Buquoy's Problem: Lifting a String from a Table

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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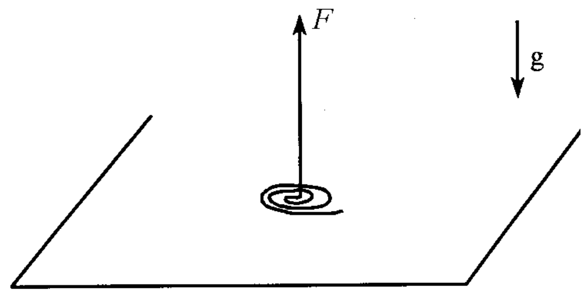
1 Problem

What force F is required to lift an inextensible string at constant acceleration a from a table, if effects of motion of the string on the table can be ignored.



Discuss also the cases of a constant upward force F, and of free fall of the string onto the table.

This example was first considered by Buquoy (1815) [1], in one of the earliest analyses of a variable-mass problem. See also [2]. However, the only reference to Buquoy in the 1800's may be that of Poisson (1819) [3].

2 Solution

For a string of mass λ per unit length whose upper end is at height h above the table, the total (vertical) force on the portion of the string above the table is $F - \lambda g h$, where F is the force on the upper end of the string. In this, we have ignored any possible force of the table on the string that has been lifted. The momentum of the string is $p = \lambda h \dot{h}$, such that,

$$F_{\rm tot} = F - \lambda g h = \dot{p} = \lambda h \ddot{h} + \lambda \dot{h}^2, \qquad F = \lambda h(g + \ddot{h}) + \lambda \dot{h}^2. \tag{1}$$

For constant acceleration a beginning at $t = 0$ when $h = h_0$ and $\dot{h} = v_0$, we have for $t > 0$ that,

$$\dot{h} = v_0 + at, \qquad h = h_0 + v_0 t + \frac{at^2}{2}, \qquad F = \lambda \left[\left(h_0 + v_0 t + \frac{at^2}{2} \right) (a + g) + (v_0 + at)^2 \right]. \tag{2}$$

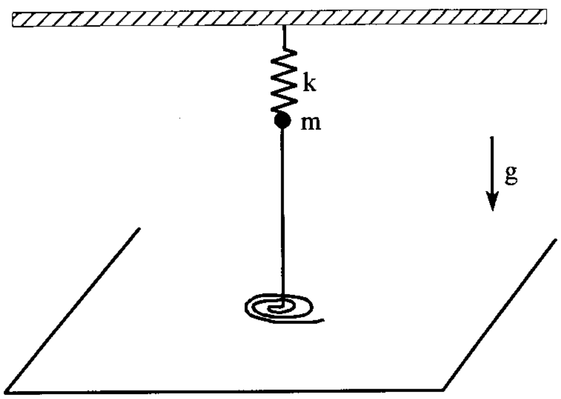
A nonzero initial velocity v_0 implies that a large impulse was applied at $t = 0$; in this scenario the force required to lift the string at this constant velocity for $t > 0$ would be $F = \lambda[g(h_0 + v_0 t) + v_0^2]$.

A delicacy in the above argument concerns the tension T in the string just above the table. This must be nonzero, since this is the force that accelerates length $\dot{h}\,dt$ of the string from rest on the table to vertical velocity $\dot{h}$ during time dt. The change in momentum during time dt of this segment of the string is $dp = \lambda\dot{h}^2\, dt$, which implies that $T = \lambda\dot{h}^2$.

If we suppose that there exists an equal and opposite force on the string, due to the string on the table, then in eq. (1) the total force on the portion of the string off the table would be $F - \lambda gh - T$, which would imply that $F = \lambda h(g + \ddot{h}) + 2\lambda\dot{h}^2$. However, this would be "double counting" of the term $\lambda\dot{h}^2$, and eqs. (1)-(2) are correct as is.

Moral: it is delicate to use Newton's third law in variable-mass problems.[1]

An elaboration of the present example is discussed in sec. 4 of [4], in which the upper end of the string is attached to a mass that is suspended from a ceiling by a spring.



