# Thermodynamics of a Tire Pump

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# 1 Problem

Although a tire pump is a "simple" device, making a satisfactory, simple thermodynamic model of its behavior is challenging.

Suppose the tire has volume $V$ that remains constant as its (absolute) pressure is increased by the tire pump from $P_0 \geq P_A$, where $P_A$ = atmospheric pressure, to a final pressure $P$ at the ambient (absolute) temperature $T_0$. The process of pumping may be assumed to be adiabatic, so the pressure and temperature in the tire reach maximum values of $P_{\rm max}$ and $T_{\rm max}$ before the tire cools slowly back to temperature $T_0$ and the pressure drops to the desired pressure $P$. You may assume that air is a diatomic ideal gas.

First discuss how a tire pump at a gas station works, before turning to the slightly more complicated case of a hand pump.

At a gas station, a compressor first compresses a lot of air to some desired pressure $P_C$ and stores this in a tank. Before this air is injected into a tire, it comes back to room temperature $T_0$. You may ignore the details of the thermodynamics of the compressor. Then, we connect the storage tank to the tire via a "one-way" valve. So long as the pressure in the tank is greater than the pressure in the tire, air flows from the tank into the tire.

The simple model is that the pressure in the compressor's tank stays constant during this process.

# 2 Solution

The tire of volume $V$ must end up with $n = PV/RT_0$ moles of air (assumed to be an ideal gas) to be at pressure $P$ at ambient temp $T_0$, where $R$ is the gas constant.

Suppose the tire has pressure $P_0$ to start with. Then it already contains $n_0 = n(P_0/P) = P_0V/RT_0$ moles of air. So, we must add $n' = n - n_0$ moles from the tank into the tire, which we can also express as,

$$n' = n\frac{P - P_0}{P}\,. \tag{1}$$

## 2.1 The Tire Is Pumped by a Compressor

The volume $V'$ of these $n'$ moles in the tank at pressure $P_C$ and temperature $T_0$ is,

$$V' = \frac{n'RT_0}{P_C} = nRT_0\frac{P - P_0}{PP_C}\,. \tag{2}$$

As air flows from the tank into the tire through the "one-way" valve, the tank does work $W_C$ at constant (by assumption) pressure $P_C$ given by,

$$W_C = P_CV' = nRT_0\frac{P - P_0}{P} \tag{3}$$

on the $n'$ moles of air. This work appears as internal energy of the air that is injected into the tire.

Given the assumption that air is an ideal gas, the motion of air through the "one-way" valve into the tire is a free expansion, and no work is done in the latter process. So, the internal energy $U$ of the air in the tire rises by amount $W_C$ (assuming that the air transfer is adiabatic).

Therefore, the temperature of the air in the tire rises to $T_{\rm max} = T_0 + \Delta T$, where $nC_V \Delta T = \Delta U = W_C$. Here, $C_V = 5R/2$ is the molar specific heat of air at constant volume. Thus,

$$\Delta T = \frac{W_C}{nC_V} = \frac{RT_0}{C_V}\frac{P - P_0}{P} = \frac{2}{5}\left(1 - \frac{P_0}{P}\right) T_0, \tag{4}$$

and,

$$T_{\rm max} = T_0 + \Delta T = \left[1 + \frac{2}{5}\left(1 - \frac{P_0}{P}\right)\right] T_0. \tag{5}$$

As a result, the pressure in the tire rises temporarily to,

$$P_{\rm max} = \frac{nRT_{\rm max}}{V} = \frac{T_{\rm max}}{T_0} P = \left[1 + \frac{2}{5}\left(1 - \frac{P_0}{P}\right)\right] P. \tag{6}$$

After a while, the air cools down to temperature $T_0$, and the pressure will be the desired value $P$.

However, the filling process won't work if $P_{\rm max} > P_C$.

Example: $P_0 = 1$ atm, $P = 3$ atm (these are absolute pressures, not"gauge" pressures), and $T_0 = 300$K $= 23$C. Then, $P_{\rm max} = 1.2.7P = 3.8$ atm, and $T_{\rm max} = 1.27T = 380$K $= 107$C. The pressure in the compressor must be greater than 3.8 atm (2.8 atm gauge pressure) to be able to pump the tire up to 3 atm (2 atm gauge pressure) quickly.

## 2.2 The Tire Is Pumped by a Hand Pump

Now we turn to the case of a hand pump, which has a working volume $V_p$ that is small compared to the volume $V$ of the tire.

