamma rays were detected from a recent binary-neutron-star merger [10], which provides evidence that the speed of gravitational and electromagnetic waves differs by less than one part in 10^{16}.

[8]The cosmological constant was introduced by Einstein in [19].

[9]If we relate our "classical" estimate (2) to a mass density $\rho \approx Y/c^2$, we find that $\rho \approx 4 \times 10^{10} f^2$. If we suppose this mass density corresponds to that of "dark energy", $\approx 10^{-26}$ kg/m^2 ($\approx$ critical density of the Universe), then there would be a characteristic frequency to the Universe, $f \approx 0.5 \times 10^{-18}$ Hz, and a characteristic period $T \approx 2 \times 10^{18}$ s $\approx$ 60 billion years. This suggests that the expansion of the Universe can be regarded as a very low frequency gravitational wave.

where B is the bulk modulus, and the second form holds for an ideal gas of mass m and adiabatic index γ at temperature T. Taking this mass density in intergalactic space to be mainly hydrogen at temperature around 3 K, the sound speed would be about 100 m/s. Then, the bulk modulus of the dilute hydrogen gas of the Universe is, supposing that $\rho_H \approx 0.1\rho_{\rm critical}$,

$$B_{\rm intergalactic} = v^2 \rho_H \approx 10^{-24} \text{ Pa} \approx 10^{-35} Y_{\rm steel} \approx 10^{-55} Y_{\rm spacetime}(f = 100 \text{ Hz}). \tag{10}$$

While most of the matter in the Universe is dark matter, its presumably very weak interactions imply that dark-matter sound waves are a negligible phenomenon.

Sound waves in the baryon-photon plasma of the early Universe ($\approx 400,000$ year old) led to so-called baryon acoustic oscillations whose after-effects are observable today [21]. At that time, the sound speed was $v_{\rm BAO} \approx c/\sqrt{3}$ and the baryon-matter density was $\rho_{\rm baryon} \approx 10^{-18}$ kg/m^3 (1,000 nucleons/cm^3), so the bulk modulus of these sound waves was,

$$B_{\rm BAO} = v_{\rm BAO}^2 \rho_{\rm baryon} \approx 0.01 \text{ Pa} \approx 10^{-13} Y_{\rm steel} \approx 10^{-33} Y_{\rm spacetime}(f = 100 \text{ Hz}). \tag{11}$$

B Appendix: Electromagnetic Waves

As remarked in footnote 2 above, in the wave theory of light of the 1800's, the Universe was considered to be filled with an æther of very low density, while being very stiff so as to support waves of speed c. Here, we explore use of the argument of footnote 5 above for electromagnetic (light) waves in Maxwell's theory [22]. That is, we seek to relate the energy density $u = \epsilon_0 E^2$ of a plane electromagnetic wave with electric field $\mathbf{E}$ to a modulus of elasticity Y according to,

$$u = Y h^2, \tag{12}$$

where h is the relevant strain (see, for example, p. 495 of [11]).

Maxwell considered (for example, in sec. 66 of [22]) the electric displacement $\mathbf{D}$ ($= \epsilon_0 \mathbf{E}$ in vacuum) as a kind of strain in response to the stress of the electric field $\mathbf{E}$ (such that he called $1/\epsilon_0$ the coefficient of electrical elasticity in Art. 60 of [23]). However, Maxwell did not identify a modulus of elasticity associated with electric displacement. This is perhaps because a strain h involving electric displacement $D = |\mathbf{D}|$ would need to be with respect to some reference displacement D_0, such that,

$$h = \frac{D}{D_0} = \frac{E}{E_0}\,, \tag{13}$$

where E_0 is the reference electric field that corresponds to the reference displacement D_0.

While there is no reasonable candidate for the reference displacement D_0 in classical electrodynamics, in quantum electrodynamics there exists a critical electric field strength,

$$E_{\rm crit} = \frac{m_e^2 c^3}{e\hbar} = 1.8 \times 10^{18} \text{ V/m}, \tag{14}$$

where e and m_e are the charge and mass of an electron. As first noted by Sauter [25], Klein's paradox [26] in Dirac's theory of the electron [27] arises only when the potential gradient is larger than the critical field (14),[10] and the resolution of the paradox is due to Heisenberg and Euler [28], who remarked that electrons and positrons can be spontaneously produced in supercritical static electric fields.[11]

Here, we identify the reference displacement D_0 with $\epsilon_0 E_{\rm crit}$. Then, we can identify a modulus $Y_{\rm QED}$ of elasticity of (quantum) electromagnetism as,[12]

$$\begin{aligned} Y_{\rm QED} &= \frac{u_{\rm EM\ wave}}{h^2} = \frac{u_{\rm EM\ wave}}{(D_{\rm wave}/D_0)^2} = \frac{\epsilon_0 E_{\rm wave}^2}{(E_{\rm wave}/E_{\rm crit})^2} = \epsilon_0 E_{\rm crit}^2 \approx 3 \times 10^{25}\ {\rm Pa} \\ &\approx 10^{14} Y_{\rm steel} \approx 10^{-6} Y_{\rm spacetime}(f = 100\ {\rm Hz}). \end{aligned} \tag{15}$$

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[10] Sauter [25] attributed the concept of the critical field to Bohr.

[11] In the USA, the critical field (14) is often called the Schwinger field, from eq. (6.41) of [29].

[12] Since $E_{\rm crit}$ is related to "breakdown" of the vacuum, the quantity $Y_{\rm QED} = \epsilon_0 E_{\rm crit}^2$ of eq. (15), with dimensions of pressure, might better be called the ultimate tensile strength of the QED vacuum, following the spirit of Art. 111 of Maxwell's *Treatise* [23]. However, the ultimate tensile strength of an "ordinary" material is always approximately equal to its modulus of elasticity, so eq. (15) can be considered as providing an estimate of the modulus of elasticity of the quantum-electrodynamic vacuum.

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