2.1 Constant Upward Force

The equation of motion follows from eq. (1) as,

$$\frac{d\,h\dot{h}}{dt} = \frac{F}{\lambda} - gh, \qquad h\dot{h}\frac{d\,h\dot{h}}{dt} = \frac{F}{\lambda}h\dot{h} - gh^2\dot{h}, \tag{3}$$

which integrates to,

$$\frac{1}{2}(h\dot{h})^2 = \frac{F}{\lambda}\frac{h^2}{2} - \frac{gh^3}{3}, \qquad \dot{h}^2 = \frac{F}{\lambda} - \frac{2gh}{3}, \tag{4}$$

supposing that $h = 0 = \dot{h}$ at time $t = 0$. Note that if F is small enough, the increasing weight of the string off the table will reduce the upward velocity to zero when $h = 3F/2g\lambda$. If this h is less than the length l of the string, *i.e.*, $F < 2g\lambda/3$, the string falls back onto the table (as will be confirmed after eq. (6) below).

Supposing that F is larger than this, eq. (4) leads to,

$$t = \int_0^h \frac{dh}{\sqrt{F/\lambda - 2gh/3}} = \frac{3}{g}\left(\sqrt{\frac{F}{\lambda}} - \sqrt{\frac{F}{\lambda} - \frac{2gh}{3}}\right), \qquad h = \sqrt{\frac{F}{\lambda}}t - \frac{gt^2}{6}, \tag{5}$$

up to time

$$t = \frac{3}{g}\sqrt{\frac{F}{\lambda}}\left(1 - \sqrt{1 - \frac{2g\lambda l}{3F}}\right), \tag{6}$$

[1]This point has been emphasized in [5, 6], and is illustrated in the author's examples [7, 8].

when the string is lifted completely off the table.

However, if $F < 2g\lambda/3$, the string reaches a maximum height $h_{\rm max} = 3F/2g\lambda$ at time $t = 3\sqrt{F/\lambda}/g$, after which it falls freely back onto the table. The time of the free fall is $\Delta t = \sqrt{2h_{\rm max}/g} = \sqrt{3F/\lambda}/g$, and the motion of the strings lasts only for a total time $(3+\sqrt{3})\sqrt{F/\lambda}/g$.

2.2 Free Fall of the String onto the Table

If the lower end of a freely falling string reaches the table with velocity v, the potion of the string not yet on the table continues to fall freely for additional time $(v/g)(\sqrt{1+2gl/v^2}-1)$, after which the string is at rest on the table. For $v = 0$, the additional fall time is $\sqrt{2l/g}$, while for large v it is just l/v.

The above argument supposes that as the string strikes the table the kinetic energy of the string is absorbed by the table and/or the portion of the string on the table. Another possible view is that no such energy transfer occurs. If so, then the falling portion of the string has kinetic energy equal to the change in gravitational potential energy as the string falls. As a consequence, the kinetic energy of the falling string is finite as the length of that portion of the string goes to zero, with the implication that the velocity of the "tip" of the falling string diverges, and exceeds the speed of sound. This whip-like behaviour would imply that a "sonic boom" occurs as the string falls. However, such "sonic booms" are not observed in the lab.[2]

A variant of this problem is to ask what is the force/pressure of the chain on the table, particularly at the moment when the chain is first entirely on the table.[3]

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[2]The example of a spool of tape rolling down an incline does lead to a "slap" as the last bit of tape unwinds [11, 12].

[3]This problem was posed as Ex. 6, p. 149 of [19]. Related problems of falling chains were often considered at Cambridge U., as recounted in a footnote on p. 80 of [9]: *Problems on infinitesimal impulses were solved in the lecture room of the late Mr Hopkins as long ago as 1850. A problem of this kind was set in the Smith's Prize examination in 1853 by Prof. Challis, and a solution given in Tait and Steele's* Dynamics. For the latter, see pp. 250-251 of [10].

Also of note are papers from this time by Cayley [13, 14] and by Airy [20]

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