The number $n_p$ of moles or air delivered in each stroke of the pump is,

$$n_p = \frac{P_A V_p}{RT_0} \ll n, \tag{7}$$

where $P_A$ = atmospheric pressure. So, a large number $N$ of strokes of the hand pump will be required,

$$N = \frac{n'}{n_p} = \frac{P - P_0}{P_A}\frac{V}{V_p}, \tag{8}$$

recalling eq. (1), and each stroke will make only a tiny difference in the pressure of the tire.

At the beginning of the $i$th stroke, the tire has pressure $P_{i-1}$, where for $i = 1$, $P_0$ = initial pressure in the tire.

My model is that during the $i$th stroke the pressure in the pump increases until it reaches pressure $P_i$, the pressure in the tire at the end of the stoke, after which the pump pressure is held constant at this value while the air in the pump is transferred into the tire.

I will again assume that all this happens quickly (*i.e.*, adiabatically).

We need to know the volume $V_p'$ of air in the pump when it first reaches pressure $P_i$. For, this, we use the law of adiabatic compression of an ideal gas: $PV^\gamma = PV^{7/5} =$ constant, where $\gamma = C_P/C_V = (7R/2)/(5R/2) = 7/5$ for air. Thus,

$$V_p' = V_p \left(\frac{P_A}{P_i}\right)^{5/7}. \tag{9}$$

The temperature of the air in the pump has now risen to,

$$T' = T_0 \frac{P_i V_p'}{P_A V_p} = T_0 \left(\frac{P_i}{P_A}\right)^{2/7}. \tag{10}$$

The temperature rise of the air in the pump is,

$$\Delta T' = T' - T_0 = T_0 \left[\left(\frac{P_i}{P_A}\right)^{2/7} - 1\right]. \tag{11}$$

To accomplish this, we must have done work $W'$ equal to the increase in the internal energy of the air in the pump, namely,

$$W' = n_p C_V \Delta T' = \frac{P_A V_p}{RT_0} \frac{5R}{2} T_0 \left[\left(\frac{P_i}{P_A}\right)^{2/7} - 1\right] = \frac{5}{2} P_A V_p \left[\left(\frac{P_i}{P_A}\right)^{2/7} - 1\right]. \tag{12}$$

Next, we push the hot gas of volume $V_p'$ into the tire at constant pressure $P_i$. This requires that we do additional work,

$$W'' = P_i V_P' = P_i V_p \left(\frac{P_A}{P_i}\right)^{5/7} = P_A V_p \left(\frac{P_i}{P_A}\right)^{2/7}. \tag{13}$$

So, the total work required to accomplish the $i$th stroke is,

$$W_i = W' + W'' = P_A V_p \left[\frac{7}{2}\left(\frac{P_i}{P_A}\right)^{2/7} - \frac{5}{2}\right]. \tag{14}$$

Not only is the energy $W_i$ added to that of the gas in the tire during the $i$th stroke, but also the initial energy $U_p = C_V n_p R T_0 = (5/2) P_A V_p$ of the gas in the pump. Thus, the total energy added to the gas during the $i$th stroke is,

$$\Delta U_i = W_i + \frac{5}{2} P_A V_p = \frac{7}{2} P_A V_p \left(\frac{P_i}{P_A}\right)^{2/7} = \frac{7}{2} P_A^{5/7} P_i^{2/7} V_p. \tag{15}$$

The internal energy $U$ of volume $V$ of air at pressure $P$ is related by,

$$U = C_V nRT = C_V PV = \frac{5}{2} PV, \tag{16}$$

so the change in pressure $\Delta P_i$ in the tire during the $i$th stroke is related by,

$$\Delta P_i = \frac{2}{5}\frac{\Delta U_i}{V} = \frac{7}{5} P_A^{5/7} P_i^{2/7} \frac{V_p}{V}. \tag{17}$$

As $\Delta P_i$ is small, we treat eq. (17) as a differential equation,

$$\frac{dP(i)}{P^{2/7}} = \frac{7}{5} P_A^{5/7} \frac{V_p}{V}\, di, \tag{18}$$

which integrates to,

$$P^{5/7} = \left( P_0^{5/7} + P_A^{5/7} \frac{V_p}{V}\, i \right), \tag{19}$$

or

$$P(i) = \left( P_0^{5/7} + P_A^{5/7} \frac{V_p}{V}\, i \right)^{7/5}. \tag{20}$$

After the $N$ strokes needed, according to eq. (8), to fill the tire so that it has pressure $P$ when it returns to temperature $T_0$, the maximum pressure in the tire is,

$$P_{\rm max} = \left( P_0^{5/7} + P_A^{5/7} \frac{P - P_0}{P_A} \right)^{7/5} = P_0 \left[ 1 + \left( \frac{P_A}{P_0} \right)^{5/7} \frac{P - P_0}{P_A} \right]^{7/5}, \tag{21}$$

which is independent of the volume $V_p$ of the hand pump. The maximum temperature of the air in the tire is,

$$T_{\rm max} = \frac{P_{\rm max}}{P} T_0, \tag{22}$$

The final internal energy of the air in the tire (at temperature $T_{\rm max}$) is $U_f = (5/2) P_{\rm max} V$, while the initial internal energy of the air in the tire plus the $N$ volumes $V_p$ of pumped air is $U_0 = (5/2)(P_0 V + N P_A V_p)$. Hence, the total work done while pumping is,[1]

$$\begin{aligned} W &= U_f - U_0 = \frac{5}{2}(P_{\rm max} V - P_0 V - N P_A V_p) \\ &= \frac{5}{2} P_0 V \left\{ \left[ 1 + \left( \frac{P_A}{P_0} \right)^{5/7} \frac{P - P_0}{P_A} \right]^{7/5} - \frac{P}{P_0} \right\}. \end{aligned} \tag{23}$$

Example: $P_A = P_0 = 1$ atm, $P = 3$ atm, $V = 0.1$ m$^3$, $V_p = 0.001$ m$^3$, and $T_0 = 300$K. Then, $N = 200$ strokes, $P_{\rm max} = 4.65$ atm, $T_{\rm max} = 465$K $= 192$C, and $W \approx 41{,}000$ J. To pump up the tire in 10 minutes requires $\approx 68$ W of power delivered into the air of the tire.

*An estimate of the efficiency of the engine that provides this power is beyond the scope of this problem.*

[1] July 20, 2022. Thanks to Kevin Zondervan for spotting several typos in and around eq. (23).

### 2.2.1 Entropy Change

The free expansion during filling of the tire, and the cooling at constant volume after filling, are both irreversible processes, so we expect that the entropy of the system has increased.

In classical thermodynamics, the entropy $S$ of an ideal gas can be calculated only up to an additive constant. Namely, $n$ moles of gas at temperature $T$ in a volume $V$ have entropy,

$$S = nR\left(A + \ln\frac{VT^{C_V}}{n}\right), \tag{24}$$

where $A$ is an undetermined constant.

In the present example, $n = PV/RT_0$ moles of air end up in the tire of volume $V$ at pressure $P$ and temperature $T_0$. Initially, this air was also at temperature $T$, with $n_0 = nP_0/P$ moles in volume $V$ at pressure $P_0$, and $n' = n(P - P_0)/P$ moles at pressure $P_A$ where they occupied volume $NV_p = V(P - P_0)/P_A$, recalling eqs. (1) and (8). Hence, the change in entropy of the air that ends up inside the tire during the pumping is,

$$\begin{aligned}
\Delta S_{\rm tire} &= nR\ln\frac{VT_0^{5/2}}{n} - n_0R\ln\frac{VT_0^{5/2}}{n_0} - n'R\ln\frac{V(P-P_0)T_0^{5/2}}{n'P_A} \\
&= nR\ln\frac{1}{n} - n_0R\ln\frac{P}{nP_0} - n'R\ln\frac{P}{nP_A} = -n_0R\ln\frac{P}{P_0} - n'R\ln\frac{P}{P_A} \\
&= -nR\left(\frac{P_0}{P}\ln\frac{P}{P_0} + \frac{P-P_0}{P}\ln\frac{P}{P_A}\right) = -nR\left(\ln\frac{P}{P_A} + \frac{P_0}{P}\ln\frac{P_A}{P_0}\right),
\end{aligned} \tag{25}$$

which is negative. However, this is not the total entropy change of the Universe, because as the tire cooled from temperature $T_{\rm max}$ back to $T_0$, heat $\Delta Q$, given by,

$$\Delta Q = C_V nR(T_{\rm max} - T_0) = \frac{5}{2}nRT_0\left(\frac{P_{\rm max}}{P} - 1\right), \tag{26}$$

was transferred into the environment at temperature $T_0$, so the entropy of the environment increased by $\Delta Q/T_0$. Thus, the total entropy change of the Universe during the pumping is,

$$\Delta S = nR\left\{\frac{5}{2}\frac{P_0}{P}\left[1 + \left(\frac{P_A}{P_0}\right)^{5/7}\frac{P-P_0}{P_A}\right]^{7/5} - \frac{5}{2} - \ln\frac{P}{P_A} - \frac{P_0}{P}\ln\frac{P_A}{P_0}\right\}. \tag{27}$$

For the example with $P_0 = P_1 = 1$ atm and $P = 3$ atm, $\Delta S = 0.28nR$.